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On Exact Solutions of Duffing Equation using the Modified Simple Equation Method

Ziad Rached
Department of Mathematics, Notre Dame University-Louaize, Lebanon.

Abstract

Constructing exact solutions of nonlinear ordinary and partial differential equations is an important topic in various disciplines such as Mathematics, Physics, Engineering, Astronomy, since many problems and experiments can be modeled via these equations. Various methods were applied to obtain exact solutions of Duffing differential equation In this correspondence, the modified simple equation method (MSEM) is applied to the Duffing equation, and thus adding more solutions to existing ones.

Keywords: Nonlinear differential equation, exact solutions, MSEM, Duffing equation**.**

1. INTRODUCTION

Many problems in mathematical physics are modeled using nonlinear differential equations. Finding exact solutions, if possible, is certainly favorable to approximate analytical solutions and to numerical solutions.

Several techniques were developed and successfully applied by many scientists such as the Exp-function method [1,2], the tanh-function method [3,4], the homogeneous balance method [5,6], the (G'/G)-expansion method [7,8], the Backlund transformation method [9], the Jacobi elliptic function method [10], the MSEM [11,12], the enhanced modified simple equation method [13].

In the sequel, we obtain exact solutions of Duffing DE

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(wt) \tag{1}$$

using the MSEM. Exact solutions using elliptic functions were obtained in [14]. Approximate solutions were obtained in [15]. Let us first describe the MSEM in section II. In Section 3, we apply the method to Duffing DE. Finally, we discuss the results in Section 4.

2. THE MSEM

Suppose we have a nonlinear ODE of the form

$$F(x, x', x'', x''', \dots) = f(t) \tag{2}$$

where F is a polynomial of $x(t)$ and its derivatives. The method involves the following steps.

Step 1: Suppose that equation (2) has the solution

$$u(\zeta) = \sum_{k=0}^{n} A_k \left[\frac{\varphi'}{\varphi}\right]^k \tag{3}$$

where A_k are constants and φ are unknown expressions to be obtained, $A_n \neq 0$.

Step 2: Compute n in equation (3). This is accomplished by balancing the highest order derivative and nonlinear term in equation (2).

Step 3: Substitute equation (3) into equation (2). Combine all the terms of the same power of $\varphi(\zeta)^{-j}$, where $j \geq 0$, and equate their coefficients to zero. This results in a system of algebraic and differential equations which can be solved to find A_k and φ. Consequently, we get a closed form solution of equation (2).

3. APPLICATION TO DUFFING DE

In this section, we solve the Duffing equation

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(wt)$$

Taking the balance between $\frac{d^2x}{dt^2}$ and x^3 , $(n + 2 = 3n)$, we get $n = 1$. Therefore, the solution has the form

$$x = A_0 + A_1\left(\frac{\varphi'}{\varphi}\right) \tag{4}$$

We have

$$x' = A_1\left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) \tag{5}$$

$$x'' = A_1\left(\frac{\varphi'''}{\varphi} - 3\frac{\varphi'\varphi''}{\varphi^2} + 2\left(\frac{\varphi'}{\varphi}\right)^3\right) \tag{6}$$

Substituting equations (4)-(6) in (1) and setting all the coefficients of φ^0, φ^{-1}, φ^{-2}, and φ^{-3} to zero, we get respectively

$$\alpha A_0 + \beta A_0^3 = \gamma cos(wt) \tag{7}$$

$$\varphi''' + \mu\varphi'' + \alpha\varphi' + 3\beta A_0^2\varphi' = 0 \tag{8}$$

$$-3\varphi'\varphi'' - \mu(\varphi')^2 + 3\beta A_0 A_1(\varphi')^2 = 0 \tag{9}$$

$$2 + \beta A_1^2 = 0 \tag{10}$$

From (7), by Cardano formula, one solution is given by the following formula

$$A_0 = \sqrt[3]{\frac{\gamma cos(wt)}{2\beta} + \sqrt{\left(\frac{\gamma cos(wt)}{2\beta}\right)^2 + \left(\frac{\alpha}{3\beta}\right)^3}} + \sqrt[3]{\frac{\gamma cos(wt)}{2\beta} - \sqrt{\left(\frac{\gamma cos(wt)}{2\beta}\right)^2 + \left(\frac{\alpha}{3\beta}\right)^3}}$$

The other two solutions may be obtained easily by the division algorithm and then solving a quadratic equation.

From (10), $A_1 = \pm\sqrt{-\frac{2}{\beta}}$. Hence, various solutions may be obtained depending on the values of A_0 and A_1.

From (9), we get

$$\varphi' = \left(\frac{3}{3\beta A_0 A_1 - \mu}\right)\varphi'' \tag{11}$$

From (11) and (8), we get

$$\frac{\varphi'''}{\varphi''} = -\left[\frac{\mu(3\beta A_0 A_1 - \mu) + 3(\alpha + 3\beta A_0^2)}{3\beta A_0 A_1 - \mu}\right] \tag{12}$$

Integrating equation (12) with respect to t yields

$$\varphi'' = B_1 exp\left(-\left[\frac{\mu(3\beta A_0 A_1 - \mu) + 3(\alpha + 3\beta A_0^2)}{3\beta A_0 A_1 - \mu}\right]t\right) \tag{13}$$

From (13) and (11), we get

$$\varphi' = \left(\frac{3}{3\beta A_0 A_1 - \mu}\right) B_1 exp\left(-\left[\frac{\mu(3\beta A_0 A_1 - \mu) + 3(\alpha + 3\beta A_0^2)}{3\beta A_0 A_1 - \mu}\right] t\right)$$

Or

$$\varphi' = CB_1 exp(-Dt) \tag{14}$$

Therefore

$$\varphi = B_2 - \frac{CB_1}{D} exp(-Dt) \tag{15}$$

Where B_1 and B_2 are arbitrary constants.

From (4), (14), and (15), it follows that

$$x = A_0 + A_1 \left(\frac{CB_1 exp(-Dt)}{B_2 - \frac{CB_1}{D} exp(-Dt)}\right) \tag{16}$$

4. CONCLUSION

The MSEM is applied successfully to Duffing DE without aid of a mathematical software. An explicit exact solution is obtained which is by no means easy to get using other approaches.

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