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Portfolio Optimization in Jump Model under Inefficiencies in the Market by Conditioning on the Information Flow

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Abstract

This paper considers an inefficient financial market with one bond and one stock where the dynamics of the stock price process modeled by jump-Diffusion process. We assume in the market there are two investors i.e. the investor with incomplete information who is only able to see the stock price process and the investor with full information about the market scenario. The aim of the investors is to maximize the expected utility from terminal wealth. We solve the maximum expected power utility problem by means of dynamic programming techniques by conditioning on the information flow and stated the Hamilton-Jacobi-Bellman equation as Integro-partial differential equation.

Keywords: Dynamic programming, Optimal portfolio, Jump model, Information flow, Power utilities

1. INTRODUCTION

We consider an inefficient financial market with one bond and one stock where the dynamics of the stock price process modeled by jump-Levy process. We assume there are two investors in the market i.e the investor with partial information who is only able to observe the stock price process and the investor with full information about the

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market scenario. In this paper we will treat these two aspects in one model. Most papers on problems with partial observation deal with the case of an unobserved (stochastic) appreciation rate process (μ_t). [1, 2] for example treats the case where the appreciation rate follows a linear Gaussian model. The most recent papers by [3] and [4] consider a Hidden Markov Model for (μ_t).

We solved the maximum expected power utility problem by means of dynamic programming techniques by conditioning on the information flow of each investors in which the the investors with partial information has sub optimal portfolio and stated the Hamilton-Jacobi-Bellman equation as Integro-partial differential equation.

2. Itô-Levy Diffusion

To study a portfolio of assets in a continuous time setting we need to describe the dynamics of the given assets by using the theory of stochastic differential equations.The risk asset will be modeled using a jump-diffusion process. Our theorems will be stated in the the one dimensional Itô-Levy Diffusion. The multidimensional versions of the theorems we use in this section can be found in an introductory stochastic calculus books such as [5].

Theorem 2.1. *Consider the Itô-Levy process stochastic differential equation on [s, T] of the form*

$$dX_t = \mu(t,w)dt + \sigma(t,w)dB(t) + \int_R \gamma(t,w,z)\tilde{N}(dt,dz) \quad Xs = x \tag{1}$$

where $\mu : [s,T] \times R \to R, \sigma : [s,T] \times R \to R$ *and* $\gamma : [s,T] \times R \times R \to R$ *satisfy the following Lipschitz and at most linear growth conditions*

- *There exists constant* $C_1 > 0$ *such that for all* $x, y \in R$

$$|\mu(t,x) - \mu(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \\ + \int_R |\gamma(t,x,z) - \gamma(t,y,z)|^2 \nu(dz) \leq C_1 |x-y|^2$$

- *There exists constant* $C_2 > 0$ *such that for all* $x \in R$

$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 + \int_R |\gamma(t,x,z)|^2 \nu(dz) \leq C_2(1 + |x|^2)$$

Then there exists a unique cadlag adapted solution X_t such that

$$E^{s,x}\left[\int_s^T |X_t|^2\right] < \infty$$

Through out this paper we will be working with the so called Geometric Levy process which satisfies the conditions for the existence and uniqueness for solution of the corresponding stochastic differential equation. The general form of a Geometric Levy process in differential form is given by the stochastic differential equation (SDE):

$$dX_t = \mu X_t dt + \sigma X_t dB(t) + X_{t^-}\int_R \gamma(t,z)\tilde{N}(dt,dz); \quad X_s = x. \tag{2}$$

whose unique solution can be obtained by applying the Itô-Levy theorem to $\ln(X_t)$.

Theorem 2.2. *Let $X_t \in R$ is an Itô-Levy process of the form given by equation (25) for some $R \in [0,\infty]$.*
Let $f \in C_0^{1,2}([0,\infty),R)$ and define $F(t) = f(t,X_t)$, then F_t is an Itô-Levy process and

$$\begin{aligned} dF(t) = {} & \frac{df}{dt}(t,X_t)dt + \frac{df}{dx}(t,X_t)\left\{\mu dt + \sigma dB(t)\right\} + \frac{1}{2}\frac{d^2 f}{dx^2}(t,X_t)dt \\ & + \int_{|z|<R}\left\{f(t+X_{t^-}+\gamma(t,z,w)) - f(t,X_{t^-}) - \frac{d^2 f}{dx^2}(t,X_t)\gamma(t,z,w)\right\}\nu(dz)dt \\ & + \int_R \left\{f(t,X_t+\gamma(t,X_{t^-},z)) - f(t,X_{t^-})\right\}\tilde{N}(dt,dz) \end{aligned}$$

we may write down the solution of the Geometric Levy process given by Equation (25)

$$\begin{aligned} X_t = x\exp\bigg\{ & (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \int_0^t \int_{|z|<R} \left\{\ln(1+\gamma(s,z)) - \gamma(s,z)\right\}\nu(dz)ds \\ & + \int_0^t \int_R \ln(1+\gamma(s,z))\tilde{N}(ds,dz)\bigg\} \end{aligned}$$

we must have the restriction that $1 + \gamma(s,z) > 0$ so that the logarithmic term is well defined.

Definition 2.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, a function $\tau : \Omega \to [0,\infty)$ is called a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ if $\omega : \tau(\omega) \leq t \in \mathcal{F}_t$.*
A stopping time is a random variable for which the set of all paths(events) $\omega \in \Omega$with $\tau(\omega) \leq t$ can be decided given the filtration $\mathcal{F}_t$.

once we have defined this random time we may state the strong Markov property.

Theorem 2.3. *(Strong Markov property for Itô jump diffusion) Let $(X_t)_{t\geq 0}$ be an Ito jump diffusion,τ a stopping time and $f : R^n \to R$ be a Borel measurable function, then for $\omega \in \Omega, h \geq 0$*

$$E^x[f(X_{\tau+h})|F_\tau](\omega) = E^{X_\tau(\omega)}[f(X_h)]$$

The generator of Itô-Levy diffusion.

Definition 2.2. *Let X_t Itô-Levy diffusion of the form*

$$dX_t = \alpha X_t dt + \sigma X_t dW(t) + X_{t^-}\int_R \gamma(t,z)\tilde{N}(dt,dz); \quad X_s = x,$$

if $f \in C_0^{1,2}([0,\infty), R)$ then the generator $\mathcal{A}$ of X_t for all $x \in R$ and $s \in [0,\infty)$ and is given by

$$\mathcal{A}f(s,x) = \lim_{t\to s}\frac{E^{s,x}[f(t,X_t)] - f(s,x)}{t-s}$$

To perform the analysis in the jump case using jump-diffusion theory we need to be able to use an equivalent version of the Hamilton Jacobi Bellman theorem. In order to write down the Hamilton Jacobi Bellman equation for the case with jumps we need to compute the infinitesimal generator of an Itô-Levy process, more specifically we need the generator of an Itô-Levy diffusion. The infinitesimal generator for a Levy process with jumps is found by using the Itô-Levy theorem with the fact that $\tilde{N}$ is a martingale, provided in the following theorem.

Theorem 2.4. *Let X_t Itô-Levy diffusion of the form*

$$dX_t = \alpha X_t dt + \sigma X_t dW(t) + X_{t^-}\int_R \gamma(t,z)\tilde{N}(dt,dz); \quad X_s = x,$$

if $f \in C_0^{1,2}([0,\infty), R)$ then $\mathcal{A}f(s,x)$ exist for all $x \in R$ and $s \in [0,\infty)$ and is given by

$$\begin{aligned}\mathcal{A}f(s,x) = &\frac{df}{dt}(s,x) + \alpha(s,x)\frac{df}{dx}(s,x) + \frac{1}{2}\sigma^2(s,x)\frac{d^2f}{dx^2}(s,x) + \\ &\int_R\left[f(s,x+\gamma(s,x,z)) - f(s,x) - \frac{df}{dx}\gamma(s,x,z)\right]\nu(dz)\end{aligned} \tag{3}$$

We have now the Hamilton Jacobi Bellman equation theorem for the case with jumps.

$$
\begin{aligned}
dV(t, X_t) = \frac{dV}{dt}(t, X_t)dt + \frac{dV}{dx}(t, X_t)\left\{\alpha dt + \sigma dW(t)\right\} + \frac{1}{2}\frac{d^2V}{dx^2}(t, X_t)dt \\
+ \int_{|z|<R} \left\{ V(t, X_{t^-} + \gamma(t, z, w)) - V(t, X_{t^-}) - \frac{d^2V}{dx^2}(t, X_t)\gamma(t, z, w) \right\} \nu(dz)dt \\
+ \int_R \left\{ V(t, X_t + \gamma(t, X_{t^-}, z)) - V(t, X_{t^-}) \right\} \tilde{N}(dt, dz)
\end{aligned}
$$

Theorem 2.5. *Let $f \in C^2(G) \in C \cap (\bar{G})$, suppose the following conditions hold*

- $\mathcal{A}^u f(y) + g^u(y) \leq 0$ *for all* $y \in G, u \in U$
- $f^-(Y_\tau)_{\tau \leq \tau_G}$ *is uniformly integrable for all* $u \in \mathcal{A}[0, \tau_G]$ *and* $y \in G$.
- $E^y[|f(Y_\tau)| + \int_0^{\tau_G}(|\mathcal{A}^u f(Y_t)| + |\sigma(Y_t)\frac{\partial f}{\partial y}(Y_t)|^2 + \int_R |f(Y_t + \gamma(Y_t, u_t, z)) - f(Y_t)|^2 \nu(dz))dt] < \infty$
- $Y_{\tau_G} \in \partial S$ *a.s. on* $\chi_{\tau_S < \infty}$ *and* $\lim_{t \to \tau_{S-}} f(Y_t) = h(Y_{\tau_S})\chi\tau_S < \infty$ *a.s for all* $u \in \mathcal{A}[0, \tau_G]$ *then* $f(y) \geq J^u(y)$ *for all Markov controls* $u \in \mathcal{A}$ *and* $y \in G$. *Moreover if for all* $y \in G$ *we find a Markov control* $u = u_0(y)$ *such that*

  $$g^{u_0(y)}(y) + \mathcal{A}^{u_0(y)} f(y) = 0$$

  then $u_t^* = u_0(y_t)$ *is optimal and* $f(y) = F(y) = J^{u^*}(y)$.

2.3 Verification Theorem for Levy Process with Jumps

we use the Hamilton Jacobi Bellman equation indirectly to solve the optimal control problem, the solution will come from an equivalent verification theorem for Levy jump processes.

Theorem 2.6. *Let* $u \in \mathcal{A}[s, \tau_G] and (s, x) \in G$ *and suppose the following conditions are satisfied for all* $s \in [0, \tau_G]$ *and* $x \in R$

- $f \in C^{1,2}([0, \tau_G) \times R)$ *is continuous on* $[0, \tau_G] \times R$ *and satisfies the quadratic growth condition* $|f(s, x)| \leq C_f(1 + |x|^2)$
- f *satisfies the Hamilton Jacobi Bellman equation* $\sup_{u \in \mathcal{A}[s, \tau_G]}[fu(s, x) + \mathcal{A}^u f(s, x)] = 0 s \in [0, \tau_G) f(\tau_G, x) = g(\tau_G, x)$
- f^u *is continuous with* $|f^u(s, x)| \leq C_f(1 + |x|^2 + ||u||^2)$ *for some constant* $C_f > 0$.
- $|\sigma_u(s, x)|^2 \leq C_\sigma(1 + |x|^2 + ||u||^2)$ *for some constant* $C_\sigma > 0$.
- $\int_R |\gamma(s, x, z)|^2 \nu(dz) \leq C_\gamma(1 + |x|^2 + ||u||^2)$, *then* $f(s, x) \geq F(s, x)$ *for all* $(s, x) \in G$. *Moreover if* $u_0(s, x)$ *is the max of* $u \to f^u(s, x) + \mathcal{A}^u f(s, x)$ *and* $u^* = u_0(s, X_s)$ *is admissible then* $f(s, x) = F(s, x)$ *for all* $(s, x) \in G$*and* u^* *is and optimal strategy i.e.* $F(s, x) = J^{u^*}(s, x)$.

3. Presenting the problem

Proposition 3.1. *We solve a finite horizon $T < \infty$ stochastic control problem of a rational and small investor endowed with a positive initial capital $W(0) = x > 0$ and described by a generic utility function $U(.)$, here a power utility function, whose goal is to maximize her expected utility from terminal wealth, i.e.: $E^P[U(.)(T)]$ by investing continuously in a risky, S, and in a risk-free asset, B. The optimal value of the problem, denoted by, $V_{\mathcal{I}_t}(t,x)$, is valid only if, for each $t \in [0,T]$ and $\omega \in \Omega$, there exists an optimal portfolio process, $\eta(t,\omega) = \eta^*_{\mathcal{I}_t}(t,\omega)$, which belongs to the set of admissible portfolios, $\mathcal{A}_{\mathcal{I}_t}$, s.t.:*

$$V_{\mathcal{I}_t}(t,x) = sup_{\eta \in \mathcal{A}_{\mathcal{I}_t}} E^P[\frac{1}{\theta}(W^{\eta(t)}(T))^{\theta}] = E^P[\frac{1}{\theta}(W^{\eta(t)^*}(T))^{\theta}] \tag{4}$$

where the value function of the problem is assumed to be:

$$V_{\mathcal{I}_t}(t,x) < \infty \quad \forall x \in [0,\infty) \tag{5}$$

Given proposition (2.1) the solution of the problem is strongly related to the admissibility of the portfolio with respect to the information set, $\mathcal{I}_t$. To better underline its importance, the same portfolio problem will be analyzed under two different information sets:

1. when the information set is complete (i.e $\mathcal{I}_t = \mathcal{F}_t$) in this case the investor is able to set up an asset pricing model that fully capture all past, present and future relevant pricing information, hence also and above all her forward looking beliefs with respect to the future outcome.
2. when some relevant information is missing, hence cannot be reflected in the final asset price (i.e $\mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t$) and the investor pertains to a more realistic suboptimal case.

Misleading unexisting arbitrages may be the simple consequence of an investor not being fully aware to relay to the former or the latter group.
To answer the above problem we model a simple economy made of two assets and we study the impact of a smaller information set onto a rational investor that, endowed with a positive initial capital, wants to maximize her final welfare choosing among the set of admissible portfolios:

$$V(x) = \sup_{\eta = \eta_t \in \mathcal{A}_{\mathcal{I}_t}} E^P[U(X^{\eta_t}(T))], t \in [0,T] \tag{6}$$

Under this framework, the conditionality of the expectation in (5) is represented by the set of feasible portfolios $\mathcal{A}_{\mathcal{I}_t}$ where $\mathcal{A}$ represents the set of admissible portfolios and its subscript $\mathcal{I}_t$ restricts this set to the information available at time t is defined as,

$$\mathcal{A}_{\mathcal{I}_t}(x) = \Big\{\eta = \eta_t \in [0,T] : \eta \quad \text{admissible} \quad, W(0)^\eta = 0, E^P[U(X^{\eta_t}(T))^-] < \infty\Big\}$$

We will compute the same optimization problem conditional to the complete and a suboptimal information set. Both random variables are then $\mathcal{I}_t$-adapted stochastic processes. The resolution of the optimization problem is linked to the arbitrary choice of a regular utility function $U(.) : (0,\infty] \to [-\infty,\infty)$. For simplicity we start with a power utility function:

$$U(x) = \frac{1}{\theta}x^\theta \quad x > 0 \tag{7}$$

The resolution of the asset pricing problem (5) leads to the optimal expected power utility of the investor terminal wealth $X^{\mu_T}(T)$. The entire optimization is performed under the physical measure, P, and conditional to the filtration set of the investor, $\mathcal{I}_t$. This justifies our partial information approach. For the problem it assumed a continuous time frame $0 \le t \le T \le \infty$ such that, for each time $t \ge 0$:

$$\mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t \subset \mathcal{F} \tag{8}$$

It follows that, depending on the filtration in use, the degree of adeptness (measurability) of the stochastic process may change thus impacting in various form on the final outcomes of the optimization. As an assumption: both filtration's lie in filtered probability spaces:

$$(\Omega, (\mathcal{H}_t)_{0\le t\le T}, P, H) \subset (\Omega, (\mathcal{F}_t)_{0\le t\le T}, P, F) \tag{9}$$

and satisfy the usual hypotheses. The goal of the presented framework is to model the economy of a single investor with a suboptimal information set, where the stochastic processes that affect the optimal portfolio choice are $\mathcal{H}_t$ and not $\mathcal{F}_t$-adapted, thus reflecting the poorer decisional power of the investor due to the lack of forward looking information.

4. The portfolio optimization problem

Probably as a direct consequence of the much larger theoretical literature relative to the enlargement of filtration with respect to the one relative to the shrinkage of filtration, the same degree of richness is reflected on their different applications, i.e.portfolio optimization. Equation (3) reflects the familiar stochastic control problems linked to the insider information [6]. Following [7] our starting point is a Levy-Itô market model

composed by two assets: one risky and one risk-free. Given this simple economy, the investor implements her portfolio through a dynamic trading of the two assets. The choice of modeling the stochastic part of the risky investment by means of a Brownian motion (Itô process) and a pure jump process (Levy process) instead of just using the classical diffusion-Itô model is justified by the higher descriptive power of the former. For modelling details we refer to [8] and and [9].
The two assets that compose our economy are:
A risk-free asset is represented by a risk-free bond B_0, whose unit price $B_0(t)$ at time t is:

$$dB_0(t) = r(t)B_0(t)dt \quad t \in [0, T] \quad B_0(0) = 1 \tag{10}$$

where $r \geq 0$ is a constant. The risky asset $S(t)$ evolves according to a Geometric Levy A risky asset, represented by a stock S driven by a one dimensional Brownian motion and a pure jump process (Levy-Itô process), whose unit price $S(t)$ at time t is:

$$dS(t) = S(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_R \gamma(t,z)\tilde{N}(dt,dz)\right] \quad S(0) > 0 \quad t \in [0, T] \tag{11}$$

By the Itô formula the solution of (10) is:

$$S(t) = S(0)\exp\{\int_0^t \left\{\mu(s) - \frac{1}{2}\sigma^2(s) - \int_R (\gamma(s,z) - ln(1+\gamma(s,z)))\nu_{\mathcal{I}_t}(dz)\right\} ds \\ + \int_0^t \sigma(s)dB(s) + \int_0^t \int_R ln(1+\gamma(s,z))\tilde{N}(dt,dz)\}; t \in [0, T] \tag{12}$$

Assumptions
For each $t \in [0, T]$, $\omega \in \Omega$ and $z \in R - \{0\}$ we assume that the parameters of the continuous part of the process $\mu(t, \omega)$, $\sigma(t, \omega)$ and $W(t, \omega)$ satisfy the following assumptions:

1 $\mathcal{I}_t$-progressively measurable, hence time dependent and non-anticipating

2 bounded on $[0, T] \times \Omega$

3 parameters $\mu(t) = \mu(t, \omega)$ and $\sigma(t) = \sigma(t, \omega)$ represent respectively the investors' expected returns form $S(t)$ and the volatility of $S(t)$ and

4 $W(t) = W(t, \omega)$ is an $\mathcal{I}_t$-adapted one dimension Brownian Motion

5 $\tilde{N}(dt, dz) = N(dt, dz) - \nu\mathcal{I}_t(dz)dt$ is the $\mathcal{I}_t$-compensated Poisson random measure of $\alpha(t) = \alpha(t, \omega) : [0, T] \times \Omega \to R$ where:

$$\alpha(t) = \int_0^t \int_R z\tilde{N}(dt, dz) \tag{13}$$

6 to prevent the process to be ≤ 0, we set: $z > -1$. This means we may only assume jump sizes that are larger than -1.

7 $E^P[\alpha^2(t)] < \infty$ for all $t \geq 0$

Now, suppose that $\eta(t) = \eta(t, \omega) : [0, T] \times \Omega \to R, \omega \in \Omega$ is an $\mathcal{I}_t$-measurable stochastic process representing the fraction of wealth $W(t)$ invested the investor in the risky asset and $(1 - \eta(t))$ is invested in the risk-free asset, then the evolution of the total wealth process, $W(t) = W^{\eta(t)}(t)$, of the investor by assuming we have a self financing trading strategy is:

$$\begin{gathered} dW(t) = (1 - \eta(t))W(t)r(t)dt + \eta(t)W(t-)\left\{\mu(t)dt + \sigma(t)dB(t) + \int_R z\tilde{N}(dt, dz)\right\} \\ W(0) = x > 0 \quad \text{(Initial capital)} \quad \text{or, collecting terms:} \end{gathered} \tag{14}$$

$$\begin{aligned} dW(t) &= W(t-)\left\{r(t) + (\mu(t) - r(t))\eta(t)\right\}dt + \sigma(t)\eta(t)dB(t) \\ &+ \eta(t)\int_R z\tilde{N}(dt, dz) t \in [0, T] W(0) = x > 0 \quad \text{(Initial capital)} \end{aligned} \tag{15}$$

A key element for the analysis of the paper is how to define an admissible portfolio under the different filtration's in use.

Definition 4.1. *Given a small and rational investor, a porfolio process $\eta(t)$ is assumed to be $\mathcal{I}_t$-admissible for each $t \in [0, T]$ if:*

i is $\mathcal{I}_t$-adapted for each t, where $0 < t < T < \infty$

ii $E^P \int_0^T [|\mu(t) - r(t)||\eta(t)| + \sigma^2(t)\eta^2(t) + \eta^2(t)\int_R \alpha^2(t)\nu_{\mathcal{I}_t}(dz)dt] < \infty$

iii $\alpha(t) \geq -1$ *a.s. for* $dt \times \nu_{\mathcal{I}_t}dz$ *for a.a.* t *and* z

Given this framework we analyze the stochastic control problem of a small investor whose goal is to maximize $E^P(U^{\eta(t)}(T))$ over a finite horizon in continuous time $t \in [0, T]$ and over the class of all possible time (t) admissible portfolios $\mathcal{A}_{\mathcal{I}_t}(t)$.
While is at least since Merton (1969)[10] and Samuelson (1969) [11] that the stochastic control problem for $(\mathcal{F}_t) \quad t > 0 = F$-adapted portfolios is a well-known problem in literature (a good review of the subject is, among the others, Cvitanic and Karatzas (1992) [12]), the one for an Itô-Lévy market model with a suboptimal information set is not.

5. Optimal Portfolio Under full information(Theoretical case($\mathcal{I}_t = \mathcal{F}_t$))

In this chapter we solve the portfolio optimization problem for an investor with power utility function under the theoretical case: when the filtration set is the complete one by using dynamic programming method.

5.0.1 Hamilton-Jacobi-Bellman equation

The stochastic control approach is based on showing that under certain conditions the value function equation (3) satisfies the Hamilton-Jacobi-Bellman (HJB) equation by using the following Levy-itô formula.

$$
\begin{aligned}
dV(t,W_t) = &\left\{\frac{dV}{dt}(t,W_t) + [\eta_t(\mu_t - r_t) + r_t]W_t\frac{dV}{dw}(t,W_t) + \frac{1}{2}W_t^2\eta_t^2\sigma_t^2\frac{d^2V}{dw^2}(t,W_t)\right\}dt \\
&+ \eta_t W_t \sigma \frac{dV}{dw}dB(t) + \int_R \{V(t, W_{t^-} + \eta_t W_{t^-}\gamma(t,z,W_t)) - V(t,W_{t^-}) \\
&- \eta_t W_{t^-}\frac{dV}{dw}(t,W_t)\gamma(t,z,W_{t^-})\}\nu(dz)dt \\
&+ \int_R \{V(t, W_t + \eta_t\gamma(t,W_{t^-},z)) - V(t,W_{t^-})\}\tilde{N}(dt,dz)
\end{aligned}
$$

Showing $V_{\mathcal{I}_t}(t,x) = sup_{\eta\in\mathcal{A}_{\mathcal{I}_t}} E[(\tau, W^{\eta(t)}(T))|W^\eta = x]$ for stopping time $\tau \geq t$, substituting in for $\tau = t$ the representation $V(s, W_s)$ above, and taking expectation in which the term term with dB_t and $\tilde{N}(dt,dz)$ have expectation 0 under some conditions ,and letting $s \to t$ by mean-value theorem we get HJB equation.

$$
\begin{aligned}
&\sup_{\eta\in\mathcal{F}_s} E^P\{\frac{dV}{dt}(t,x) + [\eta_{\in\mathcal{F}_s}(\mu - r) + r]x\frac{dV}{dw}(t,x) + \frac{1}{2}x^2\eta^2_{\in\mathcal{F}_s}\sigma^2\frac{d^2V}{dw^2}(t,x) \\
&+ \int_R \{V(t, x + \eta_{\in\mathcal{F}_s}x\gamma(t,z)) - V(t,x) - \eta_{\in\mathcal{F}_s}x\frac{dV}{dw}(t,z)\gamma(t,z)\}\nu_{\mathcal{F}_s}(dz)\} = 0
\end{aligned}
\tag{16}
$$

The HJB approach then consists of three steps:

- Finding the maximizer η^* in (14) as a function of t, x, and of the partial derivatives of V .
- Plugging η^* back into the HJB equation yields a partial differential equation for V. The solution V can be put into η^* and using $x = W_t^*$ at t one obtains candidate ${\eta^*}_t$ for the optimal strategy.
- Finally it has to be verified that V obtained from solving the HJB equation is indeed

the value function. Typical verification results require appropriate smoothness and growth conditions.

Often, step (2) can only be solved numerically and thus step (3) may be difficult.Then this approach only yields a candidate for the optimal policy. An important step towards a solution is a reduction of the dimension. Since we consider power utility we expect that V scales as

$$V(t,x) = \frac{x^{\theta}}{\theta} h(t), x > 0.$$

Under sufficient smoothness of V , partial derivatives then are

$$V_t = \frac{x^{\theta}}{\theta} h_t, \quad V_x = x^{\theta-1} h(t), \quad V_{xx} = (\theta-1) x^{\theta-2} h(t)$$
$$\text{and } \ V(t, x + \eta_{\in\mathcal{F}_s} x \gamma(t,z)) = \frac{x^{\theta}}{\theta} h(t)(1 + \eta_{\in\mathcal{F}_s}\gamma)^{\theta}$$

By substituting this in equation (14) and dividing $\frac{x^{\theta}}{\theta}$ we get

$$\begin{aligned} &\sup_{\eta\in\mathcal{F}_s}\{h_t + [\eta_{\in\mathcal{F}_s}(\mu - r) + r]h(t) + \frac{1}{2}\eta^2_{\in\mathcal{F}_s}\sigma^2(\theta-1)h(t) \\ &+ h(t)\int_R \{(1+\eta_{\in\mathcal{F}_s}\gamma(t,z))^{\theta} - 1 - \eta_{\in\mathcal{F}_s}\gamma(t,z)\}\nu_{\mathcal{F}_s}(dz)\} = 0 \end{aligned} \tag{17}$$

So the optimal portfolio under full information $\eta^*_{\in\mathcal{F}_s}$ satisfy

$$\frac{x^{\theta}}{\theta} h(t)\{(\mu - r) + \eta^*_{\in\mathcal{F}_s}\sigma^2(\theta-1) + \int_R\{\theta\gamma(1+\eta^*_{\in\mathcal{F}_s}\gamma(t,z))^{\theta-1} - \gamma(t,z)\}\nu_{\mathcal{F}_s}(dz)\} = 0 \tag{18}$$

and exists if $\mu - r \leq \sigma^2(1-\theta) + \int_R[1-\theta(1+\gamma)^{\theta-1}]\gamma\nu_{\mathcal{F}_s}(dz)$
The Hamilton-Jacobi-Bellman equation as Integro-partial differential equation is written as follows,

$$\begin{aligned} h_t + \{\eta^*_{\in\mathcal{F}_s}(\mu - r) + r + \frac{1}{2}\eta^{*2}_{\in\mathcal{F}_s}\sigma^2(\theta-1) + \int_R\{(1+\eta^*_{\in\mathcal{F}_s}\gamma(t,z))^{\theta} - 1 \\ -\eta^*_{\in\mathcal{F}_s}\gamma(t,z)\}\nu_{\mathcal{F}_s}(dz)\}h(t) = 0 \end{aligned} \tag{19}$$

Hence $h(t)$ satisfy the ODE with terminal value

$$h'(t) + Lh(t) = 0 \quad , \quad h(T) = 1$$

So we have $V(x,t) = \frac{x^\theta}{\theta} \exp L(T-t)$ where,

$$L = \eta^*_{\in\mathcal{F}_s}(\mu - r) + r + \frac{1}{2}\eta^{*2}_{\in\mathcal{F}_s}\sigma^2(\theta-1) + \int_R \{(1+\eta^*_{\in\mathcal{F}_s}\gamma(t,z))^\theta - 1 - \eta^*_{\in\mathcal{F}_s}\gamma(t,z)\}\nu_{\mathcal{F}_s}(dz)$$

6. Optimal Portfolio Under partial information(real-word case ($\mathcal{I}_t = \mathcal{H}_t) \subset \mathcal{F}_t$)

we solve the same portfolio (3) optimization problem for an investor with power utility function with partial observation i.e the trader at time t does not have access to all the information $\mathcal{F}_t$ that can be obtained by observing the underlying Brownian motion $B(s); s \le t$ and jump process $\tilde{N}([0,t], F); \bar{F} \in R - 0$. This will be the situation if, for example, the trader only observes the stock prices $S(s); s \le t$ and not the underlying processes. Therefore,we focus on both theoretically and conceptually, on how the missing information of the filtration set propagates and affect onto the final profit of the investor. Except $\eta \in \mathcal{H}_t$ and $\nu_{\mathcal{H}_s}(dz)$ the other parameters are affected under partial information contents given sub optimal filtration set and we use the same procedure as the complete one.

$$\begin{aligned}\sup_{\eta\in\mathcal{H}_s} E^{\hat{P}}\{\frac{dV}{dt}(t,x) + [\eta_{\in\mathcal{F}_s}(\mu-r)+r]x\frac{dV}{dw}(t,x) + \frac{1}{2}x^2\eta^2_{\in\mathcal{H}_s}\sigma^2\frac{d^2V}{dw^2}(t,x) \\ + \int_R\{V(t,x+\eta_{\in\mathcal{H}_s}x\gamma(t,z)) - V(t,x) - \eta_{\in\mathcal{H}_s}x\frac{dV}{dw}(t,z)\gamma(t,z)\}\nu_{\mathcal{H}_s}(dz)\} = 0\end{aligned} \tag{20}$$

Given the evolution of the total wealth of the investor and assuming that:

$$E^{\hat{P}}[\frac{1}{\theta}(W^{\eta(t)}(T))^\theta] < \infty$$

To find the optimal portfolio we use extra conditioning of the expectation with respect to $\mathcal{H}_t$:

$$\begin{aligned}\sup_{\eta\in\mathcal{H}_s} E^{\hat{P}}[E^P\{\frac{dV}{dt}(t,x) + [\eta_{\in\mathcal{H}_s}(\mu-r)+r]x\frac{dV}{dw}(t,x) + \frac{1}{2}x^2\eta^2_{\in\mathcal{H}_s}\sigma^2\frac{d^2V}{dw^2}(t,x) \\ + \int_R\{V(t,x+\eta_{\in\mathcal{H}_s}x\gamma(t,z)) - V(t,x) - \eta_{\in\mathcal{H}_s}x\frac{dV}{dw}(t,z)\gamma(t,z)|\mathcal{H}_s\}\nu_{\mathcal{H}_s}(dz)\}] = 0\end{aligned} \tag{21}$$

By reduction of the dimension as equation (15) we have

$$\sup_{\eta\in\mathcal{H}_s} E^{\hat{P}}[E^P\{h_t+[\eta_{\in\mathcal{H}_s}(\mu-r)+r]h(t)+\frac{1}{2}\eta^2_{\in\mathcal{H}_s}\sigma^2(\theta-1)h(t)+ \\ h(t)\int_R\{(1+\eta_{\in\mathcal{H}_s}\gamma(t,z))^\theta-1-\eta_{\in\mathcal{H}_s}\gamma(t,z)|\mathcal{H}_s\}\nu_{\mathcal{H}_s}(dz)\}]=0 \tag{22}$$

which is given as below after conditioning,

$$\sup_{\eta\in\mathcal{H}_s} E^{\hat{P}}[h_t+[\eta_{\in\mathcal{H}_s}(\hat{\mu}-r)+r]h(t)+\frac{1}{2}\eta^2_{\in\mathcal{H}_s}\widehat{\sigma^2}(\hat{\theta}-1)h(t)+ \\ h(t)\int_R\{E^P[(1+\eta_{\in\mathcal{H}_s}\gamma(t,z))^\theta-1-\eta_{\in\mathcal{H}_s}\gamma(t,z)]|\mathcal{H}_s\}\nu_{\mathcal{H}_s}(dz)]=0 \tag{23}$$

where $\hat{\mu}=E^{\hat{P}}[\mu_t|\mathcal{H}_s]$, $\hat{\theta}=E^{\hat{P}}[\theta|\mathcal{H}_s]$.
So the optimal portfolio under partial information $\eta^*_{\in\mathcal{H}_s}$ satisfy

$$\frac{x^{\hat{\theta}}}{\hat{\theta}}h(t)\{(\hat{\mu}-r)+\eta^*_{\in\mathcal{H}_s}\widehat{\sigma^2}(\hat{\theta}-1)+\int_R\{E^P[\theta\gamma(1+\eta^*_{\in\mathcal{H}_s}\gamma(t,z))^{\theta-1} \\ -\gamma(t,z)]|\mathcal{H}_s\}\nu_{\mathcal{H}_s}(dz)\}=0 \tag{24}$$

We can write the Hamilton-Jacobi-Bellman equation as the following Integro-partial differential equation.

$$h_t+\{\eta^*_{\in\mathcal{H}_s}(\hat{\mu}-r)+r+\frac{1}{2}\eta^{*2}_{\in\mathcal{H}_s}\widehat{\sigma^2}(\hat{\theta}-1)+\int_R\{(1+\eta^*_{\in\mathcal{F}_s}\gamma(t,z))^\theta-1 \\ -\eta^*_{\in\mathcal{H}_s}\gamma(t,z)\}\nu_{\mathcal{H}_s}(dz)\}h(t)=0 \tag{25}$$

Hence $h(t)$ satisfy the ODE with terminal value

$$h'(t)+Lh(t)=0 \quad , \quad h(T)=1$$

So we have $V(x,t)=\frac{x^\theta}{\theta}\exp L(T-t)$ where,

$$L=\eta^*_{\in\mathcal{F}_s}(\hat{\mu}-r)+r+\frac{1}{2}\eta^{*2}_{\in\mathcal{H}_s}\widehat{\sigma^2}(\hat{\theta}-1)+\int_R\{(1+\eta^*_{\in\mathcal{H}_s}\gamma(t,z))^\theta-1 \\ -\eta^*_{\in\mathcal{H}_s}\gamma(t,z)\}\nu_{\mathcal{H}_s}(dz)$$

7. CONCLUSIONS

In this paper,we solved the maximum expected utility problem by means of dynamic programming techniques by conditioning on the information flow and stated the Hamilton-Jacobi-Bellman equation as Integro-partial differential equation for the investors who wants to maximize the expected utility from terminal wealth where we assume there two investors in the market i.e the investor with partial information who is only able to observe the stock price process and the investor with full information about the market scenario.We considered an inefficient financial market with one bond and one stock where the dynamics of the stock price process modeled by jump-Levy process.

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