Bifurcation and Stability in Discrete Fractional Order Prey Predator Model with Harvesting

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Abstract

This work investigates the dynamical behavior of a harvesting in fractional order prey - predator model at discrete time. Here the prey population is continuously harvested at a linear function rate by a harvesting agency. Discretization process is applied to the original system, in order to obtain the discrete version. Existence of fixed points is established and a detailed analysis of the stability of fixed points is carried out with the linearization process and the impact of harvesting on the ecosystem is discussed for the fractional order discretized system. Simulations are used to verify the correctness of the analytical results. Time plots, phase portraits and bifurcation figures are plotted to study the dynamics of the system. Finally the system is found to be sensitive to the initial conditions.

Keywords: Predator - prey system, stability, discretization, harvesting, bifurcation.

1. INTRODUCTION

Fractional calculus is a generalization of integration and differentiation to non-integer order (fractional) operators. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference associated with Leibniz and L’Hospital in 1695. In recent years, numerous studies and applications of fractional order systems in many areas of science and engineering have been carried out [2, 7, 8].

Mathematical modeling is one of the most important applications of fractional calculus which also has applications in different fields of sciences such as economics, engineering and biological systems. Recently, both mathematical modeling and simulation of the ecological systems have received a great attention due to the development in technology. In order to interpret the phenomena in nature and provide a guide to analyze, various types of predator-prey models are described by system of
differential equations, difference equations, partial differential equations and stochastic differential equations. Many mathematical models have been constructed based on more realistic biological assumptions. Stability and dynamical analysis of fractional order Lotka Volterra models can be found in [1, 3].

The development of subsequent predator-prey models is mostly based on the Lotka-Volterra model. The development may be a change of functional response (e.g., Holling type I, II, and III classifications) or numerical response to describe different ecological processes in the predator-prey interaction. In order to incorporate the influence of seasonal variation into the predator-prey interaction, predator-prey models with periodically varying parameters were also proposed and investigated [4, 5, 6].

From the point of view of human needs, the exploitation of biological resources and the harvesting of populations are commonly practiced in fishery, forestry and wildlife management. There is a wide range of interest in the use of bio-economic models to gain insight into the scientific management of renewable resources like fisheries and forestry’s. It is necessary to consider the harvesting of populations in some models. However, human need is not invariable for a long time. In detail, human need increases as biological resource become abundant, while human need decreases as biological resource is exiguous. Thus, we focus on harvesting rate in the form of linear function of $x$ (prey) for the predator prey model.

Let us consider the fractional order prey - predator model with harvesting, describing the interactions among two species by the following system of equations:

$$
D^\alpha x(t) = rx(t)(1-x(t)) - ax(t)y(t) - hx(t)
$$
$$
D^\alpha y(t) = bx(t)y(t) - cy(t)
$$

(1)

where $r$ is an intrinsic birth rate of prey, $a$ is the death rate of prey, $b$ growth rate of predator in the presence of prey, $c$ is the natural death rate of predator and $h$ represents the harvesting of prey population. While $x$ and $y$ represent the densities of the prey and predator populations respectively. The harvesting activity does not affect the predator population directly. It is obvious that the harvesting activity does reduce the predator population indirectly by reducing the availability of the prey to the predator. Here the parameters $r, a, b, c, h$ are positive and $\alpha$ is the fractional order satisfying $\alpha \in (0,1]$.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

The following lemma will be useful to study the existence and uniqueness of the solution for system (1).

**Lemma 1** Consider the system

$$
D^\alpha x(t) = f(t,x), t > t_0
$$

(2)

with initial condition $x_0$, where $\alpha \in (0,1]$, $f:[t_0, \infty) \times \Omega \rightarrow \mathbb{R}^n, \Omega \in \mathbb{R}^n$, if $f(t,x)$
satisfies the locally lipschitz condition with respect to \( x \), then there exists a unique solution of (2) on \([t_0, \infty) \times \Omega\).

Let \( \Omega = \{(x, y) : \max \{|x|, |y|\} \leq \beta\} \) and \( \Omega \times (0, T) \) is taken as a region for existence and uniqueness of the solution of the system (1).

We denote \( X = (x, y), \bar{X} = (\bar{x}, \bar{y}) \). Consider a mapping \( G(X) = (G_1(X), G_2(X)) \) and

\[
G_1(X) = rx(t)(1 - x(t)) - ax(t)y(t) - hx(t)
\]

\[
G_2(X) = bx(t)y(t) - cy(t)
\]

For any \( X, \bar{X} \in \Omega \), it follows from (3) that

\[
\|G(X) - G(\bar{X})\| = |G_1(X) - G_1(\bar{X})| + |G_2(X) - G_2(\bar{X})|
\]

\[
= |rx - rx^2 - axy - hx - r\bar{x} + r\bar{x}^2 + a\bar{x}\bar{y} + h\bar{x}|
\]

\[
+ |bx - cy - b\bar{x}\bar{y} + cy|
\]

\[
= |r(x - \bar{x}) - r(x^2 - \bar{x}^2) - a(xy - \bar{x}\bar{y}) - h(x - \bar{x})|
\]

\[
+ b(xy - \bar{x}\bar{y}) - c(y - \bar{y})|
\]

\[
\leq (r + 2r\beta + h + ab\beta)\|x - \bar{x}\|
\]

\[
+ (c + ab\beta)\|y - \bar{y}\|
\]

\[
\leq B\|X - \bar{X}\|
\]

where \( B = \max \{r + 2r\beta + h + ab\beta, c + ab\beta\} \)

Thus \( G(X) \) satisfies the Lipschitz condition with respect to \( X \). It follows from the lemma 1 that there exists a unique solution \( X(t) \) of system (1) with initial condition \( X_0 = (x_0, y_0) \). Consequently we have the following theorem.

**Theorem 2.** For each \( X_0 = (x_0, y_0) \in \Omega \), there exists a unique solution \( X(t) \in \Omega \) of system (1) with initial condition \( X_0 \), which is defined for all \( t \geq t_0 \).

**3. MATHEMATICAL ANALYSIS OF THE SYSTEM**

Now, assume that \( x(0) = x_0, y(0) = y_0 \) are the initial conditions of system (1). Then discretization of system (1) with piecewise constant argument [9] is:

\[
x_{n+1} = x_n + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( rx_n \left(1 - x_n\right) - ax_n y_n - hx_n\right)
\]

\[
y_{n+1} = y_n + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( bx_n y_n - cy_n\right).
\]

The fixed points of system (4) are:

- both populations are extinct (trivial point) \( E_0 = (0, 0) \)
• only prey population survies (axial point) \( E_1 = \left( \frac{r-h}{r}, 0 \right) \),

• coexistence (interior point) \( E_2 = \left( \frac{c}{b}, \frac{br-bh-cr}{ab} \right) \).

Dynamics of the discretized fractional-order prey-predator model (4) has been investigated here. The system matrix of model (4) evaluated at any fixed point \((x^*, y^*)\) is given by

\[
J(x^*, y^*) = \begin{bmatrix}
1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( r(1-2x^*) - ay^* - h \right) & -\frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( ax^* \right) \\
\frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( by^* \right) & 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( bx^* - c \right)
\end{bmatrix}
\]

(5)

The characteristic equation of the system matrix can be written as

\[
\lambda^2 - Tr\lambda + Det = 0
\]

(6)

where \( Tr \) is the trace and \( Det \) is the determinant of the system matrix \( J(x^*, y^*) \), and they are

\[
Tr(J) = 2 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( r(1-2x^*) - ay^* - h + bx^* - c \right)
\]

\[
Det(J) = \left( 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( r(1-2x^*) - ay^* - h \right) \right) \left( 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( bx^* - c \right) \right) + \left( \frac{s^\alpha}{\alpha \Gamma(\alpha)} \right)^2 \left( by^* \right) \left( ax^* \right)
\]

(7)

The following lemma which can be easily proved by relation between roots and coefficient of quadratic equation (6) is useful.

**Lemma 3.** If the characteristic polynomial is \( P(\lambda) = \lambda^2 - a_1\lambda + a_2 \) where \( a_1 \) is the trace and \( a_2 \) the determinant of the system matrix. Then we have the following conditions

- \(|\lambda_1| < 1 \) and \(|\lambda_2| < 1\) if and only if \( p(-1) > 0 \) and \( a_2 < 1 \).
- \(|\lambda_1| < 1 \) and \(|\lambda_2| > 1 \) (or \(|\lambda_1| > 1 \) and \(|\lambda_2| < 1\)) if and only if \( p(-1) < 0 \).
- \(|\lambda_1| > 1 \) and \(|\lambda_2| > 1 \) if and only if \( p(-1) > 0 \) and \( a_2 > 1 \).
- \(|\lambda_1| = -1 \) and \(|\lambda_2| \neq 1\) if and only if \( p(-1) = 0 \) and \( a_i \neq 0,2 \).
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\[ \lambda_1 \text{ and } \lambda_2 \text{ are complex and } |\lambda_1| = |\lambda_2| \text{ if and only if } a_1^2 - 4a_2 < 0 \text{ and } a_2 = 1. \]

**Theorem 4.** If \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{r-h}} \) and \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{c}} \) then the fixed point \( E_0 \) is stable. If \( s > \sqrt{\frac{2a\Gamma(\alpha)}{r-h}} \) and \( s > \sqrt{\frac{2a\Gamma(\alpha)}{c}} \), then \( E_0 \) is unstable.

**Proof:** The system matrix at \( E_0 \) is

\[
J(E_0) = \begin{bmatrix}
1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}(r-h) & 0 \\
0 & 1 - \frac{s^\alpha}{\alpha \Gamma(\alpha)}(c)
\end{bmatrix}.
\]

Hence, the eigenvalues are \( \lambda_1 = 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}(r-h) \) and \( \lambda_2 = 1 - \frac{s^\alpha}{\alpha \Gamma(\alpha)}(c) \). Thus \( E_0 \) is stable, if \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{r-h}} \) and \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{c}} \). Otherwise \( E_0 \) is unstable when \( s > \sqrt{\frac{2a\Gamma(\alpha)}{r-h}} \) and \( s > \sqrt{\frac{2a\Gamma(\alpha)}{c}} \).

**Theorem 5.** If \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{h-r}} \) and \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)(r)}{br-bh-cr}} \) then \( E_1 \) is stable. If \( s > \sqrt{\frac{2a\Gamma(\alpha)}{h-r}} \) and \( s > \sqrt{\frac{2a\Gamma(\alpha)(r)}{br-bh-cr}} \) then \( E_1 \) is unstable.

**Proof:** The system matrix at \( E_1 \) is

\[
J(E_1) = \begin{bmatrix}
1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}(h-r) & -\frac{s^\alpha}{\alpha \Gamma(\alpha)}(a)\left(\frac{r-h}{r}\right) \\
0 & 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}\left(\frac{br-bh-cr}{r}\right)
\end{bmatrix}.
\]

The eigenvalues are \( \lambda_1 = 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}(h-r) \) and \( \lambda_2 = 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)}\left(\frac{br-bh-cr}{r}\right) \).

Therefore the fixed point \( E_1 \) is stable when \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)}{h-r}} \) and \( 0 < s < \sqrt{\frac{2a\Gamma(\alpha)(r)}{br-bh-cr}} \). \( E_1 \) is unstable if \( s > \sqrt{\frac{2a\Gamma(\alpha)}{h-r}} \) and \( s > \sqrt{\frac{2a\Gamma(\alpha)(r)}{br-bh-cr}} \).
Theorem 6. The positive fixed point \((x^*, y^*)\) of the prey-predator system (4) is stable if and only if
\[
B < \frac{c[br - bh - cr]}{4 + 2\left(\frac{s^a}{a\Gamma(a)}(\delta)^2 + \left(\frac{a}{a\Gamma(a)}\right)^2\right)} \quad \text{and} \quad B > \frac{c[br - bh - cr]}{4 + 2\left(\frac{s^a}{a\Gamma(a)}(\delta)^2 + \left(\frac{a}{a\Gamma(a)}\right)^2\right)}.
\]

Proof. The system matrix evaluated at the fixed point \(E_2\) has the form
\[
J(E_2) = \begin{bmatrix}
1 + \frac{s^a}{a\Gamma(a)}(\delta) & -\frac{s^a}{a\Gamma(a)}\left(\frac{a}{b}\right) \\
\frac{s^a}{a\Gamma(a)}\left(\frac{br - bh - cr}{a}\right) & 1
\end{bmatrix}.
\]

where \(\delta = \left(\frac{r - 2rc}{b} - \left[\frac{br - bh - cr}{a}\right] - h\right).\)

The trace and determinant of the system matrix \(J(E_2)\) are given by
\[
TrJ(E_2) = 2 + \frac{s^a}{a\Gamma(a)}(\delta) \quad \text{and} \quad DetJ(E_2) = 1 + \frac{s^a}{a\Gamma(a)}(\delta) + \beta.
\]

where \(\beta = \left(\frac{s^a}{a\Gamma(a)}\right)^2\left(\frac{c}{b}\right)[br - bh - cr].\)

According to the Jury conditions [8], we must have that \(P(1) > 0\) if and only if \(s < \sqrt{\frac{a\Gamma(a)}{2(\delta)}}(\beta)\). Similarly, \(P(-1) > 0\) and \(DetE_2 < 1\) holds if and only if
\[
b < \frac{c[br - bh - cr]}{4 + 2\left(\frac{s^a}{a\Gamma(a)}(\delta)^2 + \left(\frac{a}{a\Gamma(a)}\right)^2\right)} \quad \text{and} \quad b > \frac{c[br - bh - cr]}{4 + 2\left(\frac{s^a}{a\Gamma(a)}(\delta)^2 + \left(\frac{a}{a\Gamma(a)}\right)^2\right)},
\]
respectively.

4. NUMERICAL SIMULATIONS

Dynamic behavior of the system is examined numerically in this section. Time plots, phase portraits and bifurcation figures are used to investigate the dynamics of the system (4).

Example 1. With the initial conditions \(x(0) = 0.5, y(0) = 0.2\), the parameters \(\alpha = 0.95, r = 0.26, a = 0.4, b = 1.8, c = 2.37, h = 0.02\), and \(s = 0.1\), the eigenvalues are \(\lambda_1 = 0.9725\) and \(\lambda_2 = 0.9189\). So that \(|\lambda_{1,2}| < 1\). Here the prey population becomes
stable at $x = 0.9230$ and predator approaches zero. Hence the axial fixed point $E_1$ is locally asymptotically stable (see figure-1).

**Example 2.** For the values $\alpha = 0.9; r = 0.26, a = 0.4, b = 0.8, c = 0.15, h = 0.02,$ and $s = 0.1$ with the initial conditions $x(0) = 0.5, y(0) = 0.2$. Prey and predator populations approaches the equilibrium points and becomes stable at $x = 0.1875, y = 0.4781$. Here the eigenvalues are satisfied the conditions that $|\lambda_1| < 1$. Hence the interior fixed point $E_2$ is locally asymptotically stable (see figure-2).

**5. BIFURCATIONS**

In this section, we plot the bifurcation figures of the system (2) for different bifurcation parameters in some particular ranges. The parameters are assigned for different set of values.

Figure 3 shows the sudden changes happening in the system (2) and 3(a) is the local
magnification of period doubling bifurcation in the range 5.5 to 6 where $r$ is the bifurcation parameter. Fig 3(b) is periodic halving bifurcation. A period halving bifurcation in a dynamical system is a bifurcation in which the system switches to a new behavior with half the period of the original system. A series of period-halving bifurcations leads the system from chaos to order. Here we fix the fractional order $\alpha$ as a bifurcation parameter. Fig 3(c) is another period doubling bifurcation when $s$ is the bifurcation parameter.

In the absence of harvesting the bifurcation occurs in the range 3 to 5.5, it is shown in figure 4(a). Local magnifications of periodic windows are shown in 4(b) to 4(d). Hence from the period doubling bifurcation we see that the dynamic behavior of the system changes from stability to chaos then after certain period again it undergoes the process of stability to chaos. Also we see that with harvesting the dynamic behavior of the system appearing in the range 4 to 6 and in the absence of harvesting, it occurs in the range 3 to 5.5.

Fig 3 & 3(a). Period doubling bifurcation for prey population with harvesting.

Fig 3(b) Periodic halving bifurcation & Fig 3(c) Period doubling bifurcation for different bifurcation parameter.
6. SENSITIVE DEPENDENCE TO INITIAL CONDITIONS

Now, we see the changes occurring in the dynamical behavior of prey population as there are changes to initial conditions. The sensitivity to initial conditions is a characteristic of chaos. Here we take two set of values to analyze the behavior of the system to show its dependence to initial conditions.

1. In the presence of harvesting, the values are \( a = 0.375, b = 0.514, c = 0.734, h = 0.9, \alpha = 0.85, \) and \( s = 0.45, \) with initial conditions \( x(0) = 0.4, \ y(0) = 0.7 \) and \( x(0) = 0.4001, \ y(0) = 0.7 \)

2. In the absence of harvesting, the values are \( a = 0.375, b = 0.514, c = 0.734, \alpha = 0.85, \) and \( s = 0.45, \) with initial conditions \( x(0) = 0.4, \ y(0) = 0.7 \) and \( x(0) = 0.4001, \ y(0) = 0.7 \). The changes occurring
in the dynamical behavior of prey population as there is a sensitive change in the initial conditions are plotted in the following figures 5 and 6. From the figures we observed that, at the beginning the time plots are indistinguishable but after a number of iterations, the difference between them builds up rapidly.

Figure 5: Sensitive dependence to initial conditions of prey population with harvesting

Figure 6: Sensitive dependence to initial conditions of prey population without harvesting
7. CONCLUSION

In this paper, the dynamical behavior of a discrete fractional order prey - predator model with harvesting is investigated. Discretization process is applied to the original system, in order to obtain the discrete version. The fixed points are computed and a detailed analysis of the stability of fixed points is carried out with the linearization process and the impact of harvesting on the ecosystem is discussed for the fractional order discretized system. Numerical simulations are presented to illustrate the stability occurring in the system. Time plots, phase portraits and bifurcation figures are plotted to study the dynamics of the system. Finally the system is found to be sensitive to the initial conditions.

REFERENCES


