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On Almost strongly Lindelöf and Weakly strongly Lindelöf spaces

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Abstract

In this paper we introduce the notion of almost strongly Lindelöf and weakly strongly Lindelöf spaces as a generalization of strongly Lindelöf spaces depending on preopen cover with countable subfamily whose closures covers the space, or whose union is dense in the space, and give some characterizations and properties of these spaces. We study their subspaces and the relation between these topological properties and the subspaces.

Keywords : Almost strongly Lindelof, Weakly strongly Lindelof, preopen sets, preopen cover, dense union, subspaces, topological properties.

Subjclass : 54D20; 54B05

1. INTRODUCTION

A lot of attention has been given to covering properties on topological spaces specially those covers by open sets and covers by preopen sets. In [9] Mashhour defined the notion of strongly compact (resp. Lindelöf) by requiring that each preopen cover has a finite (resp. a countable) subcover, and many papers study these spaces such as M. Ganster [4, 5], Al. Omari, T. Noiri and M. Noorani in [1], and H. Z. Hdeib and M. Sarsak in [6].

In this work we define a characterizations of strongly Lindeöf spaces witch we will call almost strongly Lindeöf, where each preopen cover has a countable subfamily at which the union of its closures covers the space, weakly strongly Lindeöf) spaces, where each preopen cover has a countable subfamily with dense union. Many characterizations will be introduced to those properties and many properties will be given. Also we will study the subspaces of these spaces and investigate their properties. Also we introduce the relation between these properties and some kinds of their subspaces.

Throughout this work, a space X means a topological space (X, τ) on which no separation axioms are assumed. The interior and the closure of a subset A of X will be denoted by $int(A)$ and $cl(A)$ respectively.

2. PRELIMINARIES

Let X be a topological space, then a subset A of X is called preopen (resp. preclosed) if $A \subseteq int(cl(A))$ (resp. $cl(int(A)) \subseteq A$). A is called regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $cl(int(A) = A$), and A is called α-open if $A \subseteq int(cl(int(A)))$. Note that the complement of a preopen subset is preclosed and the complement of a regular open subset is regular closed. Also a subset A of X is called semi-open if $A \subseteq cl(int(A))$, the complement of a semi-open subset is called semi-closed.

The preclosure of a subset A of X is the smallest preclosed set containing A, denoted by $pcl(A)$, and the preinterior of a subset A of X is the largest preopen subset contained in A, and will be denoted by $pint(A)$. It is well known that $pcl(A) = A \cup cl(int(A))$ and $pint(A) = A \cap int(cl(A))$, see [2].

Definition 2.1. [7]. A subset A of a topological space (X, τ) is called pre-regular p-open if $A = pint(pcl(A))$.

One observes that $A \subseteq X$ is pre-regular p-open if and only of A is the pre-interior of some pre-closed subset, and if $S \subseteq X$ is preopen and $A = pint(pcl(S))$ then $pcl(S) = pcl(A)$.

Theorem 2.2. *[10]. Let A and B be subsets of a topological space (X, τ). Then*

(i) If A is preopen in X and B is semi-open in X then $A \cap B$ is preopen in B.

(ii) If A is preopen in B and B is preopen in X then also A is preopen in X.

Lemma 2.3. *[3]. Let $B \subseteq A \subseteq X$ and A be semi-open in X. Then $pcl_A(B) \subseteq pcl_X(B)$.*

Definition 2.4. [9]. A topological space (X, τ) is called strongly Lindelöf if every preopen cover $\{A_\alpha \ ; \ \alpha \in \Delta\}$ has a countable subcover.

3. ON ALMOST STRONGLY LINDELÖF SPACES AND SUBSPACES

Definition 3.1. A topological space (X, τ) is called almost strongly Lindelöf if every preopen cover $\{A_\alpha \ ; \ \alpha \in \Delta\}$ has a countable subfamily $\{A_{\alpha_n} \ ; \ n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} pcl(A_{\alpha_n})$.

Its clear that every strongly Lindelöf space is almost strongly Lindelöf, but the converse is not true as in the next example.

Example 3.2. Let $I = [0, 1]$ and A be an uncountable set disjoint from I, and let $X = I \cup A$. A topology τ on X is defined as : for each $x \in I$, $\{x\}$ is open and a basis open neighborhood of $a \in A$ has the form $\{a\} \cup I$. Since (X, τ) is not Lindelöf then it is not strongly Lindelöf (recall that every strongly Lindelöf space is Lindelöf).

Let $S \subseteq X$ be a preopen subset of X and let $a \in S$ for some $a \in A$, then $\{a\} \cup I \subseteq S$, so $pcl(\{a\} \cup I) \subseteq pcl(S)$. But $pcl(\{a\} \cup I) = (\{a\} \cup I) \cup cl(int(\{a\} \cup I)) = (\{a\} \cup I) \cup cl(\{a\} \cup I) = cl(\{a\} \cup I) = X$ because I is dense in X, so $pcl(S) = X$. Let $\{S_\alpha \ ; \ \alpha \in \Delta\}$ be a preopen cover of X, then there exists $a \in A$ and $\alpha_a \in \Delta$ such that $a \in S_{\alpha_a}$, so $pcl(S_{\alpha_a}) = X$. Therefore X is almost strongly Lindelöf.

Theorem 3.3. *Let (X, τ) be a topological space. The following are equivalent :*

(i) X almost strongly Lindelöf.

(ii) For any family $\{C_\alpha \ ; \ \alpha \in \Delta\}$ of preclosed subsets of X such that $\cap_{\alpha\in\Delta} C_\alpha = \phi$, there exists a countable subfamily $\{C_{\alpha_n} \ ; \ n \in \mathbb{N}\}$ such that $\cap_{n\in\mathbb{N}} pint(C_{\alpha_n}) = \phi$.

(iii) For every family $\{C_\alpha \ ; \ \alpha \in \Delta\}$ of preclosed subsets of X for which every countable subfamily $\{C_{\alpha_n} \ ; \ n \in \mathbb{N}\}$ satisfies $\cap_{n\in\mathbb{N}} pint(C_{\alpha_n}) \neq \phi$, the intersection $\cap_{\alpha\in\Delta} C_\alpha \neq \phi$.

Proof. (ii) $\Leftrightarrow$ (iii) is clear by contraposition.

(i) $\Rightarrow$ (ii). Suppose that X is almost strongly Lindelöf and let $\{C_\alpha \ ; \ \alpha \in \Delta\}$ be a family of preclosed subsets of X such that $\cap_{\alpha\in\Delta} C_\alpha = \phi$, so $X = X - \cap_{\alpha\in\Delta} C_\alpha = \cup_{\alpha\in\Delta}(X - C_\alpha)$. Hence $\{X - C_\alpha \ ; \ \alpha \in \Delta\}$ is a preopen cover of X and since X is almost strongly Lindelöf then there exists a countable subfamily $\{X - C_{\alpha_n} \ ; \ n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} pcl(X - C_{\alpha_n}) = \cup_{n\in\mathbb{N}}(X - pint(C_{\alpha_n})) = X - \cap_{n\in\mathbb{N}} pint(C_{\alpha_n})$. Therefore, $\cap_{n\in\mathbb{N}} pint(C_{\alpha_n}) = \phi$.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds and let $\{A_\alpha\ ;\ \alpha \in \Delta\}$ be a preopen cover of X, then $X = \cup_{\alpha\in\Delta} A_\alpha$, so $X - \cup_{\alpha\in\Delta} A_\alpha = \phi$ hence $\cap_{\alpha\in\Delta}(X - A_\alpha) = \phi$, so $\{X - A_\alpha\ ;\ \alpha \in \Delta\}$ is a family of preclosed subsets of X with empty intersection, from (ii), there exists a countable subfamily $\{X - A_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $\cap_{n\in\mathbb{N}} pint(X - A_{\alpha_n}) = \phi$. So $\phi = \cap_{n\in\mathbb{N}} pint(X - A_{\alpha_n}) = \cap_{n\in\mathbb{N}} X - pcl(A_{\alpha_n}) = X - \cup_{n\in\mathbb{N}} pcl(A_{\alpha_n})$. Thus $X = \cup_{n\in\mathbb{N}} pcl(A_{\alpha_n})$. Therefore, X is almost strongly Lindelöf. □

Theorem 3.4. *A topological space (X, τ) is almost strongly Lindelöf if and only if every cover $\{V_\alpha\ ;\ \alpha \in \Delta\}$ of X by pre-regular p-open subsets of X has a countable subfamily $\{V_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} pcl(V_{\alpha_n})$.*

Proof. $\Rightarrow$) is clear.

$\Leftarrow$) Let $\{A_\alpha\ ;\ \alpha \in \Delta\}$ be a preopen cover of X. Let $V_\alpha = pint(pcl(A_\alpha))$ for all $\alpha \in \Delta$. Then $\{V_\alpha\ ;\ \alpha \in \Delta\}$ is a cover of X by pre-regular p-open subsets of X, so there exists a countable subfamily $\{V_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} pcl(V_{\alpha_n}) = \cup_{n\in\mathbb{N}} pcl(pint(pcl(A_{\alpha_n}))) \subseteq \cup_{n\in\mathbb{N}} pcl(pcl(A_{\alpha_n})) = \cup_{n\in\mathbb{N}} pcl(A_{\alpha_n})$. Therefore, X is almost strongly Lindelöf. □

Definition 3.5. [3] A space X is called strongly p-regular if for every $x \in X$ and each preclosed subset F not containing x, there exists disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$.

It is clear that X is strongly p-regular if for every $x \in X$ and every preopen set V containing x, there exists a preopen set U such that

$$x \in U \subseteq pcl(U) \subseteq V.$$

Theorem 3.6. *If a topological space X is almost strongly Lindelöf and strongly p-regular then X is strongly Lindelöf.*

Proof. Let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ be a preopen cover of X. For each $x \in X$ let $\{V_{\alpha_x}\ ;\ \alpha_x \in \Delta\}$ such that $x \in V_{\alpha_x}$, so there exits a preopen set U_x such that $x \in U_x \subseteq pcl(U_x) \subseteq V_{\alpha_x}$, hence $\{U_x\ ;\ x \in X\}$ is a preopen cover of X, there exists a countable subfamily $\{U_{x_n}\ ;\ n \in \mathbb{N}\}$ such that

$$X = \cup_{n\in\mathbb{N}} pcl(U_{x_n}) \subseteq \cup_{n\in\mathbb{N}} (V_{\alpha_{x_n}}).$$

Therefore, X is strongly Lindelöf. □

Corollary 3.1. *Let X be a strongly p-regular space. Then X is almost strongly Lindelöf if and only if X is strongly Lindelöf.*

Definition 3.7. A subset A of a topological space X is said to be almost strongly Lindelöf if A is almost strongly Lindelöf as a subspace of X, i.e. A is almost strongly Lindelöf with respect to the induced topology on A by X.

Definition 3.8. A subset A of a topological space X is said to be almost strongly Lindelöf relative to X if for every cover $\{U_\alpha\ ;\ \alpha \in \Delta\}$ of A by preopen subsets of X there exists a countable subfamily $\{U_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $A \subseteq \cup_{n\in\mathbb{N}} pcl(U_{\alpha_n})$.

Theorem 3.9. *Let X be a topological space. If every preopen subset of X is almost strongly Lindelöf relative to X then every subset of X is almost strongly Lindelöf relative to X.*

Proof. Let B be any subset of X and $\{U_\alpha\ ;\ \alpha \in \Delta\}$ ba a cover of B by preopen subsets of X. Since the union of preopen subsets is preopen, then $\{U_\alpha\ ;\ \alpha \in \Delta\}$ is a cover of $\cup_{\alpha\in\Delta} U_\alpha$ by preopen subsets of X, so there exists a countable subfamily $\{U_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $\cup_{\alpha\in\Delta} U_\alpha \subseteq \cup_{n\in\mathbb{N}} pcl(U_{\alpha_n})$, so $B \subseteq \cup_{n\in\mathbb{N}} pcl(U_{\alpha_n})$. Therefore, B is almost strongly Lindelöf relative to X. □

Theorem 3.10. *Let X be a topological space and A be a subset of X. Then the following are equivalent :*

(i) A is almost strongly Lindelöf relative to X.

(ii) For every family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X such that $(\cap_{\alpha\in\Delta} C_\alpha) \cap A = \phi$, there exists a countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $(\cap_{n\in\mathbb{N}} pint(C_{\alpha_n})) \cap A = \phi$.

(iii) For every family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X for which every countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ satisfies $(\cap_{n\in\mathbb{N}} pint(C_{\alpha_n})) \cap A \neq \phi$, the intersection $(\cap_{\alpha\in\Delta} C_\alpha) \cap A \neq \phi$.

Proof. (ii) is equivalent to (iii) by contraposition.

(i) $\Rightarrow$ (ii). Suppose that A is almost strongly Lindelöf relative to X and let $\{C_\alpha\ ;\ \alpha \in \Delta\}$ be a family of preclosed subsets of X such that $(\cap_{\alpha\in\Delta} C_\alpha) \cap A = \phi$, so $A \subseteq X - \cap_{\alpha\in\Delta} C_\alpha = \cup_{\alpha\in\Delta}(X - C_\alpha)$, hence $\{X - C_\alpha\ ;\ \alpha \in \Delta\}$ is a cover of A by preopen subsets of X, so there exists a countable subfamily $\{X - C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $A \subseteq \cup_{n\in\mathbb{N}} pcl(X - C_{\alpha_n})$, hence $\phi = (X - \cup_{n\in\mathbb{N}} pcl(X - C_{\alpha_n})) \cap A = (\cap_{n\in\mathbb{N}}(X - pcl(X - C_{\alpha_n}))) \cap A = (\cup_{n\in\mathbb{N}}(X - (X - pint(C_{\alpha_n})))) \cap A = (\cap_{n\in\mathbb{N}} pint(C_{\alpha_n})) \cap A$.

(ii) $\Rightarrow$ (i). Suppose that (iii) holds and let $\{U_\alpha\ ;\ \alpha \in \Delta\}$ be a cover of A by preopen subsets of X, so $A \subseteq \cup_{\alpha\in\Delta} U_\alpha$, hence $(X - \cup_{\alpha\in\Delta} U_\alpha) \cap A = \phi$, so $(\cap_{\alpha\in\Delta}(X - U_\alpha)) \cap A = \phi$, hence $\{X - U_\alpha\ ;\ \alpha \in \Delta\}$ is a family of preclosed subsets of

X and $\cap_{\alpha\in\Delta}(X-U_\alpha)\cap A=\phi$ then by the contraposition of (iii) there exists a countable subfamily $\{X-U_{\alpha_n}\ ;\ n\in\mathbb{N}\}$ such that $\cap_{n\in\mathbb{N}}pint(X-A_{\alpha_n})=\phi$. From (ii) there exists a countable subfamily $\{X-A_{\alpha_n}\ ;\ n\in\mathbb{N}\}$ such that $\cap_{n\in\mathbb{N}}pint(X-U_{\alpha_n})\cap A=\phi$, so $A\subseteq X-\cap_{n\in\mathbb{N}}pint(X-U_{\alpha_n})=X-\cap_{n\in\mathbb{N}}(X-pcl(U_{\alpha_n}))=X-(X-\cup_{n\in\mathbb{N}}pcl(U_{\alpha_n}))=\cup_{n\in\mathbb{N}}pcl(U_{\alpha_n})$ So $\phi=\cap_{n\in\mathbb{N}}pint(X-U_{\alpha_n})=\cap_{n\in\mathbb{N}}X-pcl(A_{\alpha_n})=X-\cup_{n\in\mathbb{N}}pcl(A_{\alpha_n})$. So we obtain $X=\cup_{n\in\mathbb{N}}pcl(A_{\alpha_n})$. Therfore, A is almost strongly Lindelöf relative to X. □

Theorem 3.11. *Let X be a topological space such that every proper preclosed subset of X is almost strongly Lindelöf relative to X. Then X is almost strongly Lindelöf.*

Proof. Let $\{U_\alpha\ ;\ \alpha\in\Delta\}$ ba a preopen cover of X and let U_{α_0} be one them. Then $\{U_\alpha\ ;\ \alpha\in\Delta\}-\{U_{\alpha_0}\}$ is a cover of the proper preclosed subset $X-U_{\alpha_0}$ by preopen subsets of X. Since $X-U_{\alpha_0}$ is almost strongly Lindelöf relative to X, there exists a countable subfamily $\{U_{\alpha_n}\ ;\ n\in\mathbb{N}\}$ such that $X-U_{\alpha_0}\subseteq\cup_{n\in\mathbb{N}}pcl(U_{\alpha_n})$. Since $U_{\alpha_0}\subseteq pcl(U_{\alpha_0})$ then $X=(\cup_{n\in\mathbb{N}}pcl(U_{\alpha_n}))\cup pcl(U_{\alpha_0})$. Therefore X is almost strongly Lindelöf. □

Theorem 3.12. *Let A be a semi-open subset of a topological X. If A is an almost strongly Lindelöf subspace of X then A is almost strongly Lindelöf relative to X.*

Proof. Let $\{V_\alpha\ ;\ \alpha\in\Delta\}$ ba a cover of A by preopen subsets of X. Since A is semi-open in X then by theorem 2.3, $A\cap U_\alpha=W_\alpha$ is preopen in A, now $\{W_\alpha\ ;\ \alpha\in\Delta\}$ is a preopen cover of A. Since A is an almost strongly Lindelöf subspace there exists a countable subfamily $\{W_{\alpha_n}\ ;\ n\in\mathbb{N}\}$ such that $A=\cup_{n\in\mathbb{N}}pcl_A(W_{\alpha_n})\subseteq\cup_{n\in\mathbb{N}}pcl(W_{\alpha_n})\subseteq\cup_{n\in\mathbb{N}}pcl(V_{\alpha_n})$. Therefore, A is almost strongly Lindelöf relative to X. □

Theorem 3.13. *Let A be a preopen subset of a topological X. If A is an almost strongly Lindelöf relative to X then A is almost strongly Lindelöf subspace of X.*

Proof. Let $\{V_\alpha\ ;\ \alpha\in\Delta\}$ ba a cover of A by preopen subsets of A. Since A is preopen in X then by theorem 2.3, V_α is preopen in X for every $\alpha\in\Delta$, so $\{V_\alpha\ ;\ \alpha\in\Delta\}$ is a cover of A by preopen subsets of X. There exists then a countable subfamily $\{V_{\alpha_n}\ ;\ n\in\mathbb{N}\}$ such that $A\subseteq\cup_{n\in\mathbb{N}}pcl(V_{\alpha_n})$. Therefore, A is an almost strongly Lindelöf subspace of X. □

Theorem 3.14. *Let (X,τ) be a topological space. If there exsits a proper semi-closed and semi-open subset A of X such that A and $X-A$ are almost strongly Lindelöf subspaces of X, then X is almost strongly Lindelöf.*

Proof. Let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ ba a preopen cover of X. Since A is semi-open in X then by theorem 2.3, $A \cap V_\alpha = U_\alpha$ is preopen in A, so $\{U_\alpha\ ;\ \alpha \in \Delta\}$ is a preopen cover of (A, τ_A). Since A is an almost strongly Lindelöf subspace there exists a countable subset Δ_1 of Δ such that $A = \cup_{n\in\Delta_1} pint_A(pcl_A(U_{\alpha_n})) \subseteq \cup_{n\in\Delta_1} pint(pcl(U_{\alpha_n})) \subseteq \cup_{n\in\Delta_1} pint(pcl(V_{\alpha_n}))$. Since $X - A$ is semi-open in X and almost strongly Lindelöf, by a similar argumentation we can find a countable subset Δ_2 of Δ such that $X - A \subseteq \cup_{n\in\Delta_2} pint(pcl(V_{\alpha_n}))$. Thus $X = \cup_{n\in\Delta_1\cup\Delta_2} pint(pcl(V_{\alpha_n}))$. Therefore X is almost strongly Lindelöf. □

Theorem 3.15. *Let (X, τ) be a topological space. If there exsits a proper preclosed and preopen subset A of X such that A and $X - A$ are almost strongly Lindelöf relative to X. Then X is almost strongly Lindelöf.*

Proof. Let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ ba a preopen cover of X. Since A and $X - A$ are almost strongly Lindelöf relative to X, then there are a countable subsets Δ_1 and Δ_2 of Δ such that $A = \cup_{n\in\Delta_1} pcl(V_{\alpha_n})$ and $X - A \subseteq \cup_{n\in\Delta_2} pcl(V_{\alpha_n})$. Thus $X = \cup_{n\in\Delta_1\cup\Delta_2} pcl(V_{\alpha_n})$. Therefore X is almost strongly Lindelöf. □

Theorem 3.16. *Let X be an almost strongly Lindelöf space, and A be a preopen and preclosed subset of X. Then A is an almost strongly Lindelöf subspace of X.*

Proof. Let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ ba a cover of A by preopen subsets of X, then $\{V_\alpha\ ;\ \alpha \in \Delta\} \cup \{X - A\}$ is a preopen cover of X then there exists a countable subfamily $\{V_{\alpha_n}\ ;\ n \in \mathbb{N} \cup \{X - A\}\}$ such that $X = \cup_{n\in\mathbb{N}} pcl(V_{\alpha_n}) \cup pcl(X - A)$. Since $X - A$ is preclosed, $pcl(X - A) = X - A$, hence $X = \cup_{n\in\mathbb{N}} pcl(V_{\alpha_n}) \cup (X - A)$, so $A \subseteq \cup_{n\in\mathbb{N}} pcl(V_{\alpha_n})$. Thus A is almost strongly Lindelöf relative to X. By theorem 3.13, we conclude that A is an almost strongly Lindelöf subspace of X. □

4. WEAKLY STRONGLY LINDELÖF SPACES AND SUBSPACES

Definition 4.1. A topological space (X, τ) is called weakly strongly Lindelöf if every preopen cover $\{A_\alpha\ ;\ \alpha \in \Delta\}$ has a countable subfamily $\{A_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $X = pcl(\bigcup_{n\in\mathbb{N}} A_{\alpha_n})$.

Theorem 4.2. *Let (X, τ) be a topological space. The following are equivalents :*

(i) X is weakly strongly Lindelöf.

(ii) For any family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X such that $\cap_{\alpha\in\Delta} C_\alpha = \phi$, there exists a countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $pint(\cap_{n\in\mathbb{N}} C_{\alpha_n}) = \phi$.

(iii) For every family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X for which every countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ satisfies $pint(\cap_{n\in\mathbb{N}} C_{\alpha_n}) \neq \phi$, the intersection $\cap_{\alpha\in\Delta} C_\alpha \neq \phi$.

Proof. Notice that (ii) is equivalent to (iii) so we are going to prove that (i) $\Rightarrow$ (ii) and then (ii) $\Rightarrow$ (i).

Suppose that X is weakly strongly Lindelöf and let $\{C_\alpha\ ;\ \alpha \in \Delta\}$ be a family of preclosed subsets of X such that $\cap_{\alpha\in\Delta} C_\alpha = \phi$, so $X = X - \cap_{\alpha\in\Delta} C_\alpha = \cup_{\alpha\in\Delta}(X - C_\alpha)$. Hence $\{X - C_\alpha\ ;\ \alpha \in \Delta\}$ is a preopen cover of X and since X is weakly strongly Lindelöf then there exists a countable subfamily $\{X - C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $X = pcl(\cup_{n\in\mathbb{N}} X - C_{\alpha_n})$, so $X - pcl(\cup_{n\in\mathbb{N}} X - C_{\alpha_n}) = \phi$, hence $\phi = pint(X - \cup_{n\in\mathbb{N}}(X - (C_{\alpha_n})) = pint(\cap_{n\in\mathbb{N}} C_{\alpha_n})$.

Conversely, let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ be a preopen cover of X, then $X - \cup_{\alpha\in\Delta} V_\alpha$, so $X - \cup_{\alpha\in\Delta} V_\alpha = \phi$. Since $(X - V_\alpha)$ is preclosed for every $\alpha \in \Delta$, there exists a countable subfamily $\{X - V_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $pint(\cap_{n\in\mathbb{N}}(X - V_{\alpha_n})) = \phi$. So $X = X - pint(\cap_{n\in\mathbb{N}}(X - V_{\alpha_n})) = pcl(X - \cap_{n\in\mathbb{N}}(X - V_{\alpha_n})) = X - pcl(\cup_{n\in\mathbb{N}} V_{\alpha_n})$. Therefore, X is weakly strongly Lindelöf. □

Definition 4.3. A subset A of a topological space X is said to be weakly strongly Lindelöf if A is weakly strongly Lindelöf as a subspace of X, i.e. A is weakly strongly Lindelöf with respect to the induced topology on A by X.

Definition 4.4. A subset A of a topological space X is said to be weakly strongly Lindelöf relative to X if for every cover $\{U_\alpha\ ;\ \alpha \in \Delta\}$ of A by preopen subsets of X there exists a countable subfamily $\{U_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $A \subseteq pcl(\cup_{n\in\mathbb{N}} U_{\alpha_n})$.

The proof of the following is similar to the previous one.

Theorem 4.5. *Let X be a topological space and A be a subset of X. Then the following are equivalent :*

(i) A is weakly strongly Lindelöf relative to X.

(ii) For every family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X such that $(\cap_{\alpha\in\Delta} C_\alpha) \cap S = \phi$, there exists a countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $(\cap_{n\in\mathbb{N}} pint(C_{\alpha_n})) \cap S = \phi$.

(iii) For every family $\{C_\alpha\ ;\ \alpha \in \Delta\}$ of preclosed subsets of X for which every countable subfamily $\{C_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ satisfies $(\cap_{n\in\mathbb{N}} pint(C_{\alpha_n})) \cap A \neq \phi$, the intersection $(\cap_{\alpha\in\Delta} C_\alpha) \cap A \neq \phi$.

Theorem 4.6. *Let S be a semi-open subset of a topological X. If S is weakly strongly Lindelöf subspace of X then S is weakly strongly Lindelöf relative to X.*

Proof. Let $\{U_\alpha\ ;\ \alpha \in \Delta\}$ ba a cover of S by preopen subsets of X. Since S is semi-open in X then by theorem 2.3, $S \cap U_\alpha = W_\alpha$ is preopen in S, so $\{W_\alpha\ ;\ \alpha \in \Delta\}$ is a preopen cover of S. Since S is a weakly strongly Lindelöf subspace, there exists a countable subfamily $\{W_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $S = pcl_S(\cup_{n\in\mathbb{N}} W_{\alpha_n})$, so $S \subseteq pcl(\cup_{n\in\mathbb{N}} W_{\alpha_n}) \subseteq pcl(\cup_{n\in\mathbb{N}} U_{\alpha_n})$. Therefore, S is weakly strongly Lindelöf relative to X. □

Theorem 4.7. *Let S be a preopen subset of a topological X. If A is weakly strongly Lindelöf relative to X then S is a weakly strongly Lindelöf subspace of X.*

Proof. Let $\{V_\alpha\ ;\ \alpha \in \Delta\}$ ba a cover of S by preopen subsets of S. Since A is preopen in X then, by theorem 2.3, V_α is preopen in X for every $\alpha \in \Delta$, so $\{V_\alpha\ ;\ \alpha \in \Delta\}$ is a cover of S by preopen subsets of X, hence there exists a countable subfamily $\{V_{\alpha_n}\ ;\ n \in \mathbb{N}\}$ such that $S \subseteq pcl(\cup_{n\in\mathbb{N}} V_{\alpha_n})$. Therefore, S is a weakly strongly Lindelöf subspace of X. □

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