Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 6 (2019), pp. 981-995

http://www.ripublication.com/gjpam.htm

External Vertex Edge Domination

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Abstract

A non empty subset S of V is said to be an external vertex edge (*eve*) dominating set for the graph G if and only if for each uv in G, there is a vertex $w \in S - \{u, v\}$ such that uw or vw is an edge in G. An eve dominating set of a graph G of minimum cardinality is said to be a minimum eve dominating set and its cardinality, eve domination number of G, denoted by $\gamma_{eve}(G)$. By a $\gamma_{eve}(G) - set$, we mean an eve dominating set of minimum cardinality. Bounds for this parameter in terms of various graph theoretic parameters are obtained. We proved that for a tree of order $p \geq 3$ having l leaves and s support vertices, $\frac{p-2l-s+8}{3} \leq \gamma_{eve}(T) \leq \frac{2p}{3}$ and characterized the trees attaining the bounds. We obtained the necessary and sufficient condition for all the trees with diameter at least 6 to have the same $\gamma_{ve}(T), \gamma_{stve}(T), \gamma_{eve}(T)$ and $\gamma_{cve}(T)$ numbers.
Subject Classification: 05C69
Key Words: *vertex edge domination*, *semi total domination*,*connected vertex edge domination*.

1. INTRODUCTION AND PRELIMINARIES

A graph G consists of a finite non empty set V of p vertices together with a set E of q edges joining pairs of distinct vertices in V. By the open neighbourhood of a vertex v of G, we mean the set $N_G(v) = \{u \in V : uv \in E\}$. The closed neighbourhood of a vertex v of G, $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. The degree $d_G(v)$ of a vertex v in a graph G is the number of edges of G incident with v.

We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G, respectively. The distance between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of the shortest $u - v$ path in G. The diameter of a graph G, denoted by $diam(G)$, is the maximum of eccentricities of the vertices in G. A leaf is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. Finally, a support vertex is said to be strong, if it is adjacent to at least two leaves, else it is said to be weak. Graph theoretic terminology not defined here can be found in [3].

A set $S(\subseteq V)$ is called a dominating set for G, provided each vertex of $V - S$ is adjacent to a member of S. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of the smallest dominating set in G. For a comprehensive survey of domination in graphs, refer [10].

A set $D \subseteq V$ is said to be a total global neighbourhood dominating set(*tgnd - set*) of G if and only if D is a total dominating set for G and G^N. The total global neighbourhood domination number is the minimum cardinality of a total global neighbourhood dominating set of G and is denoted by $\gamma_{tgn}(G)$ [7].

In [2], connected domination number has been introduced. Analogous to connected domination in graphs, connected vertex edge domination has been introduced in [8].

A subset S of the vertex set V is said to be a *vertex edge dominating set* of the graph G if for each edge uv in G there is a vertex w in S such that $w \in \{u, v\}$ or w dominates at least one of u, v. The *vertex edge domination number* $\gamma_{ve}(G)$ is the minimum cardinality of the vertex edge dominating set of G. Vertex edge domination in graphs was introduced in [5], and further studied in [4].

A set S of vertices in a graph G without isolated vertices is said to be *semi total dominating set*, if S is a dominating set and each vertex in S is within a distance two of another vertex in S. The semi total domination number of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi total dominating set of G [11]. Analogous to semi total domination, semi total vertex edge domination has been introduced in [9] as follows. A set S of vertices in a graph G without isolated vertices is said to be *semi total vertex edge dominating set*, if S is a vertex edge dominating set and each vertex in S is within a distance two of another vertex in S. The semi total vertex edge domination number of G, denoted by $\gamma_{stve}(G)$, is the minimum cardinality of a semi total vertex edge dominating set of G.

Suppose two people are in friendship or handling a business jointly. To avoid confusions or misunderstandings between them, it's always better to have an advisor (probably an *elderly person) with whom both the partners involved in business or friendship have familiarity. Motivated by this, the concept of external vertex edge domination has been introduced in this paper. An edge between two vertices denotes friendship or business between two people. A vertex which externally dominates the edge is supposed to be the advisor for the two people involved in business or friendship.*

A vertex edge dominating set $S(\subseteq V)$ is said to be an external vertex edge dominating set of G, if and only if for each uv in G, there is a vertex $w \in S - \{u, v\}$ such that uw or vw is an edge in G. An eve dominating set of the graph G of minimum cardinality is said to be minimum eve dominating set and its cardinality, eve domination number of G, denoted by $\gamma_{eve}(G)$. By a $\gamma_{eve}(G)-set$, we mean an eve dominating set of minimum cardinality.

In this paper, we obtained the bounds for eve domination number in terms of various other graph theoretic parameters. For a tree with order at least 3 and having l leaves and s support vertices, we proved that $\frac{p-2l-s+8}{3} \leq \gamma_{eve}(T) \leq \frac{2p}{3}$. We obtained that $\gamma_{ve}(T)+\gamma_{stve}(T) \leq \gamma_{ve}(T)+\gamma_{eve}(T) \leq p$ and as a consequence shown that $\gamma_{ve}(T) \leq \frac{p}{2}$.

By the definition, it is obvious that for any graph to have an eve dominating set, it should be of size at least 2. So, throughout this paper, by a graph, we mean a connected graph of size 2. Note that for such graphs order is at least 3.

2. RESULTS

We give the *eve domination numbers* of some standard graphs.

Proposition 2.1. *1. For a cycle C_n,*

$$\gamma_{eve}(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \frac{n+1}{2}, & n \text{ is odd} \end{cases}$$

2. *For a complete graph K_n, $\gamma_{eve}(K_n) = 3$.*

3. *For a complete bipartite graph $K_{m,n}$, $\gamma_{eve}(K_{m,n}) = 2$.*

4. *For a star graph $K_{1,n}$, $\gamma_{eve}(K_{1,n}) = 2$.*

5. *For a wheel graph W_n,*

$$\gamma_{eve}(W_n) = \begin{cases} 2, & n = 5 \\ 3, & n \neq 5 \end{cases}$$

6. *For a petersen graph P, $\gamma_{eve}(P) = 3$.*

Now, we give a necessary and sufficient condition for an eve dominating set to be a minimal eve dominating set.

Theorem 2.2. *Suppose that S is an eve dominating set for the graph G. Then, S is minimal eve dominating set for G if and only if for each vertex $v \in S$, there is a private edge (w.r.t S) which is not incident with v.*

Proof. Assume that S is a minimal eve dominating set for G. Then, for each $v \in S$, $S - \{v\}$ is not an eve dominating set for G. This implies, for each $v \in S$, there is an edge $xy(x, y \in V - \{v\})$ which is uniquely dominated by v. Thus for each v in S, there is a private neighbour w.r.t. S, which is not incident with v.

Assume that the converse holds. If S is not minimal, then there is a vertex v in S such that $S - \{v\}$ is an eve dominating set for G. This implies that any edge that is eve dominated by v is also eve dominated by a vertex in $S - \{v\}$. Thus for $v \in S$ there is no private edge(w.r.t S), which is not incident with v. ■

Observe that, $2 \leq \gamma_{eve}(G) \leq p$. Now, we characterize the graphs attaining these bounds.

Theorem 2.3. *For a graph G, $\gamma_{eve}(G) = p$ if and only if $G = K_3$.*

Proof. Assume that $\gamma_{eve}(G) = p$.

Suppose $p \geq 4$. Let $S = V - \{v\}$, where v is an end vertex in a diammeteral path of G. Observe that S is an eve dominating set of G of cardinality $p - 1$, which is a contradiction. Hence $G = K_3$.

The converse part is clear. ■

Theorem 2.4. *For a tree T, $\gamma_{eve}(T) = 2$ if and only if $T = K_{1,n}(n \geq 2)$.*

Proof. Assume that $\gamma_{eve}(T) = 2$. Let $S = \{u, v\}$ be the $\gamma_{eve}(T) - set$. Observe that S is independent, also $d_T(u, v) = 2$. Let $< uwv >$ be the $u - v$ path in T. If there is a vertex $x \in V - \{u, v, w\}$ satisfying any one of the following properties:

1. adjacent to u or v

2. not adjacent to w

then $\gamma_{eve}(T) > 2$, a contrary to our assumption. This implies, $d_T(u) = d_T(v) = 1$ and any vertex in $V - \{u, v, w\}$ is adjacent to w. Also, if there is an edge between any two neighbours of w, then $S \cup \{w\}$ is a $\gamma_{eve}(T) - set$, a contrary to the hypothesis. Hence $T = K_{1,n}(n \geq 2)$. ■

Theorem 2.5. *Let G be a cyclic graph. Then, $\gamma_{eve}(G) = 2$ if and only if there is a pair of vertices u, v in G such that $d_G(u, v) = 2$ and each x in $V - \{u, v\}$ satisfies one of the following conditions:*

1. $\{u, v\} \subseteq N(x)$
2. *for each* $y \in N(x)$, $\{u, v\} \subseteq N(y)$

Proof. Assume that $\gamma_{eve}(G) = 2$. Let $S = \{u, v\}$ be the $\gamma_{eve}(G) - set$. It is easy to observe that S is independent and $d_G(u, v) = 2$. Let $x \in V - S$.

If x is adjacent to both u and v, then first condition holds.

Suppose x is adjacent to one of u, v. Without loss of generality assume that x is adjacent to u. Since S is a $\gamma_{eve}(G) - set$, x is adjacent to v. Hence the first condition holds.

Suppose x is adjacent to neither of u, v. Let $y \in N(x)$. Since S is a $\gamma_{eve}(G) - set$, y is dominated by u and v. This implies, $S \subseteq N(y)$. Hence the second condition holds.

The converse part is clear. ■

Theorem 2.6. *For a graph G, $\gamma_{eve}(G) = p - 1$ if and only if $G = P_3$ or $G = P_4$.*

Theorem 2.7. *For a graph G, $\gamma_{eve}(G) = p - 2$ if and only if $G = P_5$ or G is obtained from P_3 by attaching a leaf to the internal vertex of P_3 or $G = K_4 - \{uv\}$ or $G = C_4$.*

Now, we give the bounds for eve domination number in terms of various other graph theoretic parameters.

In [11], Wayne Goddard etal., proved that for a graph with n vertices and maximum degree Δ, $\gamma_{t2}(G) \geq \frac{2n}{2\Delta+1}$.

Theorem 2.8. *For a graph G,*

$$\left\lceil \frac{3q}{6(\Delta - 1)^2 + 1} \right\rceil \leq \gamma_{eve}(G).$$

Proof. Let S be an eve - dominating set for G. For edge uv in G, define $f : E \to [0, 1]$ by

$$f(uv) = \frac{1}{l(u) + m(v)},$$

where

$$l(u) = |N[u] \cap S|, m(v) = |N[v] \cap S|$$

Let $v \in S$. For S is an eve - dominating set of G, any arbitrary edge, av, in $< N[v] >$ is dominated by a vertex of S different from v. So, $f(av) \leq \frac{1}{3}$. Then,

$$\begin{aligned}\sum_{vx \in <N[v]>} f(vx) &= \sum_{vx \in <N[v]> - \{va\}} f(vx) + f(va) \\ &\leq \sum_{vx \in <N[v]> - \{va\}} d_G(vx) + \frac{1}{3} \\ &= \sum_{vx \in <N[v]> - \{va\}} [d_G(v) + d_G(x) - 2] + \frac{1}{3} \\ &\leq 2(\Delta - 1)^2 + \frac{1}{3}.\end{aligned}$$

Hence, each vertex in S dominates atmost $[6(\Delta - 1)^2 + 1]/3$ edges in G. So, S dominates atmost $|S|[6(\Delta - 1)^2 + 1]/3$ edges in G. This implies,

$$q \leq |S|[6(\Delta - 1)^2 + 1]/3.$$

Hence the result. ■

Note: The bound is sharp, as it is attained in the case of C_4.

Theorem 2.9. *For a graph G,*

$$\frac{q}{\Delta(G)(\Delta(G) - 1)} \leq \gamma_{eve}(G).$$

Proof. Let S be an eve dominating set for G, then each vertex in S eve dominates atmost $\Delta(G)(\Delta(G) - 1)$ edges. This implies, $q \leq | S | \Delta(G)(\Delta(G) - 1)$. Hence the result. ■

Note: The bound is sharp as it is attained in the case of C_k, k is even.

Theorem 2.10. *If S is a $\gamma_{eve}(G) - set$ having enclaves in S, then*

$$\delta(G) + 1 \leq \gamma_{eve}(G).$$

Proof. By the given hypothesis, let v be an enclave in S. Then, $N[v] \subseteq S$. Hence the proof.

The bound is attained in the case of C_6. ■

For a tree T, $\delta(T) = 1$. Observe that $\gamma_{eve}(K_{1,n}) = 2 = \delta(K_{1,n}) + 1$ and for all trees T of diameter at least 3, $\gamma_{eve}(T) \geq 3$. Also, no vertex in $\gamma_{eve}(K_{1,n}) - set$ is an enclave in $\gamma_{eve}(K_{1,n}) - set$. Hence under the hypothesis of the above theorem, no tree attains the bound in the above theorem. Likewise, there is no cyclic graph of minimum degree one, which attains the bound.

Theorem 2.11. *For a graph G with $\Delta'(G) \geq 3$,*

$$\gamma_{eve}(G) \leq p - \Delta'(G) + 3.$$

Proof. Let $d_G(uv) = \Delta'(G)$ and $A = \{u, v, w\}$, where w is a neighbour of u or v. Observe that $A \cup \{V - \{N[u] \cup N[v]\}\}$ is an eve dominating set for G of cardinality $p - \Delta'(G) + 3$. Hence the proof.

The bound is attained in the case of $K_n (n \geq 4)$. ■

Theorem 2.12. *If $\delta(G) \geq 3$ and $g(G) > 4$, then*

$$\gamma_{eve}(G) \leq p - \Delta(G) + 1.$$

Proof. Suppose that S is a total global neighborhood dominating set for G. Since $g(G) > 4$, S is an eve dominating set for G. By Theorem 2.11 in [7], the result follows. ■

Observe that under the given hypothesis,

$$\gamma_{ve}(G) \leq \gamma_{stve}(G) \leq \gamma_{eve}(G) \leq \gamma_{tgn}(G)$$

Theorem 2.13. *Let G be a graph for which $\delta(G) \geq 2$. If S is a $\gamma_{eve}(G) - set$ such that $< S >$ is a null graph, then $\gamma_{eve}(G) \leq \frac{p}{2}$.*

Proof. Suppose that the hypothesis holds. Let A be the set of all vertices in $V - S$ that dominate the vertices in S.

By the nature of S and Theorem 2.2, each vertex v in S has private edge(w.r.t. S), which is not incident with v. This implies, $\mid S \mid \leq \mid A \mid$.
If $\mid S \mid > \frac{p}{2}$, then $\mid A \mid > \frac{p}{2}$. This implies,

$$\mid S \cup A \mid = \mid S \mid + \mid A \mid > \frac{p}{2} + \frac{p}{2} = p$$

a contradiction. Hence $|S| \leq \frac{p}{2}$.
The bound is sharp, as it is attained in the case of C_4. ■

Theorem 2.14. *If G is a bipartite graph with $\delta(G) > 1$, then*

$$\gamma_{eve}(G) \leq \frac{p}{2}.$$

Proof. By the given hypothesis, any partite set in G, is an eve dominating set for G. Hence the result follows. ■

The bound is attained in the case of C_4.

Open Problem: Characterize the class of all bipartite graphs satisfying the hypothesis of the above theorem and having eve domination number $\frac{p}{2}$.

Note:

1. Any eve dominating set is a semi total vertex edge dominating set.
2. Any connected vertex edge dominating set of cardinality at least three is an eve dominating set.

Thus ,

$$\gamma_{ve}(G) \leq \gamma_{stve}(G) \leq \gamma_{eve}(G) \leq \gamma_{cve}(G).$$

For the purpose of giving necessary and sufficient condition for a tree T of diameter at least 6, to have equal ve, cve, stve, eve domination numbers, we denote $I(T)$ to be the set of all internal vertices in T and
$S(T) = \{v \mid$
v is a support vertex in T for which $T - \{v\}$has exactly one non trivial component$\}$

In [6], S. Arumugam and J. Paulraj Joseph proved the following. For a tree of order $p \geq 3$, $\gamma = \gamma_c$ if and only if every internal vertex of T is support.

Theorem 2.15. *If T is a tree with $diam(T) \geq 6$, then $\gamma_{ve}(T) = \gamma_{cve}(T)$ if and only if for each vertex v in $I(T) - S(T)$, there is a non trivial component C_i in $T - \{v\}$ such that $< V(C_i) \cup \{v\} >$ is isomorphic to $K_{1,n}$.*

Proof. Observe that $I(T) - S(T)$ is a unique minimum $\gamma_{cve}(T) - set$.

Assume that $\gamma_{ve}(T) = \gamma_{cve}(T)$. Choose a vertex v from $I(T) - S(T)$. Then, $T - \{v\}$ has at least two non trivial components. If possible, suppose that none of the non trivial components is isomorphic to $K_{1,n}$. Then, $\{I(T) - S(T)\} - \{v\}$ is a ved - set of T' of cardinality $\gamma_{cve}(T) - 1$, which is a contradiction to our assumption.

Assume that the converse holds. Clearly $\gamma_{ve}(T) \geq| I(T) - S(T) |= \gamma_{cve}(T)$. Thus, $\gamma_{ve}(T) = \gamma_{cve}(T)$. ■

Corollary 2.16. *If T is a tree with $diam(T) \geq 6$, then*

$$\gamma_{ve}(T) = \gamma_{stve}(T) = \gamma_{eve}(T) = \gamma_{cve}(T)$$

if and only if for each vertex v in $I(T) - S(T)$, there is a non trivial component C_i in $T - \{v\}$ such that $< V(C_i) \cup \{v\} >$ is isomorphic to $K_{1,n}$.

Proof. Since $\gamma_{ve}(G) \leq \gamma_{stve}(G) \leq \gamma_{eve}(G) \leq \gamma_{cve}(G)$, by Theorem 2.16, the proof follows. ■

Theorem 2.17. *If S is a $\gamma_{eve}(G) - set$ such that for each vertex v in S, there is a non trivial component C_i in $G - \{v\}$ such that $< V(C_i) \cup \{v\} >$ is isomorphic to $K_{1,n}$, then*

$$\gamma_{eve}(G) \leq \frac{p}{3}.$$

Furthermore, equality holds if and only if for each vertex v in S, there is a non trivial component C_i in $G - \{v\}$ such that $< V(C_i) \cup \{v\} >$ is isomorphic to P_3.

Proof. Assume that the hypothesis holds. Let A be the smallest set of vertices in $V - S$ that dominate the vertices in S and $B = V - S - A$. Observe that A is a dominating set for G. By hypothesis, for each vertex in S, there is a vertex in A that dominates the former. Hence $| S |\leq| A |$. Also by hypothesis corresponding to each vertex in A, there is at least one vertex in B, so $| A |\leq| B |$. Also, observe that $V = S \cup A \cup B$. This implies,

$$| V |=| S | + | A | + | B | \geq | S | + | A | + | A | \geq | S | + | S | + | S |$$

Hence the Result. ■

3. BOUNDS ON THE EVE DOMINATION NUMBER OF TREES

In [1], B. Krishnakumari etal., proved that for every tree of order $p \geq 3$ and having l leaves and s support vertices, $\frac{p-s-l+3}{4} \leq \gamma_{ve}(T) \leq \frac{p}{3}$.

Theorem 3.1. *If T is a tree of order $p \geq 3$, then $\gamma_{eve}(T) \leq \lceil \frac{2p}{3} \rceil$.*

Proof. If $diam(T) = 2$, then $T = K_{1,p-1}(p \geq 2)$. Observe that,

$$\gamma_{eve}(T) = 2 \leq \frac{2p}{3}.$$

Suppose that $diam(T) \geq 3$. Then T is of order $p \geq 4$. We prove the result by using induction on the order p of the tree T. Assume that the result is true for all trees of order $p' < p$.

Assume that a support vertex, say v, in T is strong. Then, observe that $\gamma_{eve}(T) - set$ is also a $\gamma_{eve}(T - x) - set$, where x is a leaf adjacent to v. Hence,

$$\gamma_{eve}(T) \leq \gamma_{eve}(T - x) \leq \frac{2p'}{3} \leq \frac{2p}{3}$$

Assume that no support vertex in T is strong.

Suppose that $d_T(r, t) = diam(T)$. Now root the tree at r and let T_x denote the subgraph induced by x and all its descendents in the rooted tree T. Let u, v, w be the parents of t, v, u respectively.

By our assumption u can have adjacency with atmost one leaf, say x. Clearly the $\gamma_{eve}(T) - set$ is also an eve dominating set of $< (T - T_u) \cup \{u\} >$. Also, $< (T - T_u) \cup \{u\} >$ has atmost $p - 5$ vertices. Hence

$$\gamma_{eve}(T) \leq \gamma_{eve}((T - T_u) \cup \{u\}) \leq \frac{2(p-5)}{3} < \frac{2p}{3}.$$

By the principle of mathematical induction, the result is true for any tree with p vertices. Also, the bound is sharp, as it is attained in the case of P_3. ■

In [1], Krishnakumari etal., have defined a family $\mathcal{F}$ of trees $T = T_k$, as follows. Let $T_1 = P_3$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by attaching a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_2 or P_3.

Corollary 3.2. *For any tree T of order $p \geq 3$,*

$$\gamma_{stve}(T) + \gamma_{ve}(T) \leq \gamma_{eve}(T) + \gamma_{ve}(T) \leq p.$$

Furthermore, equality holds if and only if $T = P_3$.

Proof. Assume that

$$\gamma_{stve}(T) + \gamma_{ve}(T) = \gamma_{eve}(T) + \gamma_{ve}(T) = p.$$

This implies,

$$\gamma_{stve}(T) = \gamma_{eve}(T) = p - \gamma_{ve}(T).$$

From Theorem 5[1] and Theorem 3.1, it follows that,

$$\frac{2p}{3} \geq \gamma_{eve}(T) = \gamma_{stve}(T) = p - \gamma_{ve}(T) \geq \frac{2p}{3}.$$

This implies,

$$\gamma_{stve}(T) = \gamma_{eve}(T) = \frac{2p}{3}$$

.

By our assumption, $\gamma_{ve}(T) = \frac{p}{3}$. From Theorem 5 [1], $T \in \mathcal{F}$. If $T = P_3$, then we are through.

Suppose $T \neq P_3$. Then $T = T_{k+1}(k \geq 1)$. Observe that in $T = T_{k+1}(k \geq 2)$,

$$S = \{v \mid v \text{ externally dominates a pendant edge}\}$$

is an $\gamma_{eve}(T) - set$ and $\gamma_{stve}(T) - set$, as well. Also, $|S| = \gamma_{eve}(T) = \gamma_{stve}(T) = \frac{p}{3} < \frac{2p}{3}$, a contradiction. If $T = T_2 = P_6$, then $3 = \gamma_{eve}(T) \neq \gamma_{stve}(T) = 2$, again contradiction.

The converse is clear. ■

Corollary 3.3. *For any tree T of order $p \geq 3$,*

$$\gamma_{eve}(T) + \gamma_{ve}(T) \leq p.$$

Furthermore, equality holds if and only if $T = P_3$ or $T = P_4$.

Corollary 3.4. *For any tree T of order $p \geq 3$,*

$$\gamma_{ve}(T) \leq \frac{p}{2}.$$

Proof. By Corollary 3.2, we have,

$$2\gamma_{ve}(T) \leq \gamma_{eve}(T) + \gamma_{ve}(T) \leq p.$$

This implies,

$$\gamma_{ve}(T) \leq \frac{p}{2}.$$

■

For the purpose of giving a lower bound for a tree with $p \geq 3$, we denote s, l to be the number of support vertices, leaves, respectively.

Theorem 3.5. *For a tree with $p \geq 3$ vertices,*

$$\frac{p - 2l - s + 8}{3} \leq \gamma_{eve}(T)$$

Furthermore, equality holds if and only if $T = P_7$.

Proof. By hypothesis, $diam(T) \geq 2$.

If $d(T) = 2$, then $T = K_{1,p-1}$. For $p - 1 = 2$, $T = P_3$. Observe that $\gamma_{eve}(T) = 2 = \frac{3-2(2)-1+8}{3}$.

Suppose that $diam(T) \geq 3$. Clearly $p \geq 4$. By using induction on the order p of T, the result will be proved. Assume that the result is true for all trees T' of order $p' < p$ and let s', l' be the number of support vertices, leaves in T', respectively.

Suppose that there is a strong support vertex, say x, in T and y be a leaf adjacent to x. For the tree, $T' = T - y, p' = p - 1, l' = l - 1, s' = s$. Observe that $\gamma_{eve}(T) - set$ is a $\gamma_{eve}(T') - set$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 9}{3} > \frac{p - 2l - s + 8}{3}.$$

Assume that T has no strong support vertices.

Suppose that $d_T(r, t) = diam(T)$. Root the tree at the vertex r. Let t, u, v be the children of u, v, w, respectiely and T_x denotes the tree induced by x and all its descendents in T.

Assume that $d_T(v) \geq 3$. Let x be the leaf adjacent to v. For the tree, $T' = T - x$, $p' = p - 1, l' = l - 1, s' = s - 1$. Observe that $\gamma_{eve}(T) - set$ is a $\gamma_{eve}(T') - set$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 10}{3} > \frac{p - 2l - s + 8}{3}.$$

Suppose that x is a support vertex adjacent to v. For the tree, $T' = T - T_x$, $p' = p-2, l' = l-1, s' = s-1$. Observe that $\gamma_{eve}(T) - set$ is a $\gamma_{eve}(T') - set$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 9}{3} > \frac{p - 2l - s + 8}{3}.$$

Assume that $d_T(v) = 2$ and $d_T(w) \geq 3$. Let x be the leaf adjacent to w. For the tree, $T' = T - x$, $p' = p-1, l' = l-1, s' = s-1$. Observe that $\gamma_{eve}(T) - set$ is a $\gamma_{eve}(T') - set$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 9}{3} > \frac{p - 2l - s + 8}{3}.$$

Suppose that w is adjacent to a support vertex, say x. For the tree, $T' = T - T_x$, $p' = p-2, l' = l-1, s' = s-1$. Observe that $\gamma_{eve}(T) - set$ is a $\gamma_{eve}(T') - set$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 9}{3} > \frac{p - 2l - s + 8}{3}.$$

Suppose that $< wxyz >$ is a path in T, where $\{x, y, z\} \in V(T) - \{v, u, t\}$. Observe that $\gamma_{eve}(T) - set$, say S, contains v, w, x. Now, for the tree, $T' = T - T_x$, $p' = p-3, s' = s-1, l' = l-1$. It is straight forward to see that, $(S - \{x\}) \cup \{a\}$, where $a \in (N(w) - v)$, is an eve dominating set for T'. This implies, $\gamma_{eve}(T') \leq |(S - \{x\}) \cup \{a\}| = \gamma_{eve}(T)$. Thus,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') \geq \frac{p' - 2l' - s' + 8}{3} = \frac{p - 2l - s + 8}{3}.$$

Suppose that s is the parent of w in T and $d_T(w) = 2$ and $d_T(s) \geq 3$. Observe that a $\gamma_{eve}(T) - set$, say S, contains $\{u, v, w\}$. It suffices to consider that s is adjacent to P_3 and P_4.

Assume that s is adjacent to $< xyz >, \{x, y, z\} \in V(T) - \{w, u, v\}$. Observe that $\gamma_{eve}(T) - set$, say S, contains v, w, x, s. Now, for the tree, $T' = T - T_x$, $p' = p-3, s' = s-1, l' = l-1$. Clearly, $S - \{x\}$ is an eve dominating set of T'. This implies,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') + 1 \geq \frac{p' - 2l' - s' + 8}{3} + 1 = \frac{p - 2l - s + 11}{3} > \frac{p - 2l - s + 8}{3}.$$

Assume that s is adjacent to $< xyzm >, \{x, y, z, m\} \in V(T) - \{w, u, v, t\}$. Observe that $\gamma_{eve}(T) - set$, say S, contains v, w, x, y, s. Now, for the tree, $T' = T - T_x$,

$p' = p - 4, s' = s - 1, l' = l - 1$. Clearly, $S - \{x, y\}$ is an eve dominating set of T'. This implies,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') + 2 \geq \frac{p' - 2l' - s' + 8}{3} + 1 = \frac{p - 2l - s + 13}{3} > \frac{p - 2l - s + 8}{3}.$$

Suppose that q is the parent of s in T , $d_T(s) = 2$ and $d_T(q) \geq 3$. It suffices to show that q is adjacent to P_5. Assume that q is adjacent to $< xyzmn >, \{x, y, z, m, n\} \in V(T) - \{s, w, u, v, t\}$. Observe that $\gamma_{eve}(T) - set$, say S, contains z, v, w, x, y, s. Now, for the tree, $T' = T - T_x, p' = p - 5, s' = s - 1, l' = l - 1$. Clearly, $S - \{x, y, z\}$ is an eve dominating set of T'. This implies,

$$\gamma_{eve}(T) \geq \gamma_{eve}(T') + 3 \geq \frac{p' - 2l' - s' + 8}{3} + 1 = \frac{p - 2l - s + 15}{3} > \frac{p - 2l - s + 8}{3}.$$

Suppose that h is the parent of q in T , $d_T(q) = 2$. If $d_T(h) = 1$, then $T = P_7$ and also, $\{v, w, s\}$ is an $\gamma_{eve}(T) - set$. This implies,

$$\gamma_{eve}(T) = 3 = \frac{7 - 2(2) - 2 + 8}{3}.$$

Thus, equality holds for $T = P_7$.

■

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