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Action of Direct Products of Four Symmetric Groups on Cartesian Product of Four Sets

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Abstract

In this paper, the combinatorial properties(such as transitivity, primitivity)and invariants such as ranks and subdegrees associated with the action of direct product of four symmetric groups on the Cartesian product of four sets are explored. It is shown that the action is transitive, primitive with a constant rank of 16.
Keywords: Symmetric Group, Transitivity, Primitivity, Ranks, Subdegrees, Direct Product, Cartesian Product, GAP

1. INTRODUCTION

There has been several attempts to study the action of permutation groups such as Symmetric and Alternating groups on sets, [5, 8, 2, 7, 6] but a study investigating the action on four sets has to the best of our knowledge not been explored and therefore this presents a research gap.

Definition 1.1. [Product Action][1, p.3] Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the the Cartesian product $X_1 \times X_2$ by the rule

$$(g_1, g_2)(x_1, x_2) = (g_1x_1, g_2x_2) \; \forall \, g_1 \in G_1, g_2 \in G_2 \text{ and } x_1 \in X_1, \; x_2 \in X_2.$$

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Remark 1.1. Through out this paper, the group action defined is in a similar way as in Definition 1.1 as

$(g_1, g_2, \cdots, g_4)(x_1, x_2, \cdots, x_4) = (g_1x_1, g_2x_2, \cdots, g_4x_4)\, \forall\, g_1, g_2, \cdots, g_4 \in G$ and $x_i \in X_i$

where $G = S_n \times S_n \times \ldots \times S_n$ and, $X_1 = \{1, 2, \cdots, n\}$, $X_2 = \{n+1, n+2, \cdots, 2n\}, \cdots, X_4 = \{n(n-1)+1, n(n-1)+2, \cdots, n^2\}$.

1.1 Notation and Preliminary results

Definition 1.2. Let G act on X. The orbit of $x \in X$, denoted $Orb_G(x)$ is defined as the set

$$Orb_G(x) = \{gx : g \in G\}.$$

Definition 1.3. Let G act on X, and let $x \in X$. The Stabilizer of x in G, denoted G_x (sometimes $Stab_G(x)$) is set all elements in G that fix x. Thus

$$G_x = \{g \in G : gx = x\}.$$

Definition 1.4. The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit.

Definition 1.5. Suppose that G acts transitively on X. For each subset Y of X and each $g \in G$, let $gY = \{gy : y \in Y\} \subseteq X$. A subset Y of X is said to be a block for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$; In particular, $\emptyset,\ X$, and all 1-element subsets of X are obviously blocks, called the trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise, G acts imprimitively.

Definition 1.6. Suppose G is a group acting transitively on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \cdots \triangle_{k-1}$ of G_x on X are known as suborbits of G. The rank of G in this case is k. The sizes $n_i = |\triangle_i|$ $(i = 0, 1, 2, \cdots, k-1)$ are known as the subdegrees of G. It was proved by [4] that the rank and subdegrees of the suborbits $\triangle_i$ $(i = 0, 1, 2, \cdots, k-1)$ are independent of the choices of $x \in X$.

Definition 1.7. Let G act on the set X. The set of all elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix(g)$. Thus

$$Fix(g) = \{x \in X :\ gx = x\}.$$

The character π of permutation representation of G on X is defined as

$$\pi(g) = |Fix(g)|,\ \forall\, g \in G.$$

Definition 1.8. Let $\triangle$ be an orbit of G_x on X. Define $\triangle^* = \{gx : \ g \in G, \ x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x and is called the G_x-orbit paired with $\triangle$. [10] proved that if $\triangle^* = \triangle$, then $\triangle$ is called a self-paired orbit of G_x.

Theorem 1.1. [Orbit-Stabilizer Theorem][9] Let G be a group acting on a finite set X and $x \in X$. Then $|Orb_G(x)| = |G : \text{Stab}_G(x)|$.

Lemma 1.1. [Cauchy-Frobenius Lemma][3] Let G be a finite group acting on a set X. The number of orbits of G is given by $\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$.

2. MAIN RESULTS

Lemma 2.1. *The action of* $S_2 \times S_2 \times S_2 \times S_2$ *on* $X_1 \times X_2 \times X_3 \times X_4$ *is transitive where* $X_1 = \{1,2\}, \ X_2 = \{3,4\}, \ X_3 = \{5,6\},$ *and* $X_4 = \{7,8\}$.

Proof. Let $G = S_2 \times S_2 \times S_2 \times S_2$. It suffices to show that $|Orb_G(1,3,5,7)| = |X_1 \times X_2 \times X_3 \times X_4|$. Let $M = X_1 \times X_2 \times X_3 \times X_4$. Then $M = \{(1,3,5,7),(1,3,5,8),(1,3,6,7),(1,3,6,8),$
$(1,4,5,7),(1,4,5,8),(1,4,6,7),(1,4,6,8),(2,3,5,7),(2,3,5,8),(2,3,6,7),(2,3,6,8),$
$(2,4,5,7),(2,4,5,8),(2,4,6,7),(2,4,6,8)\}$.
Also, $G =$
$\{(e_1,e_2,e_3,e_4),((1\ 2),e_2,e_3,e_4),(e_1,(3\ 4),e_3,e_4),((1\ 2),(3\ 4),e_2,e_4),(e_1,e_3,(5\ 6),$
$e_4),((1\,2),e_2,(5\,6),e_4),(e_1,(3\,4),(5\,6),e_4),((1\,2),(3\,4),(5\,6)),(e_1,e_2,e_3,(7\,8)),((1\,2),e_2,e_3,$
$(7\,8)),(e_1,(3\,4),e_3,(7\,8)),((1\,2),(3\,4),e_3,(7\,8)),(e_1,e_2,(5\,6)(7\,8)),((1\,2),e_2,(5\,6),(7\,8)),$
$(e_1,(3\ 4),(5\ 6),(7\ 8)),((1\ 2),(3\ 4),(5\ 6),(7\ 8))\}$.
By Definition 1.3, $G_{(1,3,5,7)} = \{(e_1,e_2,e_3,e_4)\}$. Using Theorem 1.1,
$|Orb_G(1,3,5,7)| = \frac{16}{1} = 16 = 2^4 = |X_1 \times X_2 \times X_3 \times X_4|$.
Moreover
$Orb_G(1,3,5,7) =$
$\{(1,3,5,7),(2,3,5,7),(1,4,5,7),(1,3,6,7),(1,3,5,8),(2,4,5,7),$
$(2,3,6,7),(2,3,5,8),(1,4,6,7),(1,4,5,8),(1,3,6,8),(2,4,6,7),(2,4,5,8),(2,3,6,8),$
$(1,4,6,8),(2,4,6,8)\}$. Thus, the action is transitive since it has only one orbit. □

Lemma 2.2. *The action of* $S_3 \times S_3 \times S_3 \times S_3$ *on* $X_1 \times X_2 \times X_3 \times X_4$ *is transitive where* $X_1 = \{1,2,3\}, \ X_2 = \{4,5,6\}, \ X_3 = \{7,8,9\},$ *and* $X_4 = \{10,11,12\}$.

Proof. Let $G = S_3 \times S_3 \times S_3 \times S_3$ and $M = X_1 \times X_2 \times X_3 \times X_4$. By using the Groups, Algorithms Programming (GAP) software, G is a permutation group with 8 generators and $|G| = 1296$.
Also,
$G_{(1,4,7,10)} = \langle\{(11 \quad 12),(8 \quad 9),(5 \quad 6),(2 \quad 3)\}\rangle =$

$\{(e_1, e_2, e_3, e_4), ((2\,3), e_2, e_3, e_4), (e_1, (5\,6), e_3, e_4), ((2\,3)(5\,6), e_3, e_4), (e_1, e_2, (8\,9), e_4),$
$((2\,3), e_2, (8\,9), e_4), (e_1, (5\,6), (8\,9), e_4), ((2\,3), (5\,6), (8\,9), e_4), (e_1, e_2, e_3, (11\,12)),$
$((2\,3), e_2, e_3, (11\,12)), (e_1, (5\,6), e_3, (11\,12)), ((2\,3), (5\,6), e_3, (11\,12)), (e_1, e_2, (8\,9),$
$(11\,12)), ((2\,3), e_2, (8\,9), (11\,12)), (e_1, (5\,6), (8\,9), (11\,12)), ((2\,3), (5\,6), (8\,9), (11\,12)\}$
so that $|G_{(1,4,7,10)}| = 16$.
By Theorem 1.1, $|Orb_G(1, 4, 7, 10)| = \frac{1296}{16} = 81 = 3^4 = |X_1 \times X_2 \times X_3 \times X_4|$.

□

Lemma 2.3. *The action of* $S_4 \times S_4 \times S_4 \times S_4$ *on* $X_1 \times X_2 \times X_3 \times X_4$ *is transitive where* $X_1 = \{1, 2, 3, 4\},\ X_2 = \{5, 6, 7, 8\},\ X_3 = \{9, 10, 11, 12\},$ *and* $X_4 = \{13, 14, 15, 16\}$.

Proof. Let $G = S_4 \times S_4 \times S_4 \times S_4$ and $M = X_1 \times X_2 \times X_3 \times X_4$. Then by using GAP software, G is a permutation group with 8 generators and $|G| = 331776$. Also, $G_{(1,5,9,13)}$ is a permutation group with 8 generators and $|G_{(1,5,9,13)}| = 1296$. Using Theorem 1.1,$|Orb_G(1, 5, 9, 13)| = \frac{331776}{1296} = 256 = 4^4 = |X_1 \times X_2 \times X_3 \times X_4|$. Thus, the action has one orbit and hence transitive. □

Lemma 2.4. *The action of* $S_5 \times S_5 \times S_5 \times S_5$ *on* $X_1 \times X_2 \times X_3 \times X_4$ *is transitive where* $X_1 = \{1, 2, 3, 4, 5\},\ X_2 = \{6, 7, 8, 9, 10\},\ X_3 = \{11, 12, 13, 14, 15\},\ X_4 = \{16, 17, 18, 19, 20\},$ *and* $X_5 = \{21, 22, 23, 24, 25\}$.

Proof. Let $G = S_5 \times S_5 \times S_5 \times S_5$ and $M = X_1 \times X_2 \times X_3 \times X_4$. Then by using GAP software, G is a permutation group with 8 generators and $|G| = 207360000$. Also, $G_{(1,6,11,16)}$ is a permutation group with 12 generators and $|G_{(1,5,9,13)}| = 331776$. Using Theorem 1.1, $|Orb_G(1, 6, 11, 16)| = \frac{207360000}{331776} = 625 = 5^4 = |X_1 \times X_2 \times X_3 \times X_4|$. Thus, the action is transitive since it has one orbit. □

Theorem 2.1. *For* $n \geq 2$*, the action of* $S_n \times S_n \times S_n \times S_n$ *on* $X_1 \times X_2 \times X_3 \times X_4$ *is transitive where* $X_1 = \{1, 2, \cdots, n\},\ X_2 = \{n + 1, n + 2, \cdots, 2n\},\ X_3 = \{2n + 1, 2n + 2, \cdots, 3n\}, \cdots,$ *and* $X_4 = \{3n + 1, 3n + 2, \cdots, 4n\}$.

Proof. We show that the cardinality of $Orb_G(1, n + 1, 2n + 1, 3n + 1)$ is equal to the cardinality of $X_1 \times X_2 \times X_3 \times X_4$ for $n \geq 2$. Now, by Definition 1.3, $g_1, g_2, g_3, g_4 \in G$ fixes $x_1, x_2, x_3, x_4 \in X_1 \times X_2 \times X_3 \times X_4$ if and only if $(g_1, g_2, g_3, g_4)(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$. By Definition 1.1, we have $g_1x_1 = x_1, g_2x_2 = x_2, g_3x_3 = x_3$ and $g_4x_4 = x_4$. Hence x_1, x_2, x_3, x_4 comes from a 1-cycle of $g_i (i = 1, 2, 3, 4)$. Therefore, $G_{(1,n+1,2n+1,3n+1)}$ is isomorphic to $S_{n-1} \times S_{n-1} \times S_{n-1} \times S_{n-1}$. Thus, $|G_{(1,n+1,2n+1),3n+1}| = ((n - 1)!)^4$. By Theorem 1.1, $|Orb_G(1, n+1, 2n+1, 3n+1)| = \frac{(n!)^4}{((n-1)!)^4} = n^4 = |X_1 \times X_2 \times X_3 \times X_4|$. Hence the action is transitive. □

Lemma 2.5. *The group* $S_2 \times S_2 \times S_2 \times S_2$ *acts imprimitively on* $X_1 \times X_2 \times X_3 \times X_4$.

Proof. Let $G = S_2 \times S_2 \times S_2 \times S_2$ and $K = X_1 \times X_2 \times X_3 \times X_4$. Using Theorem 2.1, this action is transitive. Consider a non-trivial subset $Y = \{(1,3,5,7),(1,3,5,8)\}$ of K where $\cap_{j=7}^{8}\{1,3,5,j\} = \{1,3,5\}$. Then $|Y|$ divides $|K|$. For each element of Y, if $g = (g_1, g_2, g_3, g_4) \in G$ is such that $1, 3,$ and, 5 belongs to a 1-cycle permutation of g, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. However if $1, 3,$ and, 5 is from a 2-cycles permutations of any other $g \in G$, then $g \in G$ moves an element of Y to another element not in Y. Thus, $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action of G on K. By Definition 1.5, the action is imprimitive. □

Lemma 2.6. *The group $S_3 \times S_3 \times S_3 \times S_3$ acts imprimitively on $X_1 \times X_2 \times X_3 \times X_4$.*

Proof. Let $G = S_3 \times S_3 \times S_3 \times S_3$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.2, this action is transitive. We now consider a non-trivial subset $Y = \{(2,4,7,10),(2,4,7,11),(2,4,7,12)\}$ of K where $\cap_{j=10}^{12}\{2,4,7,j\} = \{2,4,7\}$. Then $|Y| = 3$ which divides $|K|$. Now for each element of Y, if $g = (g_1, g_2, g_3, g_4) \in G$ is such that $2, 4$ and 7 belongs to a 1-cycle permutation, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other g such that $2, 4$ and 7 come from $2, 3$-cycles permutations moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action of G on K. By Definition 1.5, the action is imprimitive. □

Lemma 2.7. *Action of $S_4 \times S_4 \times S_4 \times S_4$ on $X_1 \times X_2 \times X_3 \times X_4$ is imprimitive.*

Proof. Let $G = S_4 \times S_4 \times S_4 \times S_4$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.3, this action is transitive. We now consider a non-trivial subset $Y = \{(3,5,9,13),(3,5,9,14),(3,5,9,15),(3,5,9,16)\}$ of K where $\cap_{j=13}^{16}\{3,5,9,j\} = \{3,5,9\}$. Then $|Y| = 4$ which divides $|K|$. Now for each element of Y, if $g = (g_1, g_2, g_3, g_4) \in G$ is such that $3, 5$ and 9 belongs to a 1-cycle permutation, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other g such that $3, 5$ and 9 come from $2, 3, 4$-cycles permutations moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action of G on K. By Definition 1.5, the action is imprimitive. □

Lemma 2.8. *If $S_5 \times S_5 \times S_5 \times S_5$ acts on $X_1 \times X_2 \times X_3 \times X_4$, then this action is imprimitive.*

Proof. Let $G = S_5 \times S_5 \times S_5 \times S_5$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.4, this action is transitive. We now consider a non-trivial subset $Y = \{(4,6,11,16),(4,6,11,17),(4,6,11,18),(4,6,11,19),(4,6,11,20)\}$ of K where $\cap_{j=16}^{20}\{4,6,11,j\} = \{4,6,11\}$. Then $|Y| = 5$ which divides $|K|$. Now for each element of Y, if $g = (g_1, g_2, g_3, g_4) \in G$ is such that $4, 6$ and 11 belongs to a 1-cycle permutation, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other g such that $4, 6$ and 11 come from $2, 3, 4, 5$-cycles permutations moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action of G on K. Conclusion follows from Definition 1.5. □

Theorem 2.2. *If $n \geq 2$ and $S_n \times S_n \times S_n \times S_n$ acts on $X_1 \times X_2 \times X_3 \times X_4$, then this action is imprimitive.*

Proof. Let $G = S_n \times S_n \times S_n \times S_n$ and $K = X_1 \times X_2 \times X_3 \times X_4$. Using Theorem 2.1, this action is transitive. We now consider a non-trivial subset $Y = \{(n, n+1, 2n+1, 3n+1), (n, n+1, 2n+1, 3n+2), (n, n+1, 2n+1, 3n+3), \cdots, (n, n+1, 2n+1, 4n)\}$ of K where $\cap_{j=3n+1}^{4n}\{n, n+1, 2n+1, j\} = \{n, n+1, 2n+1\}$. Then $|Y| = n$ which divides $|K|$. Now for each element of Y, if $g = (g_1, g_2, g_3, g_4) \in G$ is such that $n, n+1$ and $2n+1$ belongs to a 1-cycle permutation, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other g such that $n, n+1$ and $2n+1$ come from $2, 3, 4, \cdots, n$-cycles permutations moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action of G on K. Conclusion follows from Definition 1.5. □

Lemma 2.9. *Action of $S_2 \times S_2 \times S_2 \times S_2$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4.*

Proof. Let $G = S_2 \times S_2 \times S_2 \times S_2$ and $K = X_1 \times X_2 \times X_3 \times X_4$. Then $K = \{(1,3,5,7), (1,3,5,8), (1,3,6,7), (1,3,6,8), (1,4,5,7), (1,4,5,8), (1,4,6,7), (1,4,6,8), (2,3,5,7), (2,3,5,8), (2,3,6,7), (2,3,6,8), (2,4,5,7), (2,4,5,8), (2,4,6,7), (2,4,6,8)\}$ and by using Theorem 2.1, $G_{(1,3,5,7)} = \langle(e_1, e_2, e_3, e_4)\rangle$. Thus a permutation of $G_{(1,3,5,7)}$ is of the form (e_1, e_2, e_3, e_4) since it identity. Hence the number of elements in K fixed by each $g = (g_1, g_2, g_3, g_4) \in G$ is 16 as identity fixes all elements of K. By Lemma 1.1, the number of orbits of $G_{(1,3,5,7)}$ acting on K are;

$$\frac{1}{|G_{(1,3,5,7)}|} \sum_{g_1,g_2,g_3,g_4 \in G_{(1,3,5,7)}} |Fix(g_1, g_2, g_3, g_4)| = \frac{1}{1}(1 \times 16) = 2^4.$$

Let $B = \{1, 3, 5, 7\}$

The orbits of $G_{(1,3,5,7)}$ include those with exactly $4, 3, 2, 1$, and no elements from B and they are;

(a) Orbits with exactly four elements from B
$Orb_{G_{(1,3,5,7)}}(1, 3, 5, 7) = \{(1, 3, 5, 7)\} = \triangle_0$.

(b) Orbits with exactly three elements from B
$Orb_{G_{(1,3,5,7)}}(1, 3, 5, 8) = \{(1, 3, 5, 8)\} = \triangle_1$.

$Orb_{G_{(1,3,5,7)}}(1,3,6,7) = \{(1,3,6,7)\} = \triangle_2.$
$Orb_{G_{(1,3,5,7)}}(1,4,5,7) = \{(1,4,5,7)\} = \triangle_3.$
$Orb_{G_{(1,3,5,7)}}(2,3,5,7) = \{(2,3,5,7)\} = \triangle_4$

(c) Orbits with exactly two elements from B
$Orb_{G_{(1,3,5,7)}}(1,3,6,8) = \{(1,3,6,8)\} = \triangle_5.$
$Orb_{G_{(1,3,5,7)}}(1,4,5,8) = \{(1,4,5,8)\} = \triangle_6.$
$Orb_{G_{(1,3,5,7)}}(1,4,6,7) = \{(1,4,6,7)\} = \triangle_7.$
$Orb_{G_{(1,3,5,7)}}(2,3,5,8) = \{(2,3,5,8)\} = \triangle_8.$
$Orb_{G_{(1,3,5,7)}}(2,4,5,7) = \{(2,4,5,7)\} = \triangle_9.$

(d) Orbits with exactly one element from B;
$Orb_{G_{(1,3,5,7)}}(1,4,6,8) = \{(1,4,6,8)\} = \triangle_{10}.$
$Orb_{G_{(1,3,5,7)}}(2,3,6,7) = \{(2,3,6,7)\} = \triangle_{11}$
$Orb_{G_{(1,3,5,7)}}(2,3,6,8) = \{(2,3,6,8)\} = \triangle_{12}.$
$Orb_{G_{(1,3,5,7)}}(2,4,5,8) = \{(2,4,5,8)\} = \triangle_{13}.$
$Orb_{G_{(1,3,5,7)}}(2,4,6,7) = \{(2,4,6,7)\} = \triangle_{14}.$

(e) Orbits with no element from B;
$Orb_{G_{(1,3,5,7)}}(2,4,6,8) = \{(2,4,6,8)\} = \triangle_{15}.$

Thus rank of action of $G = S_2 \times S_2 \times S_2 \times S_2$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 with subdegrees $\underbrace{1,1,\cdots,1,1}_{16 \text{ factors}}$. □

Lemma 2.10. *Rank of* $S_3 \times S_3 \times S_3 \times S_3$ *acting on* $X_1 \times X_2 \times X_3 \times X_4$ *is* 2^4.

Proof. Let $G = S_3 \times S_3 \times S_3 \times S_3$ on $K = X_1 \times X_2 \times X_3 \times X_4$. Then $|K| = 81$ Using Theorem 2.2,

$G_{(1,4,7,10)} = \{(e_1,e_2,e_3,e_4),((2,3),e_2,e_3,e_4),(e_1,(5,6),e_3,e_4),((2,3),(5,6),e_4),(e_1,e_2,(8,9),$
$e_4),((2,3),e_2,(8,9),e_4),(e_1,(5,6),(8,9),e_4),((2,3),(5,6),(8,9),e_4),(e_1,e_2,e_3,(11,12)),$
$((2,3),e_2,e_3,(11,12)),(e_1,(5,6),e_3,(11,12)),((2,3),(5,6),e_3,(11,12)),(e_1,e_2,(8,9),$
$(11,12)),((2,3),e_2,(8,9),(11,12)),(e_1,(5,6),(8,9),(11,12)),((2,3),(5,6),(8,9),(11,12))\}.$

The group $G_{(1,4,7,10)}$ is isomorphic to $S_2 \times S_2 \times S_2 \times S_2$ with S_2 having permutations of types (I) and (ab). Thus the number of elements in $X_1 \times X_2 \times X_3 \times X_4$ fixed by each $g_1, g_2, g_3, g_4 \in G_{(1,4,7,10)}$ are given in Table 1

Table 1: Permutations in $G_{(1,4,7,10)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,4,7,10)}$	Number of quadraple permutations in $G_{(1,4,7,10)}$	$\|Fix(g_1, g_2, g_3, g_4)\|$
(e_1, e_2, e_3, e_4)	1	81
$(e_1, e_2, e_3, (ab))$	1	27
$(e_1, e_2, (ab), e_4)$	1	27
$(e_1, e_2, (ab), (ab))$	1	9
$(e_1, (ab), e_3, e_4)$	1	27
$(e_1, (ab), e_3, (ab))$	1	9
$(e_1, (ab), (ab), e_4)$	1	9
$(e_1, (ab), (ab), (ab))$	1	3
$((ab), e_2, e_3, e_4)$	1	27
$((ab), e_2, e_3, (ab))$	1	9
$((ab), e_2, (ab), e_4)$	1	9
$((ab), e_2, (ab), (ab))$	1	3
$((ab), (ab), e_3, e_4)$	1	9
$((ab), (ab), e_3, (ab))$	1	3
$((ab), (ab), (ab), e_4)$	1	3
$((ab), (ab), (ab), (ab))$	1	1

By applying Lemma 1.1, the number of orbits of $G_{(1,4,7,10)}$ acting on K is;

$$\frac{1}{|G_{(1,4,7)}|}\sum_{g_1,g_2,g_3\in G_{(1,4,7)}}|Fix(g_1,g_2,g_3)| = \frac{1}{16}\big[(1\times 81)+(1\times 27)+(1\times 27)+(1\times 9) + (1\times 27)+(1\times 9)+(1\times 9)+(1\times 3) + (1\times 27)+(1\times 9)+(1\times 9)+(1\times 3) + (1\times 9)+(1\times 3)+(1\times 3)+(1\times 1)\big] = 16 = 2^4$$

Let $B = \{1, 4, 7, 10\}$
Orbits of $G_{(1,4,7,10)}$ on K are those with exactly $4, 3, 2, 1$, and no element from B.

(a) Orbits with exactly four elements of B are;
$Orb_{G_{(1,4,7,10)}}(1,4,7,10) = \{(1,4,7,10)\} = \triangle_0$

(b) Orbits with exactly three elements of B are;
$Orb_{G_{(1,4,7,10)}}(1,4,7,11) = \{(1,4,7,11),(1,4,7,12)\} = \triangle_1$.
$Orb_{G_{(1,4,7,10)}}(1,4,8,10) = \{(1,4,8,10),(1,4,9,10)\} = \triangle_2$.
$Orb_{G_{(1,4,7,10)}}(1,5,7,10) = \{(1,5,7,10),(1,6,7,10)\} = \triangle_3$.
$Orb_{G_{(1,4,7,10)}}(2,4,7,10) = \{(2,4,7,10),(3,4,7,10)\} = \triangle_4$.

(c) Orbits containing exactly two elements of B are;
$Orb_{G_{(1,4,7,10)}}(1,4,8,11) = \{(1,4,8,11),(1,4,8,12),(1,4,9,11),(1,4,9,11)\} = \triangle_5$.
$Orb_{G_{(1,4,7,10)}}(1,5,7,11) = \{(1,5,7,11),(1,5,7,12),(1,6,7,11),(1,6,7,12)\} = \triangle_6$.

$Orb_{G_{(1,4,7,10)}}(1,5,8,10) = \{(1,5,8,10),(1,5,9,10),(1,6,8,10),(1,6,9,10)\} = \triangle_7$.
$Orb_{G_{(1,4,7,10)}}(2,4,7,11) = \{(2,4,7,11),(2,4,7,12),(3,4,7,11),(3,4,7,12)\} = \triangle_8$.
$Orb_{G_{(1,4,7,10)}}(2,5,7,10) = \{(2,5,7,10),(2,6,7,10),(3,5,7,10),(3,6,7,10)\} = \triangle_9$.
$Orb_{G_{(1,4,7,10)}}(2,4,8,10) = \{(2,4,8,10),(2,4,9,10),(3,4,8,10),(3,4,9,10)\} = \triangle_{10}$.

(d) Orbits containing exactly one element of B are;
$Orb_{G_{(1,4,7,10)}}(1,5,8,11) = \{(1,5,8,11),(1,5,8,12),(1,5,9,11),(1,6,8,11),(1,5,9,12),(1,6,8,12),(1,6,9,11),(1,6,9,12)\} = \triangle_{11}$.
$Orb_{G_{(1,4,7,10)}}(2,4,8,11) = \{(2,4,8,11),(2,4,8,12),(2,4,9,11),(3,4,8,11),(2,4,9,12),(3,4,8,12),(3,4,9,11),(3,4,9,12)\} = \triangle_{12}$.
$Orb_{G_{(1,4,7,10)}}(2,5,7,11) = \{(2,5,7,11),(2,5,7,12),(2,6,7,11),(3,5,7,11),(2,6,7,12),(3,5,7,12),(3,6,7,11),(3,6,7,12)\} = \triangle_{13}$.
$Orb_{G_{(1,4,7,10)}}(2,5,8,10) = \{(2,5,8,10),(2,5,9,10),(2,6,8,10),(3,5,8,10),(2,6,9,10),(3,5,9,10),(3,6,8,10),(3,6,9,10)\} = \triangle_{14}$.

(e) Orbits containing no element of B are;
$Orb_{G_{(1,4,7,10)}}(2,5,8,11) = \{(2,5,8,11),(2,5,8,12),(2,5,9,11),(2,6,8,11),(3,5,8,11),(2,5,9,12),(2,6,8,12),(3,5,8,12),(2,6,9,11),(3,5,9,11),(3,6,8,11),(2,6,9,12),(3,5,9,12),(3,6,8,12),(3,6,9,11),(3,6,9,12)\} = \triangle_{15}$.

Therefore the rank of $S_3 \times S_3 \times S_3 \times S_3$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 and the subdegrees are $1, \underbrace{2,\cdots,2,2}_{4\ \text{factors}}, \underbrace{4,\cdots,4,4}_{6\ \text{factors}}, \underbrace{8,\cdots,8,8}_{4\ \text{factors}}, 16$. □

Remark 2.1. Table 2 shows the respective subdegrees of $S_3 \times S_3 \times S_3 \times S_3$ acting on $X_1 \times X_2 \times X_3 \times X_4$.

Table 2: Subdegrees of $S_3 \times S_3 \times S_3 \times S_3$ on $X_1 \times X_2 \times X_3 \times X_4$

Number of suborbits	1	4	6	4	1
Subdegree	1	2	4	8	16

Lemma 2.11. *$S_4 \times S_4 \times S_4 \times S_4$ acts on $X_1 \times X_2 \times X_3 \times X_4$ with rank 2^4.*

Proof. Let $G = S_4 \times S_4 \times S_4 \times S_4$ on $K = X_1 \times X_2 \times X_3 \times X_4$. Using Theorem 2.3, $|K| = 256$ and $|G_{(1,5,9,13)}| = 1296$.
It is observed that $G_{(1,5,9,13)}$ is isomorphic to $S_3 \times S_3 \times S_3 \times S_3$ whrere S_3 have

permutations of types $(I),\ (ab)$ and (abc) with the respective numbers $1, 3$ and 2. Thus the number of elements in $X_1 \times X_2 \times X_3 \times X_4$ fixed by each $g_1, g_2, \cdots, g_4 \in G_{(1,5,9,13)}$ are given in Tables 3, 4, and 5.

Table 3: Permutations in $G_{(1,5,9,13)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,5,9,13)}$	Number of quadraple permutations in $G_{(1,5,9,13)}$	$\|Fix(g_1, \cdots, g_4)\|$
(e_1, e_2, e_3, e_4)	1	256
$(e_1, e_2, e_3, (ab))$	3	128
$(e_1, e_2, e_3, (abc))$	2	64
$(e_1, e_2, (ab), e_4)$	3	128
$(e_1, e_2, (ab), (ab))$	9	64
$(e_1, e_2, (ab), (abc))$	6	32
$(e_1, e_2, (abc), e_4)$	2	64
$(e_1, e_2, (abc), (ab))$	6	32
$(e_1, e_2, (abc), (abc))$	4	16
$(e_1, (ab), e_3, e_4)$	3	128
$(e_1, (ab), e_3, (ab))$	9	64
$(e_1, (ab), e_3, (abc))$	6	32
$(e_1, (ab), (ab), e_4)$	9	64
$(e_1, (ab), (ab), (ab))$	27	32
$(e_1, (ab), (ab), (abc))$	18	16
$(e_1, (ab), (abc), e_4)$	6	32
$(e_1, (ab), (abc), (ab))$	18	16
$(e_1, (ab), (abc), (abc))$	12	8
$(e_1, (abc), e_3, e_4)$	2	64
$(e_11, (abc), e_3, (ab))$	6	32
$(e_1, (abc), e_3, (abc))$	4	16
$(e_1, (abc), (ab), e_4)$	6	32
$(e_1, (abc), (ab), (ab))$	18	16
$(e_1, (abc), (ab), (abc))$	12	8
$(e_1, (abc), (abc), e_4)$	4	16
$(e_11, (abc), (abc), (ab))$	12	8
$(e_1, (abc), (abc), (abc))$	8	4

Table 4: Permutations in $G_{(1,5,9,13)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,5,9,13)}$	Number of quadraple permutations in $G_{(1,5,9,13)}$	$\|Fix(g_1,\cdots,g_4)\|$
$((ab), e_2, e_3, e_4)$	3	128
$((ab), e_2, e_3, (ab))$	9	64
$((ab), e_2, e_3, (abc))$	6	32
$((ab), e_2, (ab), e_4)$	9	64
$((ab), e_2, (ab), (ab))$	27	32
$((ab), e_2, (ab), (abc))$	18	16
$((ab), e_2, (abc), e_4)$	6	32
$((ab), e_2, (abc), (ab))$	18	16
$((ab), e_2, (abc), (abc))$	12	8
$((ab), (ab), e_3, e_4)$	9	64
$((ab), (ab), e_3, (ab))$	27	32
$((ab)1, (ab), e_3, (abc))$	18	16
$((ab), (ab), (ab), e_4)$	27	32
$((ab), (ab), (ab), (ab))$	81	16
$((ab), (ab), (ab), (abc))$	54	8
$((ab), (ab), (abc), e_4)$	18	16
$((ab), (ab), (abc), (ab))$	54	8
$((ab), (ab), (abc), (abc))$	36	4
$((ab), (abc), e_3, e_41)$	6	32
$((ab), (abc), e_3, (ab))$	18	16
$((ab), (abc), e_3, (abc))$	12	8
$((ab), (abc), (ab), e_4)$	18	16
$((ab), (abc), (ab), (ab))$	54	8
$((ab), (abc), (ab), (abc))$	36	4
$((ab), (abc), (abc), e_4)$	12	8
$((ab), (abc), (abc), (ab))$	36	4
$((ab), (abc), (abc), (abc))$	24	2

Table 5: Permutations in $G_{(1,5,9,13)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,5,9,13)}$	Number of quadraple permutations in $G_{(1,5,9,13)}$	$\lvert Fix(g_1, \cdots, g_4)\rvert$
$((abc), e_2, e_3, e_4)$	2	64
$((abc), e_2, e_3, (ab))$	6	32
$((abc), e_2, e_3, (abc))$	4	16
$((abc), e_2, (ab), e_4)$	6	32
$((abc), e_2, (ab), (ab))$	18	16
$((abc), e_2, (ab), (abc))$	12	8
$((abc), e_2, (abc), e_4)$	4	16
$((abc), e_2, (abc), (ab))$	12	8
$((abc), e_2, (abc), (abc))$	8	4
$((abc), (ab), e_3, e_4)$	6	32
$((abc), (ab), e_3, (ab))$	18	16
$((abc), (ab), e_3, (abc))$	12	8
$((abc), (ab), (ab), e_3)$	18	16
$((abc), (ab), (ab), (ab))$	54	8
$((abc), (ab), (ab), (abc))$	36	4
$((abc), (ab), (abc), e_4)$	12	8
$((abc), (ab), (abc), (ab))$	36	4
$((abc), (ab), (abc), (abc))$	24	2
$((abc), (abc), e_3, e_4)$	4	16
$((abc), (abc), e_3, (ab))$	12	8
$((abc), (abc), e_3, (abc))$	8	4
$((abc), (abc), (ab), e_4)$	12	8
$((abc), (abc), (ab), (ab))$	36	4
$((abc), (abc), (ab), (abc))$	24	2
$((abc), (abc), (abc), e_4)$	8	4
$((abc), (abc), (abc), (ab))$	24	2
$((abc), (abc), (abc), (abc))$	16	1

Applying Lemma 1.1 on Tables 3, 4, and 5, the number of orbits of $G_{(1,5,9,13)}$ acting on K is;

$$\frac{1}{|G_{(1,5,9,13)}|} \sum_{g_1,g_2,\cdots,g_4 \in G_{(1,5,9,13)}} |Fix(g_1, g_2, \cdots, g_4)| = \frac{20736}{1296} = 2^4$$

Let $B = \{1, 5, 9, 13\}$
Orbits of $G_{(1,5,9,13)}$ on K are those with exactly $4, 3, 2, 1$, and no element from B.

(a) Orbits with exactly four elements of B are;
$Orb_{G_{(1,5,9,13)}}(1, 5, 9, 13) = \{(1, 5, 9, 13)\} = \triangle_0$

(b) Orbits with exactly three elements of B are;
$Orb_{G_{(1,5,9,13)}}(1, 5, 9, 14) = \{(1, 5, 9, 14), (1, 5, 9, 15), (1, 5, 9, 16)\} = \triangle_1$.
$Orb_{G_{(1,5,9,13)}}(1, 5, 10, 13) = \{(1, 5, 10, 13), (1, 5, 12, 13), (1, 5, 11, 13)\} = \triangle_2$.
$Orb_{G_{(1,5,9,13)}}(1, 6, 9, 13) = \{(1, 6, 9, 13), (1, 7, 9, 13), (1, 8, 9, 13)\} = \triangle_3$.
$Orb_{G_{(1,5,9,13)}}(2, 5, 9, 13) = \{(2, 5, 9, 13), (3, 5, 9, 13), (4, 5, 9, 13), \} = \triangle_4$.

(c) Orbits containing exactly two elements of B are;
$Orb_{G_{(1,5,9,13)}}(1, 5, 10, 14) =$

$\{(1,5,10,14),(1,5,10,15),(1,5,10,16),(1,5,12,14),$
$(1,5,12,15),(1,5,12,16),(1,5,11,14),(1,5,11,15),(1,5,11,16)\} = \triangle_5.$
$Orb_{G_{(1,5,9,13)}}(1,6,9,14) = \{(1,6,9,14),(1,6,9,15),(1,6,9,16),(1,7,9,14),$
$(1,7,9,15),(1,7,9,16),(1,8,9,14),(1,8,9,15),(1,8,9,16)\} = \triangle_6.$
$Orb_{G_{(1,5,9,13)}}(1,6,10,13) =$
$\{(1,6,10,13),(1,6,11,13),(1,6,12,13),(1,7,10,13),$
$(1,7,11,13),(1,7,12,13),(1,8,10,13),(1,8,11,13),(1,8,12,13)\} = \triangle_7.$
$Orb_{G_{(1,5,9,13)}}(2,5,9,14) = \{(2,5,9,14),(2,5,9,15),(2,5,9,16),(3,5,9,14),$
$(3,5,9,15),(3,5,9,16),(4,5,9,14),(4,5,9,15),(4,5,9,16)\} = \triangle_8.$
$Orb_{G_{(1,5,9,13)}}(2,5,10,13) =$
$\{(2,5,10,13),(2,5,11,13),(2,5,12,13),(3,5,10,13),$
$(3,5,11,13),(3,5,12,13),(4,5,10,13),(4,5,11,13),(4,5,12,13)\} = \triangle_9.$
$Orb_{G_{(1,5,9,13)}}(2,6,9,13) = \{(2,6,9,13),(2,7,9,13),(2,8,9,13),(3,6,9,13),$
$(3,7,9,13),(3,8,9,13),(4,6,9,13),(4,7,9,13),(4,8,9,13)\} = \triangle_{10}.$

(d) Orbits containing exactly one element of B are;
$Orb_{G_{(1,5,9,13)}}(1,6,10,14) =$
$\{(1,6,10,14),(1,6,10,16),(1,6,12,14),(1,8,10,14),$
$(1,6,10,15),(1,6,12,16),(1,8,10,16),(1,6,11,14),(1,8,12,14),(1,7,10,14),$
$(1,6,12,15),(1,8,10,15),(1,6,11,16),(1,8,12,16),(1,7,10,16),(1,8,11,14),$
$(1,7,12,14),(1,6,11,15),(1,8,12,15),(1,7,10,15),(1,8,11,16),(1,7,12,16),$
$(1,7,11,14),(1,8,11,15),(1,7,12,15),(1,7,11,16),(1,7,11,15)\} = \triangle_{11}.$
$Orb_{G_{(1,5,9,13)}}(2,5,10,14) =$
$\{(2,5,10,14),(2,5,10,16),(2,5,12,14),(4,5,10,14),$
$(2,5,10,15),(2,5,12,16),(4,5,10,16),(2,5,11,14),(4,5,12,14),(3,5,10,14),$
$(2,5,12,15),(4,5,10,15),(2,5,11,16),(4,5,12,16),(3,5,10,16),(4,5,11,14),$
$(3,5,12,14),(2,5,11,15),(4,5,12,15),(3,5,10,15),(4,5,11,16),(3,5,12,16),$
$(3,5,11,14),(4,5,11,15),(3,5,12,15),(3,5,11,16),(3,5,11,15)\} = \triangle_{12}.$
$Orb_{G_{(1,5,9,13)}}(2,6,9,14) = \{(2,6,9,14),(2,6,9,16),(2,8,9,14),(4,6,9,14),$
$(2,6,9,15),(2,8,9,16),(4,6,9,16),(2,7,9,14),(4,8,9,14),(3,6,9,14),$
$(2,8,9,15),(4,6,9,15),(2,7,9,16),(4,8,9,16),(3,6,9,16),(4,7,9,14),$
$(3,8,9,14),(2,7,9,15),(4,8,9,15),(3,6,9,15),(4,7,9,16),(3,8,9,16),$
$(3,7,9,14),(4,7,9,15),(3,8,9,15),(3,7,9,16),(3,7,9,15)\} = \triangle_{13}.$
$Orb_{G_{(1,5,9,13)}}(2,6,10,13) =$
$\{(2,6,10,13),(2,6,12,13),(2,8,10,13),(4,6,10,13),$
$(2,6,11,13),(2,8,12,13),(4,6,12,13),(2,7,10,13),(4,8,10,13),(3,6,10,13),$
$(2,8,11,13),(4,6,11,13),(2,7,12,13),(4,8,12,13),(3,6,12,13),(4,7,10,13),$
$(3,8,10,13),(2,7,11,13),(4,8,11,13),(3,6,11,13),(4,7,12,13),(3,8,12,13),$
$(3,7,10,13),(4,7,11,13),(3,8,11,13),(3,7,12,13),(3,7,11,13)\} = \triangle_{14}.$

(e) Orbits containing no element of B are;
$Orb_{G_{(2,6,10,14)}}(2,5,8,11) =$
$\{(2,6,10,14),(2,6,10,16),(2,6,12,14),(2,8,10,14),$
$(4,6,10,14),(2,6,10,15),(2,6,12,16),(2,8,10,16),(4,6,10,16),(2,6,11,14),$
$(2,8,12,14),(4,6,12,14),(2,7,10,14),(4,8,10,14),(3,6,10,14),(2,6,12,15),$
$(2,8,10,15),(4,6,10,15),(2,6,11,16),(2,8,12,16),(4,6,12,16),(2,7,10,16),$
$(4,8,10,16),(3,6,10,16),(2,8,11,14),(4,6,11,14),(2,7,12,14),(4,8,12,14),$
$(3,6,12,14),(4,7,10,14),(3,8,10,14),(2,6,11,15),(2,8,12,15),(4,6,12,15),$
$(2,7,10,15),(4,8,10,15),(3,6,10,15),(2,8,11,16),(4,6,11,16),(2,7,12,16),$
$(4,8,12,16),(3,6,12,16),(4,7,10,16),(3,8,10,16),(2,7,11,14),(4,8,11,14),$
$(3,6,11,14),(4,7,12,14),(3,8,12,14),(3,7,10,14),(2,8,11,15),(4,6,11,15),$
$(2,7,12,15),(4,8,12,15),(3,6,12,15),(4,7,10,15),(3,8,10,15),(2,7,11,16),$
$(4,8,11,16),(3,6,11,16),(4,7,12,16),(3,8,12,16),(3,7,10,16),(4,7,11,14),$
$(3,8,11,14),(3,7,12,14),(2,7,11,15),(4,8,11,15),(3,6,11,15),(4,7,12,15),$
$(3,8,12,15),(3,7,10,15),(4,7,11,16),(3,8,11,16),(3,7,12,16),(3,7,11,14),$
$(4,7,11,15),(3,8,11,15),(3,7,12,15),(3,7,11,16),(3,7,11,15)\} = \triangle_{15}.$

Therefore the rank of $S_4 \times S_4 \times S_4 \times S_4$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 and the respective subdegrees are $1, \underbrace{3, \cdots, 3, 3}_{4 \text{ factors}}, \underbrace{9, \cdots, 9, 9}_{6 \text{ factors}}, \underbrace{27, \cdots, 27, 27}_{4 \text{ factors}}, 81$. □

Remark 2.2. Table 6 shows the respective subdegrees of $S_4 \times S_4 \times S_4 \times S_4$ acting on $X_1 \times X_2 \times X_3 \times X_4$.

Table 6: Subdegrees of $S_4 \times S_4 \times S_4 \times S_4$ on $X_1 \times X_2 \times X_3 \times X_4$

Number of suborbits	1	4	6	4	1
Subdegree	1	4	16	64	256

Lemma 2.12. *$S_5 \times S_5 \times S_5 \times S_5$ acts on $X_1 \times X_2 \times X_3 \times X_4$ with rank 2^4.*

Proof. Let $G = S_5 \times S_5 \times S_5 \times S_5$ on $K = X_1 \times X_2 \times X_3 \times X_4$. Using Theorem 2.4, $|K| = 625$ and $|G_{(1,6,11,16)}| = 625$
Let $B = \{1, 6, 11, 16\}$
Orbits of $G_{(1,6,11,16)}$ on K are those with exactly $4, 3, 2, 1$, and no element from B.

(a) Orbits with exactly four elements of B are;
$Orb_{G_{(1,6,11,16)}}(1,6,11,16) = \{(1,6,11,16)\} = \triangle_0$

(b) Orbits with exactly three elements of B are;
$Orb_{G_{(1,6,11,16)}}(1,6,11,17) = \{(1,6,11,17),(1,6,11,18),(1,6,11,19),$

$(1, 6, 11, 20)\} = \triangle_1$.
$Orb_{G_{(1,6,11,16)}}(1, 6, 12, 16) = \{(1, 6, 12, 16), (1, 6, 13, 16), (1, 6, 14, 16),$
$(1, 6, 15, 16)\} = \triangle_2$.
$Orb_{G_{(1,6,11,16)}}(1, 7, 11, 16) = \{(1, 7, 11, 16), (1, 8, 11, 16), (1, 9, 11, 16),$
$(1, 10, 11, 16)\} = \triangle_3$.
$Orb_{G_{(1,6,11,16)}}(2, 6, 11, 16) = \{(2, 6, 11, 16), (3, 6, 11, 16), (4, 6, 11, 16),$
$(5, 6, 11, 16\} = \triangle_4$.

(c) Orbits containing exactly two elements of B are;

$Orb_{G_{(1,6,11,16)}}(1, 6, 12, 17) =$
$\{(1, 6, 12, 17), (1, 6, 12, 18), (1, 6, 12, 19), (1, 6, 12, 20),$
$(1, 6, 13, 17), (1, 6, 13, 18), (1, 6, 13, 19), (1, 6, 13, 20), (1, 6, 14, 17), (1, 6, 14, 18),$
$(1, 6, 14, 19), (1, 6, 14, 20), (1, 6, 15, 17), (1, 6, 15, 18), , (1, 6, 15, 19), (1, 6, 15, 20)\}$
$= \triangle_5$.
$Orb_{G_{(1,6,11,16)}}(1, 7, 11, 17) =$
$\{1, 7, 11, 17), (1, 7, 11, 18), (1, 7, 11, 19), (1, 7, 11, 20),$
$(1, 8, 11, 17), (1, 8, 11, 18), (1, 8, 11, 19), (1, 8, 11, 20), (1, 9, 11, 17), (1, 9, 11, 18),$
$(1, 9, 11, 19), (1, 9, 11, 20), (1, 10, 11, 17), (1, 10, 11, 18), (1, 10, 11, 19), (1, 10, 11, 20)\}$
$= \triangle_6$.
$Orb_{G_{(1,6,11,16)}}(1, 7, 12, 16) =$
$\{(1, 7, 12, 16), (1, 7, 13, 16), (1, 7, 14, 16), (1, 7, 15, 16),$
$(1, 8, 12, 16), (1, 8, 13, 16), (1, 8, 14, 16), (1, 8, 15, 16), (1, 9, 12, 16), (1, 9, 13, 16),$
$(1, 9, 14, 16), (1, 9, 15, 16), (1, 10, 12, 16), (1, 10, 13, 16), (1, 10, 14, 16), (1, 10, 15, 16)\}$
$= \triangle_7$.
$Orb_{G_{(1,6,11,16)}}(2, 6, 11, 17) =$
$\{(2, 6, 11, 17), (2, 6, 11, 18), (2, 6, 11, 19), (2, 6, 11, 20),$
$(3, 6, 11, 17), (3, 6, 11, 18), (3, 6, 11, 19), (3, 6, 11, 20), (4, 6, 11, 17), (4, 6, 11, 18),$
$(4, 6, 11, 19), (4, 6, 11, 20), (5, 6, 11, 17), (5, 6, 11, 18), (5, 6, 11, 19), (5, 6, 11, 20)\}$
$= \triangle_8$.
$Orb_{G_{(1,6,11,16)}}(2, 6, 12, 16) =$
$\{(2, 6, 12, 16), (2, 6, 13, 16), (2, 6, 14, 16), (2, 6, 15, 16),$
$(3, 6, 12, 16), (3, 6, 13, 16), (3, 6, 14, 16), (3, 6, 15, 16), (4, 6, 12, 16), (4, 6, 13, 16),$
$(4, 6, 14, 16), , (4, 6, 15, 16), (5, 6, 12, 16), (5, 6, 13, 16), (5, 6, 14, 16), (5, 6, 15, 16)\}$
$= \triangle_9$.
$Orb_{G_{(1,6,11,16)}}(2, 7, 11, 16) =$
$\{(2, 7, 11, 16), (2, 8, 11, 16), (2, 9, 11, 16), (2, 10, 11, 16),$
$(3, 7, 11, 16), (3, 8, 11, 16), (3, 9, 11, 16), (3, 10, 11, 16), (4, 7, 11, 16), (4, 8, 11, 16),$
$(4, 9, 11, 16), (4, 10, 11, 16), (5, 7, 11, 16), (5, 8, 11, 16), (5, 9, 11, 16), (5, 10, 11, 16)\}$
$= \triangle_{10}$.

(d) Orbits containing exactly one element of B are;

$Orb_{G_{(1,6,11,16)}}(1,7,12,17) =$
$\{(1,7,12,17),(1,7,12,18),(1,7,12,19),(1,7,12,20),$
$(1,7,13,17),(1,7,13,18),(1,7,13,19),(1,7,13,20),(1,7,14,17),(1,7,14,18),$
$(1,7,14,19),(1,7,14,20),(1,7,15,17),(1,7,15,18),(1,7,15,19),(1,7,15,20),$
$(1,8,12,17),(1,8,12,18),(1,8,12,19),(1,8,12,20),(1,8,13,17),(1,8,13,18),$
$(1,8,13,19),(1,8,13,20),(1,8,14,17),(1,8,14,18),(1,8,14,19),(1,8,14,20),$
$(1,8,15,17),(1,8,15,17),(1,8,15,17),(1,8,15,17),(1,9,12,17),(1,9,12,18),$
$(1,9,12,19),(1,9,12,20),(1,9,13,17),(1,9,13,18),(1,9,13,19),(1,9,13,20),$
$(1,9,14,17),(1,9,14,17),(1,9,14,17),(1,9,14,17),(1,9,15,17),(1,9,15,17),$
$(1,9,15,17),(1,9,15,17),(1,10,12,17),(1,10,12,17),(1,10,12,17),(1,10,12,17),$
$(1,10,13,17),(1,10,13,17),(1,10,13,17),(1,10,13,17),(1,10,14,17),(1,10,14,17),$
$(1,10,14,17),(1,10,14,17),(1,10,15,17),(1,10,15,17),(1,10,15,17),(1,10,15,17)\}$
$= \triangle_{11}.$

$Orb_{G_{(1,6,11,16)}}(2,6,12,17) =$
$\{(2,6,12,17),(2,6,12,18),(2,6,12,19),(2,6,12,20),$
$(2,6,13,17),(2,6,13,18),(2,6,13,19),(2,6,13,20),(2,6,14,17),(2,6,14,17),$
$(2,6,14,17),(2,6,14,17),(2,6,15,17),(2,6,15,18),(2,6,15,19),(2,6,15,20),$
$(3,6,12,17),(3,6,12,18),(3,6,12,19),(3,6,12,20),(3,6,13,17),(3,6,13,18),$
$(3,6,13,19),(3,6,13,20),(3,6,14,17),(3,6,14,18),(3,6,14,19),(3,6,14,20),$
$(3,6,15,17),(3,6,15,18),(3,6,15,19),(3,6,15,20),(4,6,12,17),(4,6,12,17),$
$(4,6,12,17),(4,6,12,17),(4,6,13,17),(4,6,13,17),(4,6,13,17),(4,6,13,17),$
$(4,6,14,17),(4,6,14,17),(4,6,14,17),(4,6,14,17),(4,6,15,17),(4,6,15,17),$
$(4,6,15,17),(4,6,15,17),(5,6,12,17),(5,6,12,18),(5,6,12,19),(5,6,12,20),$
$(5,6,13,17),(5,6,13,18),(5,6,13,19),(5,6,13,20),(5,6,14,17),(5,6,14,18),$
$(5,6,14,19),(5,6,14,20),(5,6,15,17),(5,6,15,18),(5,6,15,19),(5,6,15,20)\}$
$= \triangle_{12}.$

$Orb_{G_{(1,6,11,16)}}(2,7,11,17) =$
$\{(2,7,11,17),(2,7,11,18),(2,7,11,19),(2,7,11,20),$
$(2,8,11,17),(2,8,11,17),(2,8,11,17),(2,8,11,20),(2,9,11,17),(2,9,11,17),$
$(2,9,11,17),(2,9,11,20),(2,10,11,17),(2,10,11,17),(2,10,11,17),(2,10,11,20),$
$(3,7,11,17),(3,7,11,18),(3,7,11,19),(3,7,11,20),(3,8,11,17),(3,8,11,18),$
$(3,8,11,19),(3,8,11,20),(3,9,11,17),(3,9,11,18),(3,9,11,19),(3,9,11,20),$
$(3,10,11,17),(3,10,11,18),(3,10,11,19),(3,10,11,20),(4,7,11,17),(4,7,11,18),$
$(4,7,11,19),(4,7,11,20),(4,8,11,17),(4,8,11,18),(4,8,11,19),(4,8,11,20),$
$(4,9,11,17),(4,9,11,18),(4,9,11,19),(4,9,11,20),(4,10,11,17),(4,10,11,18),$
$(4,10,11,19),(4,10,11,20),(5,7,11,17),(5,7,11,18),(5,7,11,19),(5,7,11,20),$
$(5,8,11,17),(5,8,11,18),(5,8,11,19),(5,8,11,20),(5,9,11,17),(5,9,11,18),$
$(5,9,11,19),(5,9,11,20),(5,10,11,17),(5,10,11,18),(5,10,11,19),(5,10,11,20)\}$

$= \triangle_{13}$.
$Orb_{G_{(1,6,11,16)}}(2,7,12,16) =$
$\{(2,7,12,16),(2,7,13,16),(2,7,14,16),(2,7,15,16),$
$(2,8,12,16),(2,8,13,16),(2,8,14,16),(2,8,15,16),(2,9,12,16),(2,9,13,16),$
$(2,9,14,16),(2,9,15,16),(2,10,12,16),(2,10,13,16),(2,10,14,16),(2,10,15,16),$
$(3,7,12,16),(3,7,13,16),(3,7,14,16),(3,7,15,16),(3,8,12,16),(3,8,13,16),$
$(3,8,14,16),(3,8,15,16),(3,9,12,16),(3,9,13,16),(3,9,14,16),(3,9,15,16),$
$(3,10,12,16),(3,10,13,16),(3,10,14,16),(3,10,15,16),(4,7,12,16),(4,7,13,16),$
$(4,7,14,16),(4,7,15,16),(4,8,12,16),(4,8,13,16),(4,8,14,16),(4,8,15,16),$
$(4,9,12,16),(4,9,13,16),(4,9,14,16),(4,9,15,16),(4,10,12,16),(4,10,13,16),$
$(4,10,14,16),(4,10,15,16),(5,7,12,16),(5,7,13,16),(5,7,14,16),(5,7,15,16),$
$(5,8,12,16),(5,8,13,16),(5,8,14,16),(5,8,15,16),(5,9,12,16),(5,9,13,16),$
$(5,9,14,16),(5,9,15,16),(5,10,12,16),(5,10,13,16),(5,10,14,16),(5,10,15,16)\}$
$= \triangle_{14}$.

(e) Orbits containing no element of B are;
$Orb_{G_{(1,6,11,16)}}(2,7,12,17) =$
$\{(2,7,12,17),(2,7,12,18),(2,7,12,18),(2,7,12,18),$
$(2,7,13,17),(2,7,13,18),(2,7,13,19),(2,7,13,20),(2,7,14,17),(2,7,14,18),$
$(2,7,14,19),(2,7,14,20),(2,7,15,17),(2,7,15,18),(2,7,15,19),(2,7,15,20)$
$,(2,8,12,17),(2,8,12,18),(2,8,12,18),(2,8,12,18),(2,8,13,17),(2,8,13,18),$
$(2,8,13,19),(2,8,13,20),(2,8,14,17),(2,8,14,18),(2,8,14,19),(2,8,14,20),$
$(2,8,15,17),(2,8,15,18),(2,8,15,19),(2,8,15,20),\cdots,(2,10,12,17),$
$(2,10,12,18),(2,10,12,18),(2,10,12,18),(2,10,13,17),(2,10,13,18),(2,10,13,19),$
$(2,10,13,20),(2,10,14,17),(2,10,14,18),(2,10,14,19),(2,10,14,20),(2,10,15,17),$
$(2,10,15,18),(2,10,15,19),(2,10,15,20),\cdots,(5,7,12,17),(5,7,12,18),$
$(2,7,12,18),(2,7,12,18),(5,7,13,17),(5,7,13,18),(2,7,13,19),(2,7,13,20),$
$(5,7,14,17),(5,7,14,18),(2,7,14,19),(2,7,14,20),(5,7,15,17),(5,7,15,18),$
$(2,7,15,19),(2,7,15,20),(5,8,12,17),(5,8,12,18),(5,8,12,18),(5,8,12,18),$
$(5,8,13,17),(5,8,13,18),(5,8,13,19),(5,8,13,20),(5,8,14,17),(5,8,14,18),$
$(5,8,14,19),(5,8,14,20),(5,8,15,17),(5,8,15,18),(5,8,15,19),(5,8,15,20),$
$(5,10,12,17),(5,10,12,18),(5,10,12,18),(5,10,12,18),(5,10,13,17),(5,10,13,18),$
$(5,10,13,19),(5,10,13,20),(5,10,14,17),(5,10,14,18),(5,10,14,19),(5,10,14,20),$
$(5,10,15,17),(5,10,15,18),(5,10,15,19),(5,10,15,20)\} = \triangle_{15}$.

Therefore the rank of $S_5 \times S_5 \times S_5 \times S_5$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 and the subdegrees are $1, \underbrace{4,4,\cdots,4}_{4 \text{ factors}}, \underbrace{16,16,\cdots,16,16}_{6 \text{ factors}}, \underbrace{64,\cdots,64}_{4 \text{ factors}}, 256$. □

Remark 1. Table 7 shows the respective subdegrees of $S_5 \times S_5 \times S_5 \times S_5$ acting on $X_1 \times X_2 \times X_3 \times X_4$.

Table 7: Subdegrees of $S_5 \times S_5 \times S_5 \times S_5$ on $X_1 \times X_2 \times X_3 \times X_4$

Number of suborbits	1	4	6	4	1
Subdegree	1	4	16	64	256

Theorem 2.3. *For $n \geq 2$, the rank of $S_n \times S_n \times S_n \times S_n$ acting on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4.*

Proof. Let $G = S_n \times S_n \times S_n \times S_n$ and $K = X_1 \times X_2 \times X_3 \times X_4$ where $X_1 = \{1, 2, \cdots, n\}$, $X_2 = \{n+1, n+2, \cdots, 2n\}$, $X_3 = \{2n+1, 2n+2, \cdots, 3n\}$, and $X_4 = \{3n+1, 3n+2, \cdots, 4n\}$. Then by Theorem 2.1, $|K| = n^5$ and $G_{(1,n+1,2n+1,3n+1)} = ((n-1)!)^5$.
Let $B = \{1, n+1, 2n+1, 3n+1\}$
Orbits of $G_{(1,n+1,2n+1,3n+1)}$ on K are those with exactly $4, 3, 2, 1$, and no element from B.

(a) Orbits with exactly four elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+1, 3n+1) = \{(1, n+1, 2n+1, 3n+1)\} = \triangle_0$

(b) Orbits with exactly three elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+1, 3n+2) = \{(1, n+1, 2n+1, 3n+2), (1, n+1, 2n+1, 3n+3), \cdots, (1, n+1, 2n+1, 4n)\} = \triangle_1$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+2, 3n+1) = \{(1, n+1, 2n+2, 3n+1), (1, n+1, 2n+3, 3n+1), \cdots, (1, n+1, 3n, 3n+1)\} = \triangle_2$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+1, 3n+1) = \{(1, n+2, 2n+1, 3n+1), (1, n+3, 2n+1, 3n+1), \cdots, (1, 2n, 2n+1, 3n+1)\} = \triangle_3$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+1, 3n+1) = \{(2, n+1, 2n+1, 3n+1), (3, n+1, 2n+1, 3n+1), \cdots, (n, n+1, 2n+1, 3n+1)\} = \triangle_4$.

(c) Orbits containing exactly two elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+2, 3n+2) = \{(1, n+1, 2n+2, 3n+2), (1, n+1, 2n+2, 3n+3), \cdots, (1, n+1, 2n+2, 4n), (1, n+1, 2n+3, 3n+2), (1, n+1, 2n+3, 3n+3), \cdots, (1, n+1, 2n+3, 4n), \cdots, (1, n+1, 3n, 3n+2), (1, n+1, 3n, 3n+3), \cdots, (1, n+1, 3n, 4n)\} = \triangle_5$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+1, 3n+2) = \{(1, n+2, 2n+1, 3n+2), (1, n+2, 2n+1, 3n+3), \cdots, (1, n+2, 2n+1, 4n), (1, n+3, 2n+1, 3n+2), (1, n+3, 2n+1, 3n+3), \cdots, (1, n+3, 2n+1, 4n), \cdots, (1, 2n, 2n+1, 3n+2), (1, 2n, 2n+1, 3n+3), \cdots, (1, 2n, 2n+1, 4n)\} = \triangle_6$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+2, 3n+1) = \{(1, n+2, 2n+2, 3n+1), (1, n+2, 2n+3, 3n+1), \cdots, (1, n+2, 3n, 3n+1), (1, n+3, 2n+2, 3n+1), (1, n+3, 2n+3, 3n+1), \cdots, (1, n+3, 3n, 3n+1), \cdots, (1, 2n, 2n+2, 3n+1), (1, 2n, 2n+

$3, 3n+1), \cdots, (1, 2n, 3n, 3n+1)\} = \triangle_7$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+1, 3n+2) = \{(2, n+1, 2n+1, 3n+2), (2, n+1, 2n+1, 3n+3), \cdots, (2, n+1, 2n+1, 4n), (3, n+1, 2n+1, 3n+2), (3, n+1, 2n+1, 3n+3), \cdots, (3, n+1, 2n+1, 4n), \cdots, (n, n+1, 2n+1, 3n+2), (n, n+1, 2n+1, 3n+3), \cdots, (n, n+1, 2n+1, 4n)\} = \triangle_8$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+2, 3n+1) = \{(2, n+1, 2n+2, 3n+1), (2, n+1, 2n+3, 3n+1), \cdots, (2, n+1, 3n, 3n+1), (3, n+1, 2n+2, 3n+1), (3, n+1, 2n+3, 3n+1), \cdots, (3, n+1, 3n, 3n+1), \cdots, (n, n+1, 2n+2, 3n+1), (n, n+1, 2n+3, 3n+1), \cdots, (n, n+1, 3n, 3n+1)\} = \triangle_9$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+1, 3n+1) = \{(2, n+2, 2n+1, 3n+1), (2, n+3, 2n+1, 3n+1), \cdots, (2, 2n, 2n+1, 3n+1), (3, n+2, 2n+1, 3n+1), (3, n+3, 2n+1, 3n+1), \cdots, (3, 2n, 2n+1, 3n+1), \cdots, (n, n+2, 2n+1, 3n+1), (n, n+3, 2n+1, 3n+1), \cdots, (n, 2n, 2n+1, 3n+1)\} = \triangle_{10}$.

(d) Orbits containing exactly one element of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+2, 3n+2) = \{(1, n+2, 2n+2, 3n+2), (1, n+2, 2n+2, 3n+3), \cdots, (1, n+2, 2n+2, 4n), (1, n+3, 2n+2, 3n+2), (1, n+3, 2n+2, 3n+3), \cdots, (1, n+3, 2n+2, 4n), \cdots, (1, 2n, 2n+2, 3n+2), (1, 2n, 2n+2, 3n+3), \cdots, (1, 2n, 2n+2, 4n), \cdots, (1, 2n, 3n, 3n+2), (1, 2n, 3n, 3n+3), \cdots, (1, 2n, 3n, 4n)\}$
$= \triangle_{11}$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+2, 3n+2) = \{(2, n+1, 2n+2, 3n+2), (2, n+1, 2n+2, 3n+3), \cdots, (2, n+1, 2n+2, 4n), (2, n+1, 2n+3, 3n+2), (2, n+1, 2n+3, 3n+3), \cdots, (2, n+1, 2n+3, 4n), \cdots, (2, n+1, 3n, 3n+2), (2, n+1, 3n, 3n+3), \cdots, (2, n+1, 3n, 4n), \cdots, (n, n+1, 3n, 3n+2), (n, n+1, 3n, 3n+3), \cdots, (n, n+1, 3n, 4n)\} = \triangle_{12}$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+1, 3n+2) = \{(2, n+2, 2n+1, 3n+2), (2, n+2, 2n+1, 3n+3), \cdots, (2, n+2, 2n+1, 4n), \cdots, (2, 2n, 2n+1, 3n+2), (2, 2n, 2n+1, 3n+3), \cdots, (2, 2n, 2n+1, 4n), \cdots, (n, 2n, 2n+1, 3n+2), (n, 2n, 2n+1, 3n+3), \cdots, (n, 2n, 2n+1, 4n)\} = \triangle_{13}$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+2, 3n+1) = \{(2, n+2, 2n+2, 3n+1), (2, n+2, 2n+3, 3n+1), \cdots, (2, n+2, 3n, 3n+1), \cdots, (2, 2n, 2n+2, 3n+1), (2, 2n, 2n+3, 3n+1), \cdots, (2, 2n, 3n, 3n+1), \cdots, (n, 2n, 2n+2, 3n+1), (n, 2n, 2n+3, 3n+1), \cdots, (n, 2n, 3n, 3n+1)\} = \triangle_{14}$.

(e) Orbits containing no element of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+2, 3n+2) = \{(2, n+2, 2n+2, 3n+2), (2, n+2, 2n+2, 3n+3), \cdots, (2, n+2, 2n+2, 4n), \cdots, (2, 2n, 3n, 3n+2), (2, 2n, 3n, 3n+3), \cdots, (2, 2n, 3n, 4n), \cdots, (n, 2n, 3n, 3n+2), (n, 2n, 3n, 3n+3), \cdots, (n, 2n, 3n, 4n)\}$

$= \triangle_{15}.$

Therefore the rank of $S_n \times S_n \times S_n \times S_n$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4. □

The number of suborbits of G are given in Table 8.

Table 8: Rank of action of $S_n \times S_n \times S_n \times S_n$ on $X_1 \times X_2 \times X_3 \times X_4$

Suborbit	**Number of suborbits**
Orbit containing exactly 4 elements of B	$\binom{4}{4}$
Orbit containing exactly 3 elements of B	$\binom{4}{3}$
Orbits containing exactly 2 elements of B	$\binom{4}{2}$
Orbits containing exactly 1 element of B	$\binom{4}{1}$
Orbits containing no elements of B	$\binom{4}{0}$

From Table 8, the rank of action of $S_n \times S_n \times S_n \times S_n$ on $X_1 \times X_2 \times X_3 \times X_4$ is

$$\begin{aligned} R(4) &= \binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} \\ &= 1 + 4 + 6 + 4 + 1 \\ &= 16 \\ &= 2^4 \end{aligned}$$

Remark 2.3. Table 9 shows the respective subdegrees of $S_n \times S_n \times S_n \times S_n$ acting on $X_1 \times X_2 \times X_3 \times X_4$ for $n \geq 2$.

Table 9: Subdegrees of $S_n \times S_n \times S_n \times S_n$ on $X_1 \times X_2 \times X_3 \times X_4$

Number of suborbits	1	4	6	4	1
Subdegree	1	$n-1$	$(n-1)^2$	$(n-1)^3$	$(n-1)^4$

3. CONCLUSION

For $n \geq 2$, the action of $S_n \times S_n \times S_n \times S_n$ on $X_1 \times X_2 \times X_3 \times X_4$ is transitive, imprimitive with a rank of 2^4 and subdegrees are 1, $n-1$, $(n-1)^2$, $(n-1)^3$, $(n-1)^4$.

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