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On Laplacian Matrix for Doubly Weighted Graph

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Abstract

In this paper we will introduce the Laplacian matrix of graph with doubly weights.We put a perception for the matrix of doubly weighted graph. Finally we express a polynomial as determinant of Laplacian matrix.

Keywords: Laplace matrix , weighted graph , polynomial.

1. INTRODUCTION

In graph theory, Laplace matrix can be get for un weighted graph , and weighted graph,

The traditional weighted graph consists of weights on edge only. In this paper we compute it for doubly weighted graph, in which both edges and vertices are weighted.

We know that (Kirchoff's Matrix-Tree Theorem, 1847). If G(V, E) is an undirected graph and L is its graph Laplacian, then the number N_T of spanning trees contained in G is given by the following computation.

(1) Choose a vertex v_j and eliminate the j-th row and column from L to get a new matrix L_j; (2) Compute $N_T = \det(L_j)$ Eqn. (1)

The number N_T in Eqn. (1) counts spanning trees that are distinct as subgraphs of G: equivalently, we regard the vertices as distinguishable. Thus some of the trees that contribute to N_T may be isomorphic.

2. DEFINITIONS:

Doubly-Weighted Graph: Define a doubly-weighted graph G = (V (G), ωV , E(G), ωE). The set V (G) is called the vertex set of G, and elements of this set are called vertices. The vertex weight function ωV : V (G) → R maps all vertices onto the set of real numbers. We denote the weight of a vertex v as ωV (v). The set E(G) is called the edge set of G, and elements of this set are called edges. Each edge e_i is defined by two vertices, $v_j$and v_k, such that $e_i = v_j v_k$. We say that this edge e_i is incident to vertices vi and v_j , and we denote this by e_i α v_j and e_i α v_k respectively. Let δ(v) be the degree of vertex v—that is the number of edges incident to v. The edge weight function ωE : E(G) → R maps all edges onto the set of real numbers. We denote the weight of an edge e as ωE(e). Vertex-Weighted Graph. Let graph G = (V (G), ωV , E(G), ωE), and let ωE be defined

Vertex-Weighted Graph. Let graph G = (V (G), ωV , E(G), ωE), and let ωE be defined such that for all edges e in E(G), ωE(e) = 0. Since all the weights of the edges are zero, it is as if the edges are not weighted at all. Thus we say that graph G is a vertex-weighted graph and its edge weights are not counted when considering weights. 1 Edge-Weighted Graph. Let graph G = (V (G), ωV , E(G), ωE), and let ωV be defined such that for all vertices v in V (G), vw(v) = 0. Since all the weights of the vertices are zero, it is as if the vertices are not weighted at all. Thus we say that graph G is an edge-weighted graph and its vertex weights are not counted when considering weights.

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3. MAIN RESULTS:

Definition 3.1:

The doubly weighted matrix A_{ij}

Is n × n matrix with raws and column for vertices where entry

$$A_{ij} = \begin{cases} Wvi\ vj & vertex\ weight\quad if\ i = j \\ Wvi\ vj & edge\ weight\quad if\ i \neq j \end{cases}$$

The Laplace matrix of G is: $L = NAN^{-1}$ where N is vertex-edge incident matrix with one row for each vertex and one column for each edge , the entry $N_{v,e}$ is:

$$N_{v,e} = \begin{cases} 1 & if\ v\ is\ a\ head. \\ -1 & if\ v\ is\ tail\ of\ e. \\ 0 & othewise. \end{cases}$$

Example 3.2:

Consider a graph with doubly weights , the Laplace matrix can be obtained as follows:

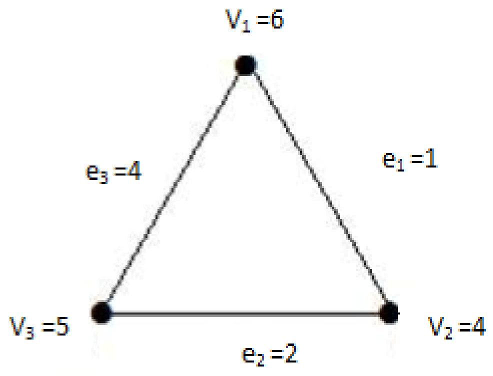

Fig.(1)

$$A = \begin{vmatrix} 6 & 1 & 4 \\ 1 & 4 & 2 \\ 4 & 2 & 5 \end{vmatrix} \qquad N = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{vmatrix}$$

Then $L = NAN^{-1}$

$$L = \begin{vmatrix} 19 & -7 & -12 \\ -7 & 8 & -1 \\ -12 & -1 & 13 \end{vmatrix}$$

By deleting the first row and column we find:

$$L = \begin{vmatrix} 8 & -1 \\ -1 & 13 \end{vmatrix}$$

det L =103.

Theorem 3.3 :

Doubly weighted Laplacian element is given by:

$$L_{ij} = \sum_k w_{ik} A_{ik}) d_{ij} - w_{ij} A_{ij}$$

(Where A_{ij} is the adjacency matrix element , w_{ij} is weight of vertices if $i = j$ and weight of edges if $i \neq j$) is not strictly positive definite.

Proof:

The constant vector has eigen value 0.But it's non negative definite.

Let D denote the diagonal matrix of total vertex weights , and denote the diagonal entries by d_i for any vector v we have:

$$V^t\,(D - W)\,V = V^t D V - V^t W V$$

$$= \sum_i V_i^2 d_i - \sum_{ij} W_{ij} V_i V_j.$$

Assume that $W_{ii}=0$, $W_{ij}=W_{ji}$.

Then the second term equal to $2\sum_{ij} W_{ij} V_i V_j$, furthermore we can rewrite the first term as :

$$(\sum_{i<j} W_{ij} + \sum_{i>j} W_{ij}) \qquad \sum_i V_i^2$$

$$= \sum_{i>j} V_i^2 W_{ij} + \sum_{i>j} V_j^2 W_{ij}$$

$$= \sum_{i<j} (V_i^2 + V_j^2)\, W_{ij}$$

Where the last inequality follow by interchanging I , j in the second sum.

Putting it together , we see :

$$V^t L V = \sum_{i<j} (V_i^2 + V_j^2 - 2 V_i V_j)\, W_{ij}$$

$$= \sum_{i<j} (V_i - V_j)^2 \geq 0 \quad \text{As desired.}$$

Example 3.4 :

For Fig. (2) we can compute Laplacian as follows:

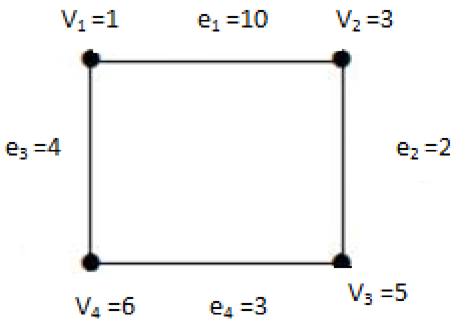

Fig.(2)

$$A = \begin{bmatrix} 1 & 10 & 0 & 4 \\ 10 & 3 & 2 & 0 \\ 0 & 2 & 5 & 3 \\ 4 & 0 & 3 & 6 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Then $L = NAN^{-1} = \begin{vmatrix} 24 & -9 & 2 & -17 \\ -9 & 6 & -6 & 9 \\ 2 & -6 & 5 & -1 \\ -17 & 9 & -1 & 9 \end{vmatrix}$

By deleting the first row and column:

Then $L = \begin{vmatrix} 6 & -6 & 9 \\ -6 & 5 & -1 \\ 9 & -1 & 9 \end{vmatrix}$

Then det L = -357 .

4. EXPRESSING A POLYNOMIAL AS A DETERMINANT LAPLACIAN MATRIX:

Let Kirchoff's matrix be n x n matrix A= a_{ij} , I , j $\in$ { 1,2,3,…..n} where:

$$a_{tij} = \begin{cases} \sum_{l=1}^{n} Xli & if\ i = j \\ -Xij & if\ i \neq j \end{cases}$$

Then by matrix tree theorem we have:

Corollary 4.1 :

F_n = det A , let G be a net on the set of vertices v with conductivities g_{vw} then polynomial X $f_G(x_1, x_2, \ldots\ldots x_n)$ = det B = det NN^{-1}.

Where B =(b_{ij}) $1 \leq I, j \leq n$ is n x n matrix

$$b_{ij} = \begin{cases} x + \sum_{l=1}^{n} xl\ gli & i = j \\ -xi\ gij & i \neq j \end{cases}$$

Example 4.2:

For graph in Fig.(1) , we can express a polynomial as determinant of Laplacian matrix as follows:

$b_{11} = x + x_1 g_{11} + x_2 g_{21} + x_3 g_{31}$.

$b_{12} = -x_1 = -x_1$

$b_{13} = -x_2 g_{21} = -x_2$

$b_{21} = -x_2 g_{21} = -x_2$

$b_{22} = x + x_1 g_{12} + x_2 g_{22} + x_3 g_{32} = x + x_1 + x_3$

$b_{23} = - x_2 g_{23} = - x_2$.

$b_{31} = - x_3 g_{31} = -x_3$.

$b_{32} = - x_3 g_{32} = -x_3$..

$b_{33} = x + x_1 g_{13} + x_2 g_{23} + x_3 g_{33} = x + x_1 + x_2$.

$$NN^{-1} = \begin{vmatrix} X+X_2+X_3 & -X_1 & -X_1 \\ -X_2 & X+X_1+X_3 & -X_2 \\ -X_3 & -X_3 & X+X_1+X_2 \end{vmatrix}$$

Delete the first row and column and take the determinant:

$$\det \begin{vmatrix} X+X_1+X_3 & -X_2 \\ -X_3 & X+X_1+X_2 \end{vmatrix}$$

From Fig.(1) $X_1=6$, $X_2=4$, $X_3 = 5$ Then the determinant will be:

$$\det \begin{vmatrix} X+11 & -4 \\ -5 & X+10 \end{vmatrix}$$

$= (X+11)(X+10)$

$= X^2 + 21X + 90$.

$= X f(wx_1, wx_2, wx_3)$.

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