Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 6 (2019), pp. 1109-1119

http://www.ripublication.com/gjpam.htm

α*-Local Closure Functions in Ideal Topological Spaces

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Abstract

In this paper we define an operator $[.]_\alpha*$ called the α*- local closure function of A with respect to the ideal ϑ and topology τ by: $[A]_\alpha*(\vartheta , \tau) = \{x \in X : Int(cl_\alpha(U)) \cap A \notin \vartheta$ for every $U \in \tau(x)\}$ in an ideal topological space (X, τ, ϑ). We investigate the basic properties and characterizations of $[A]_\alpha*(\vartheta , \tau)$. Also we investigate an operator $\Phi : P(X) \to \tau$ satisfying $\Phi(A) = X - [X - A]_\alpha*$ for each $A \in P(X)$. Also we proved the closure operator defined as $cl_\alpha*(A) = [A]_\alpha* \cup A$ is a Kuratowski closure operator and the topology obtained is $\tau_\alpha* = \{U \subseteq X / cl_\alpha*(X - U) = X - U\}$.

Keywords: Ideal Topology, Local Function, α*-local closure function

1. Introduction and Preliminaries

In [5] and [10] Kuratowski and vaidhyanathaswamy was studied the notion of ideal topological spaces, J.Dontchev, M.Ganster [3], M.Navaneethakrishnan, P.Paulraj Joseph [8], D.Jankovic, T.R.Hamlett [4], M.N.Mukherjee, R.Bishwambhar, R.Sen [6], A.A.Nasef, R.A.Mahmond [7] etc., were investigated applications to various fields of ideal topology. In a topological space (X, τ) with no separation properties assumed, for a subset A of X, cl(A) the closure of A denotes intersection of all closed set containing A and Int(A) denotes the union of all open set contained in A of (X, τ). An ideal ϑ is a non empty collection of subsets of X which satisfies: (i) $A \in \vartheta$ and $B \subseteq A$ implies $B \in \vartheta$, and (ii) $A \in \vartheta$ and $B \in \vartheta$ implies $A \cup B \in \vartheta$. Given a topological space (X, τ) with an ideal ϑ on X called ideal topological space denoted by (X, τ, ϑ) and if P(*X*) is the collection of all subsets of X a set operator $(.)^* : P(X) \rightarrow P(X)$ called a local function [4, 5] for any subset A of X with respect to ϑ and τ is defined as, $A^*(\vartheta, \tau) = \{x \in X : U \cap A \notin \vartheta$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau / x \in U\}$. A kuratowski closure operator cl*(A) for a topology $\tau^*(\vartheta, \tau)$ called *-topology finer than τ is defined by $cl^*(A) = A \cup (A)^*(\vartheta, \tau)$. In [9] O.Njasted investigated the notation of α-closed sets. A subset A of X is said to be α-open (resp. α-closed) set if $A \subseteq Int(cl(Int(A)))$ (resp.$cl(Int(cl(A)))$) $\subseteq$ A A. The set of all α-open sets forms topology and it is denoted by τ^α. The closure operator defined on τ^α for a subset A of X is denoted by cl_α $cl_\alpha(A)$. In [11] N.Velicko investigated the operator $cl_\theta(A)$ for a subset A of X, defined as $cl_\theta(A) = \{x \in X : cl(U) \cap A \neq \phi$, for each $U \in \tau(x)\}$. A subset A of X is said to be a θ-closed [11] set if $cl_\theta(A) = A$. A subset A of X is said to be δ-closed [11] set if $cl_\delta(A) = A$, where $cl_\delta(A) = \{x \in X : Int(cl(U)) \cap A \neq \phi$, for every $U \in \tau(x)\}$. The complement of δ-closed set is δ-open set. Ahmad Al-Omari and Takashi Noiri [1] introduced and investigated the operator $\Gamma(A)(\vartheta, \tau) = \{x \in X : cl(U) \cap A \notin \vartheta$, for every $U \in \tau(x)\}$. When there is no confusion, denote A* for $A^*(\vartheta, \tau)$ and $\Gamma(A)$ for $\Gamma(A)(\vartheta, \tau)$. In this paper we introduce and investigate an operator $[A]_\alpha^*(\vartheta, \tau)$ called α*-local closure function of A with respect to ϑ and τ. Also, investigate a kuratowski closure operator $cl_\alpha^*(A)$ and an operator $\Phi : P(X) \rightarrow \tau$ using $[A]_\alpha^*(\vartheta, \tau)$.

Lemma 1.1.[1] Let (X, τ, ϑ) be an ideal topological space and A a subset of X. Then

(i) If A is open, then $cl(A) = cl_\theta(A)$.

(ii) If A is closed, then $Int(A) = Int_\theta(A)$.

2. α*-Local Closure Functions

Definition 2.1. Let (X, τ, ϑ) be an ideal topological space and A a subset of X. Then $[A]_\alpha^*(X, \tau) = \{x \in X : Int(cl_\alpha(U)) \cap A \notin \vartheta$ for every $U \in \tau(x)\}$ is called α*-local closure function of A with respect to the ideal ϑ and topology τ, where $\tau(x) = \{U \in \tau / x \in U\}$. Simply denote $[A]_\alpha^*$ for $[A]_\alpha^*(\vartheta, \tau)$.

Remark 2.2. In general neither $A \subseteq [A]_\alpha^*$ nor $[A]_\alpha^* \subseteq A$. Theorem 2.3 deals the basic properties of α*-local closure function.

Theorem 2.3. Let (X, τ, ϑ) be an ideal topological space and A, B subsets of X. Then for α*-local closure functions the following properties hold.

(i) $[\phi]_\alpha^* = \phi$.

(ii) If $A \in \vartheta$ then $[A]_\alpha^* = \phi$.

(iii) $A \subseteq B$ then $[A]_\alpha^* \subseteq [B]_\alpha^*$.

(iv) $[A]_\alpha^* = cl([A]_\alpha^*) \subseteq cl_\theta(A)$ and $[A]_\alpha^*$ is closed.

(v) $[[A]_\alpha^*]_\alpha^* \subseteq [A]_\alpha^*$.

(vi) $[A \cup B]_\alpha^* = [A]_\alpha^* \cup [B]_\alpha^*$.

(vii) $[A]_\alpha^* - [B]_\alpha^* = [A - B]_\alpha^* - [B]_\alpha^* \subseteq [A - B]_\alpha^*$.

(viii) If $U \in \tau$ then $U \cap [A]_\alpha^* = U \cap [U \cap A]_\alpha^* \subseteq [U \cap A]_\alpha^*$.

(ix) If $U \in \vartheta$, then $[A - U]_\alpha^* = [A]_\alpha^*$.

(x) If $A \subseteq [A]_\alpha^*$ and A is open, then $[A]_\alpha^* = cl([A]_\alpha^*) = cl_\theta(A)$.

(x) $[A \cap B]_\alpha^* \subseteq [A]_\alpha^* \cap [B]_\alpha^*$.

Proof. (i) and (ii) is obvious.

(iii) Let $A \subseteq B$ and $x \notin [B]_\alpha^*$. Then there exists $U \in \tau(x)$ such that $Int(cl_\alpha(U)) \cap B \in \vartheta$. Since $A \subseteq B$, $Int(cl_\alpha(U)) \cap A \in \vartheta$ and hence $x \notin [A]_\alpha^*$.

(iv) We have $[A]_\alpha^* \subseteq cl([A]_\alpha^*)$. Let $x \in cl([A]_\alpha^*)$. Then $[A]_\alpha^* \cap U_x \neq \phi$ for every every $U_x \in \tau(x)$ and hence $[A]_\alpha^* \cap Int(cl_\alpha(U_x)) \neq \phi$. Therefore, there exists $y \in [A]_\alpha^* \cap Int(cl_\alpha(U_x))$ and $y \in Int(cl_\alpha(U_x))$. Since $y \in [A]_\alpha^*$, $A \cap Int(cl_\alpha(V_{xy})) \notin \vartheta$ and hence $x \in [A]_\alpha^*$ where $V_{xy} = Int(cl_\alpha(U_x))$. Now, let $x \in [A]_\alpha^*$, then $Int(cl_\alpha(U)) \cap A \notin \vartheta$, for every $U \in \tau(x)$. Hence, $cl(U) \cap A \neq \phi$ for every $U \in \tau(x)$. Hence $x \in cl_\theta(A)$.

(v) Let $x \in [[A]_\alpha^*]_\alpha^*$. Then for every $U_x \in \tau(x)$, $Int(cl_\alpha(U_x)) \cap [A]_\alpha^* \notin \vartheta$ and hence $Int(cl_\alpha(U_x)) \cap [A]_\alpha^* \neq \phi$. Let $y \in Int(cl_\alpha(U_x)) \cap [A]_\alpha^*$. Then $y \in Int(cl_\alpha(U_x))$ and $y \in A_\alpha^*$. Therefore, $Int(cl_\alpha(V_{xy})) \notin \vartheta$ and hence $x \in [A]_\alpha^*$ where $V_{xy} = Int(cl_\alpha(U_x))$.

(vi) By (iii), $[A]_\alpha^* \cup [B]_\alpha^* \subseteq [A \cup B]_\alpha^*$. Now, let $x \in [A \cup B]_\alpha^*$. Then for every $U \in \tau(x)$, $(Int(cl_\alpha (U)) \cap A) \cup (Int(cl_\alpha(U)) \cap B) = Int(cl_\alpha(U)) \cap (A \cup B) \notin \vartheta$. Therefore, $Int(cl_\alpha(U)) \cap A \notin \vartheta$ or $Int(cl_\alpha(U)) \cap B \notin \vartheta$. Consequently, $x \in [A]_\alpha^* \cup [B]_\alpha^*$.

(vii) Since $A = (A - B) \cup (B \cap A)$, then by (vi), $[A]_\alpha^* = [A - B]_\alpha^* \cup [B \cap A]_\alpha^*$. Therefore, $[A]_\alpha^* - [B]_\alpha^* = [A]_\alpha^* \cap (X - [B]_\alpha^*) = (([A - B]_\alpha^* \cup [B \cap A]_\alpha^*) \cap (X - [B]_\alpha^*)) = ([A - B]_\alpha^* \cap (X - [B]_\alpha^*)) \cup ([B \cap A]_\alpha^* \cap (X - [B]_\alpha^*)) = [A - B]_\alpha^* - [B]_\alpha^* \cup \phi \subseteq [A - B]_\alpha^*$.

(viii) Let $U \in \tau$ and $x \in U \cap [A]_\alpha^*$. Then $x \in U$ and $x \in [A]_\alpha^*$. Since for every $V \in \tau(x)$, $U \cap W \in \tau(x)$ and thus $W \cap (U \cap A) = (U \cap W) \cap A \notin \vartheta$ where $W = Int(cl_\alpha(V))$. Hence $x \in [U \cap A]_\alpha^*$. Also $U \cap [A]_\alpha^* \subseteq U \cap [U \cap A]_\alpha^*$ and by (iii) $[U \cap A]_\alpha^* \subseteq [A]_\alpha^*$ and $U \cap [U \cap A]_\alpha^* \subseteq U \cap [A]_\alpha^*$.

(ix) Since $A \cap U \subseteq U \in \vartheta$, $A \cap U \in \vartheta$ by heredity of ϑ and by (ii) $[A \cap U]_\alpha^* = \phi$. Since $A = (A - U) \cup (A \cap U)$, $[A]_\alpha^* = [A - U]_\alpha^* \cup [A \cap U]_\alpha^* = [A - U]_\alpha^*$ by (vi).

(x) B y (iv) for any $A \subseteq X$, we have $[A]_\alpha^* = cl([A]_\alpha^*) \subseteq cl_\theta(A)$. Since $A \subseteq [A]_\alpha^*$ and A is open, $cl_\theta(A) \subseteq cl_\theta([A]_\alpha^*) = cl([A]_\alpha^*)$ by Lemma 1.1. Therefore, $[A]_\alpha^* = cl([A]_\alpha^*) = cl_\theta(A)$.

(xi) Proof is clear by (iii).

The following Example shows that the reverse inclusion of Theorem 2.3(x) is not always hold. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\},\{b, c\},\{a, b, c\}\}$ and $\vartheta = \{\phi, \{b\}\}$ and $A = \{a, d\}$, $B = \{b, c\}$ then $[A]_\alpha^* = \{a, d\}$, $[B]_\alpha^* = \{b, c, d\}$ and hence clear.

Theorem 2.4. Let (X, τ, ϑ) be a topological space with ideals ϑ_1 and ϑ_2 of X and $A \subseteq X$. Then the following properties hold.

(i) If $\vartheta_1 \subseteq \vartheta_2$ then $[A]_\alpha^*(\vartheta_2) \subseteq [A]_\alpha^*(\vartheta_1)$

(ii) $[A]_\alpha^*(\vartheta_1 \cap \vartheta_2) \subseteq [A]_\alpha^*(\vartheta_1) \cup [A]_\alpha^*(\vartheta_2)$.

Proof. (i) Let $\vartheta_1 \subseteq \vartheta_2$ and $x \in [A]_\alpha^*(\vartheta_2)$. Then $A \cap Int(cl_\alpha(U)) \notin \vartheta_2$ for every $U \in \tau(x)$ and hence $A \cap Int(cl_\alpha(U)) \notin \vartheta_1$. That is, $x \in [A]_\alpha^*(\vartheta_1)$.

(ii) We have $[A]_\alpha^*(\vartheta_1) \subseteq [A]_\alpha^*(\vartheta_1 \cap \vartheta_2)$ and $[A]_\alpha^*(\vartheta_2) \subseteq [A]_\alpha^*(\vartheta_1 \cap \vartheta_2)$ by (i). Therefore, $[A]_\alpha^*(\vartheta_1) \cup [A]_\alpha^*(\vartheta_2) \subseteq [A]_\alpha^*(\vartheta_1 \cap \vartheta_2)$. Let $x \in [A]_\alpha^*(\vartheta_1 \cap \vartheta_2)$, for each $U \in \tau(x)$, $Int(cl_\alpha(U)) \cap A \notin \vartheta_1 \cap I_2$ and hence $Int(cl_\alpha(U)) \cap A \notin \vartheta_1$ or $Int(cl_\alpha(U)) \cap A \notin \vartheta_2$, and so $x \in [A]_\alpha^*(\vartheta_1)$ or $x \in [A]_\alpha^*(\vartheta_2)$. Therefore, $x \in [A]_\alpha^*(\vartheta_1) \cup [A]_\alpha^*(\vartheta_2)$.

Theorem 2.5. (i) $A^* \subseteq [A]_\alpha^*$, (ii) $[A]_\alpha^* \subseteq cl_\delta(A)$, (iii) $[A]_\alpha^* \subseteq \Gamma(A)$
Proof. The proof is clear by Definition.

The following Examples shows that Theorem 2.5 is true. (i) $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and $\vartheta = \{\phi, \{a\}\}$. Let $A = \{b\}$ then $A^* = \{b, c\}$ and $[A]_\alpha^* = X$. (ii) $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\},\{b, c\},\{a, b, c\}\}$ and $\vartheta = \{\phi, \{b\}\}$. Let $A = \{a, b, d\}$ then $[A]_\alpha^* = \{a, d\}$ and $cl_\delta(A) = \Gamma(A) = X$.

3. τ_α^* - open sets.

In this section we defined a closure operator $cl_\alpha^*(A) = A \cup [A]_\alpha^*$ and proved is a Kuratowski closure operator.

Theorem 3.1. Let (X, τ, ϑ) be an ideal topological space, $cl_\alpha^*(A) = A \cup [A]_\alpha^*$ and A, B subsets of X. Then

(i) $cl_\alpha^*(\phi) = \phi$ and $cl_\alpha^*(X) = X$.

(ii) $A \subseteq cl_\alpha^*(A)$.

(iii) if $A \subseteq B$ then $cl_\alpha^*(A) \subseteq cl_\alpha^*(A)$.

(iv) $cl_\alpha^*(A \cup B) = cl_\alpha^*(A) \cup cl_\alpha^*(B)$.

(v) $cl_\alpha^*(A \cap B) \subseteq cl_\alpha^*(A) \cap cl_\alpha^*(B)$.

(vi) $cl_\alpha^*(cl_\alpha^*(A)) = cl_\alpha^*(A)$.

(vii) if $U \in \tau(x)$ then $U \cap cl_\alpha^*(A) \subseteq cl_\alpha^*(U \cap A)$.

(viii) $cl^*(A) \subseteq cl_\alpha^*(A)$

Proof. (i) $cl_\alpha^*(\phi) = [\phi]_\alpha^* \cup \phi = \phi$ and $cl_\alpha^*(X) = [X]_\alpha^* \cup X = X$.

(ii) $A \subseteq A \cup [A]_\alpha^* = cl_\alpha^*(A)$.

(iii) Let $A \subseteq B$, $cl_\alpha^*(A) = A \cup [A]_\alpha^* \subseteq B \cup [B]_\alpha^* = cl_\alpha^*(B)$ by Theorem 2.3(iii).

(iv) By Theorem 2.3(vi), $cl_\alpha^*(A \cup B) = [A \cup B]_\alpha^* \cup (A \cup B) = ([A]_\alpha^* \cup [B]_\alpha^*) \cup (A \cup B) = ([A]_\alpha^* \cup A) \cup ([B]_\alpha^* \cup B) = cl_\alpha^*(A) \cup cl_\alpha^*(B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ then by (iii), $cl_\alpha^*(A \cap B) \subseteq cl_\alpha^*(A)$ and $cl_\alpha^*(A \cap B) \subseteq cl_\alpha^*(B)$ and hence $cl_\alpha^*(A \cap B) \subseteq cl_\alpha^*(A) \cap cl_\alpha^*(B)$.

(vi) $cl_\alpha^*(cl_\alpha^*(A)) = cl_\alpha^*([A]_\alpha^* \cup A) = [[A]_\alpha^* \cup A]_\alpha^* \cup ([A]_\alpha^* \cup A) = ([[A]_\alpha^*]_\alpha^* \cup [A]_\alpha^*) \cup ([A]_\alpha^* \cup A) = [A]_\alpha^* \cup ([A]_\alpha^* \cup A) = [A]_\alpha^* \cup A = cl_\alpha^*(A)$ by Theorem 2.3(vi) and (v).

(vii) Since $U \in \tau(x)$. By Theorem 2.3(viii) we have, $U \cap cl_\alpha^*(A) = U \cap ([A]_\alpha^* \cup A) = (U \cap A) \cup (U \cap [A]_\alpha^*) \subseteq (U \cap A) \cup [U \cap A]_\alpha^* = cl_\alpha^*(U \cap A)$.

(viii)The proof is clear by Definition.

Remark 3.2. By Theorem 3.1 (i), (ii), (iv) and (vi) $cl_\alpha^*(A) = A \cup [A]_\alpha^*$ is a Kuratoswski closure operator. The topology generated by $cl_\alpha^*(A)$ is denoted and defined by $\tau_\alpha^* = \{U \subseteq X : cl_\alpha^*(X - U) = X - U\}$ the open sets in τ_α^* is called τ_α^*-open sets and its complement is called τ_α^*- closed sets.

The following Examples shows that (i) The reveres inclusion of Theorem 3.1 (v) is not always hold, $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\vartheta = \{\phi, \{b\}\}$. Let $A = \{b\}$ and $B = \{c\}$ then $cl_\alpha^*(A) = \{b\}$ and $cl_\alpha^*(B) = \{b, c, d\}$ then it clear and (ii) Theorem 3.1 (viii) is true, $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and $\vartheta = \{\phi, \{a\}\}$. Let $A = \{b\}$ then $cl^*(A) = \{b, c\}$ and $cl_\alpha^*(A) = X$.

Corollary 3.3. Let (X, τ, ϑ) be an ideal topological space and A a subset of X. If $A \subseteq [A]_\alpha^*$ and A is open then,

(i) $cl_\theta(A) = cl_\alpha^*(A)$.

(ii) $Int_\theta(X - A) = Int_\alpha^*(X - A)$.

Proof. (i) By Theorem 2.3 and by the Definition of $cl_\alpha^*(A)$, since $A \subseteq [A]_\alpha^*$ and A is open, $cl_\theta(A) = [A]_\alpha^* = cl_\alpha^*(A)$.

(ii) $Int_\theta(X - A) = X - cl_\theta(A) = X - cl_\alpha^*(A) = Int_\alpha^*(X - A)$, by (i).

Lemma 3.4. Let (X, τ, ϑ) be an ideal topological space and A, B subsets of X. Then $[A]_\alpha^* - [B]_\alpha^* = [A - B]_\alpha^* - [B]_\alpha^*$.

Proof. By Theorem 2.3, $[A]_\alpha^* = [(A - B) \cup (B \cap A)]_\alpha^* = [A - B]_\alpha^* \cup [A \cap B]_\alpha^* \subseteq [A - B]_\alpha^* \cup [B]_\alpha^*$. Thus $[A]_\alpha^* - [B]_\alpha^* \subseteq [A - B]_\alpha^* - [B]_\alpha^*$. Also by Theorem 2.3, $[A - B]_\alpha^* \subseteq [A]_\alpha^*$ since
$A - B \subseteq A$, and hence $[A - B]_\alpha^* - [B]_\alpha^* \subseteq [A]_\alpha^* - [B]_\alpha^*$. Hence $[A]_\alpha^* - [B]_\alpha^* = [A - B]_\alpha^* - [B]_\alpha^*$.

Corollary 3.5. (X, τ, ϑ) be an ideal topological space and A, B subsets of X with $B \in \vartheta$. Then
$[A \cup B]_\alpha^* = [A]_\alpha^* = [A - B]_\alpha^*$.

Proof. Since $B \in \vartheta$, by Theorem 2.3(ii), $[B]_\alpha^* = \phi$. By Lemma 3.4 and by Theorem 2.3, $[A \cup B]_\alpha^* = [A]_\alpha^* \cup [B]_\alpha^* = [A]_\alpha^* = [A - B]_\alpha^*$.

4. α*- Local Compatibility with an Ideal

Definition 4.1. Let (X, τ, ϑ) be an ideal topological space. We say that the topology τ is α*-local compatible with the ideal ϑ, denoted by $\tau \sim [\vartheta]_\alpha^*$, if the following condition holds for every $A \subseteq X$, if for every $x \in A$ there exists a $U \in \tau(x)$ such that $Int(cl_\alpha(U)) \cap A \in \vartheta$, then $A \in \vartheta$.

Theorem 4.2. Let (X, τ, ϑ) be an ideal topological space then the following properties are equivalent:

(i) $\tau \sim [\vartheta]_\alpha^*$.

(ii) If a subset A of X has a cover of open sets each of whose interior α-closure intersection with A is in ϑ, then $A \in \vartheta$.

(iii) For every $A \subseteq X$, $A \cap [A]_\alpha^* = \phi$ implies that $A \in \vartheta$.

(iv) For every $A \subseteq X$, $A - [A]_\alpha^* \in \vartheta$.

(v) For every $A \subseteq X$, if A contains no nonempty set B with $B \subset [B]_\alpha^*$ then $A \in \vartheta$.

Proof. (i) ⇒ (ii). The proof is obvious by Definition.

(ii) ⇒ (iii). Let $A \subseteq X$ and $x \in A$. Then $x \notin [A]_\alpha^*$ and there exist $U_x \in \tau(x)$ such that $Int(cl_\alpha(U_x)) \cap A \in \vartheta$. Therefore we have, $A \subseteq \cup \{U_x : x \in A\}$ and $U_x \in \tau(x)$ and by (ii), $A \in \vartheta$.

(iii) ⇒ (iv). For any $A \subseteq X$, $A - [A]_\alpha^* \subseteq A$. Then $[(A - [A]_\alpha^*) \cap [A - [A]_\alpha^*]_\alpha^* \subseteq (A - [A]_\alpha^*) \cap [A]_\alpha^* = \phi$. Therefore by (iii), $A - [A]_\alpha^* \in \vartheta$.

(iv) ⇒ (v). By (iv), for every $A \subseteq X$, $A - [A]_\alpha^* \in \vartheta$. Let $A - [A]_\alpha^* = \vartheta^1 \in \vartheta$. Then $A = \vartheta^1 \cup (A \cap [A]_\alpha^*)$, by Theorem 2.3 (vi) and by Theorem 2.3 (ii), $[A]_\alpha^* = [\vartheta^1]_\alpha^* \cup [A \cap [A]_\alpha^*]_\alpha^* = [A \cap [A]_\alpha^*]_\alpha^*$. Therefore, $A \cap [A]_\alpha^* = A \cap [A \cap [A]_\alpha^*]_\alpha^* \subseteq [A \cap [A]_\alpha^*]_\alpha^*$ and $A \cap [A]_\alpha^* \subseteq A$. By the assumption, $A \cap [A]_\alpha^* = \phi$ and hence $A = A - [A]_\alpha^* \in \vartheta$.

(v) ⇒ (i). Let $A \subseteq X$ and assume that for every $x \in A$, there exist $U \in \tau(x)$ such that $Int(cl_\alpha(U)) \cap A \in \vartheta$. Then $A \cap [A]_\alpha^* = \phi$. Suppose that $B \subseteq A$ such that, $B \subseteq [B]_\alpha^*$. Than $B = B \cap [B]_\alpha^* \subseteq A \cap [A]_\alpha^* = \phi$. Therefore, A contains no nonempty set B such that $B \subseteq [B]_\alpha^*$. Hence $A \in \vartheta$.

Theorem 4.3. Let (X, τ, ϑ) be an ideal topological space. If τ is α^*-local compatible with ϑ, then the following equivalent properties hold:

(i) For every $A \subseteq X$, $A \cap [A]_\alpha^* = \phi$ implies that $[A]_\alpha^* = \phi$.

(ii) For every $A \subseteq X$, $[A - [A]_\alpha^*]_\alpha^* = \phi$.

(iii)For every $A \subseteq X$, $[A \cap [A]_\alpha^*]_\alpha^* = [A]_\alpha^*$.

Proof. First we prove that (i) holds if τ is α^*-local compatible with ϑ. Let A be any subset of X and $A \cap [A]_\alpha^* = \phi$, Then by Theorem 4.2, $A \in \vartheta$ and hence by Theorem 2.3, $[A]_\alpha^* = \phi$.

(i) ⇒ (ii). Assume that for every $A \subseteq X$, $A \cap [A]_\alpha^* = \phi$ implies that $[A]_\alpha^* = \phi$. Let $B = A - [A]_\alpha^*$. Then $B \cap [B]_\alpha^* = (A - [A]_\alpha^*) \cap [A - [A]_\alpha^*]_\alpha^*$. Now $B \cap [B]_\alpha^* = (A \cap (X - [A]_\alpha^*)) \cap [A \cap (X - [A]_\alpha^*)]_\alpha^* \subseteq (A \cap (X - [A]_\alpha^*)) \cap ([A]_\alpha^* \cap [X - [A]_\alpha^*]_\alpha^*) = \phi$. Therefore by (i), $[B]_\alpha^* = \phi$.

(ii) ⇒ (iii). Assume that for every $A \subseteq X$, $[A - [A]_\alpha^*]_\alpha^* = \phi$. $A = (A - [A]_\alpha^*) \cup (A \cap [A]_\alpha^*)$. Therefore, $[A]_\alpha^* = [A - [A]_\alpha^*]_\alpha^* \cup [A \cap [A]_\alpha^*]_\alpha^* = [A \cap [A]_\alpha^*]_\alpha^*$.

(iii) $\Rightarrow$ (i). Assume that for every $A \subseteq X$, $A \cap [A]_\alpha^* = \phi$ and $[A \cap [A]_\alpha^*]_\alpha^* = [A]_\alpha^*$. Therefore, $[A]_\alpha^* = \phi$.

Corollary 4.4. Let (X, τ, ϑ) be an ideal topological space and $A \subseteq X$. If τ is α^*-local compatible with ϑ, then $[A]_\alpha^* = [A_\alpha^*]_\alpha^*$.

Proof. Let $A \subseteq X$, $[A]_\alpha^* = [A \cap [A]_\alpha^*]_\alpha^* \subseteq [A]_\alpha^* \cap [[A]_\alpha^*]_\alpha^* = [[A]_\alpha^*]_\alpha^*$ by Theorem 4.3 and by Theorem 2.3. Therefore, $[A]_\alpha^* = [[A]_\alpha^*]_\alpha^*$ again by Theorem 2.3.

Theorem 4.5. Let (X, τ, ϑ) be an ideal topological space and τ is α^*- local compatible with ϑ, A a τ_α^*-closed subset of X. Then $A = B \cup \vartheta^1$, where B is closed and $\vartheta^1 \in \vartheta$.

Proof. Let $A \subseteq X$ and A is τ_α^*-closed. Then $[A]_\alpha^* \subseteq A$ implies $A = (A - [A]_\alpha^*) \cup [A]_\alpha^*$ and hence by Theorem 4.2 and by Theorem 2.3(iv), proof completes.

Theorem 4.6. Let (X, τ, ϑ) be an ideal topological space then the following properties are equivalent:

(i) $\tau \sim [\vartheta]_\alpha^*$,

(ii) For every τ_α^*-closed subset A, $A - [A]_\alpha^* \in \vartheta$,

Proof. (i) $\Rightarrow$ (ii). The proof is clear by Theorem 4.2.

(ii) $\Rightarrow$ (i) Let $A \subseteq X$ and assume that for every $x \in A$, there exists an open set $U \in \tau(x)$ such that, $A \cap \mathrm{Int}(cl_\alpha(U)) \in \vartheta$, then $A \cap [A]_\alpha^* = \phi$. Since $cl_\alpha^*(A)$ is τ_α^*-closed, $(A \cup [A]_\alpha^*) - [A \cup [A]_\alpha^*]_\alpha^* \in \vartheta$ and $(A \cup [A]_\alpha^*) - [A \cup [A]_\alpha^*]_\alpha^* = (A \cup [A]_\alpha^*) - ([A]_\alpha^* \cup [[A]_\alpha^*]_\alpha^*) = (A \cup [A]_\alpha^*) - [A]_\alpha^* = A$.

Theorem 4.7. (X, τ, ϑ) be an ideal topological space and τ be α^*-local compatible with ideal ϑ. Then for every $V \in \tau$ and any $A \subseteq X$, $[V \cap A]_\alpha^* = [V \cap [A]_\alpha^*]_\alpha^* = cl([V \cap [A]_\alpha^*]_\alpha^*) \subseteq cl_\theta(V \cap [A]_\alpha^*)$.

Proof. Let $V \in \tau$. Then by Theorem 2.3(viii), $V \cap [A]_\alpha^* = V \cap [V \cap A]_\alpha^* \subseteq [V \cap A]_\alpha^*$ and therefore $[V \cap [A]_\alpha^*]_\alpha^* \subseteq [[V \cap A]_\alpha^*]_\alpha^* \subseteq [V \cap A]_\alpha^*$ again by Theorem 2.3. Also by Theorem 2.3 and Theorem 4.3 we obtain $[V \cap (A - [A]_\alpha^*)]_\alpha^* \subseteq [V]_\alpha^* \cap [A - [A]_\alpha^*]_\alpha^* = [V]_\alpha^* \cap \phi = \phi$. Moreover, $[V \cap A]_\alpha^* - [V \cap [A]_\alpha^*]_\alpha^* \subseteq [(V \cap A) - (V \cap [A]_\alpha^*)]_\alpha^* = [V \cap (A - [A]_\alpha^*)]_\alpha^* = \phi$ and hence $[V \cap A]_\alpha^* \subseteq [V \cap [A]_\alpha^*]_\alpha^*$. Therefore we have, $[V \cap A]_\alpha^* = [V \cap [A]_\alpha^*]_\alpha^*$. Also by Theorem 2.3, $[V \cap A]_\alpha^* =$

$[V \cap [A]_\alpha^*]_\alpha^* = cl([V \cap [A]_\alpha^*]_\alpha^*) \subseteq cl_\theta(V \cap [A]_\alpha^*)$.

5. Φ-Operator

Definition 5.1. Let (X, τ, ϑ) be an ideal topological space. An operator $\Phi(A) : P(X) \to \tau$ is

defined as follows: for every $A \subseteq X$, $\Phi(A) = \{x \in X$: there exist $U \in \tau(x)$ such that

$Int(cl_\alpha(U)) - A \in \vartheta\}$ and observe that $\Phi(A) = X - [X - A]_\alpha^*$.

Theorem 5.2. Let (X, τ, ϑ) be an ideal topological space. Then the following properties hold:

(i) For every $A \subseteq X$, $\Phi(A)$ is open.

(ii) If $A \subseteq B$ then $\Phi(A) \subseteq \Phi(B)$.

(iii) If $A, B \in P(X)$, then $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

(iv) If $A \subseteq X$, then $\Phi(A) = \Phi(\Phi(A))$ iff $[X - A]_\alpha^* = [[X - A]_\alpha^*]_\alpha^*$.

(v) If $A \in \vartheta$, then $\Phi(A) = X - [X]_\alpha^*$.

(vi) If $A \subseteq X$, $\vartheta^1 \in \vartheta$, then $\Phi(A - \vartheta^1) = \Phi(A)$.

(vii) If $A \subseteq X$, $\vartheta^1 \in \vartheta$, then $\Phi(A \cup \vartheta^1) = \Phi(A)$.

(viii) If $(A - B) \cup (B - A) \in \vartheta$, then $\Phi(A) = \Phi(B)$.

Proof. (i) Proof is clear from Theorem 2.3(iv).

(ii) Proof follows from Theorem 2.3(iii).

(iii) $\Phi(A \cap B) = X - [X - (A \cap B)]_\alpha^* = X - [(X - A) \cup (X - B)]_\alpha^* = X - ([X - A]_\alpha^* \cup [X - B]_\alpha^*) = (X - [X - A]_\alpha^*) \cap (X - [X - B]_\alpha^*) = \Phi(A) \cap \Phi(B)$.

(iv) Let $A \subseteq X$. Since $\Phi(A) = X - [X - A]_\alpha^*$ and $\Phi(\Phi(A)) = X - [X - \Phi(A)]_\alpha^* = X - [X - (X - [X - A]_\alpha^*)]_\alpha^* = X - [[X - A]_\alpha^*]_\alpha^*$ proof follows.

(v) By Corollary 3.5, if $A \in \vartheta$, $[X - A]_\alpha^* = [X]_\alpha^*$.

(vi) By Corollary 3.5, $\Phi(A - \vartheta^1) = X - [X - (A - \vartheta^1)]_\alpha^* = X - [(X - A) \cup \vartheta^1)]_\alpha^* = X - [X - A]_\alpha^* = \Phi(A)$.

(vii) $\Phi(A \cup \vartheta^{1}) = X - [X - (A \cup \vartheta^{1})]_{\alpha}^{*} = X - [(X - A) - \vartheta^{1}]_{\alpha}^{*} = X - [X - A]_{\alpha}^{*} = \Phi(A)$.

(viii) Assume that , $(A - B) \cup (B - A) \in \vartheta$. Let $A - B = \vartheta^{1}$ and $B - A = \vartheta^{11}$. Observe that $\vartheta^{1}, \vartheta^{11} \in \vartheta$ by heredity. Also observe that $B = (A - \vartheta^{1}) \cup \vartheta^{11}$. Thus $\Phi(A) = \Phi(A - \vartheta^{1}) = \Phi[(A - \vartheta^{1}) \cup \vartheta^{11}] = \Phi(B)$, by (vi) and (vii).

Corollary 5.3. Let (X, τ, ϑ) be an ideal topological space. Then $U \subseteq \Phi(U)$ for every θ-open set $U \subseteq X$.

Proof. We know that $\Phi(U) = X - [X - U]_{\alpha}^{*}$. Now $[X - U]_{\alpha}^{*} \subseteq cl_{\theta}(X - U) = (X - U)$, Since
$(X - U)$ is θ-closed. Therefore, $U = X - (X - U) \subseteq X - [X - U]_{\alpha}^{*} = \Phi(U)$.

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