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Qualitative Behaviour of a Detritus-Based Prey-Predator Model of Sundarban Mangrove Forest, India

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Abstract

In this study we consider a non-spatial detritus based prey-predator model in the Sundarban mangrove forest, India. Here detritus plays the crucial role as the micro-organism pool gets its nutrient from detritus. We consider that the loss rate of detritus due to micro-organism pool follows Holling Type-II functional response and the growth rate of micro-organism is donor-controlled type function and the rate of uptake of invertebrate predator is taken as Ivlev-type response function. The mathematical analysis consists of the existence of different feasible equilibria, their local and global stability and existence of stable periodic orbit around the interior equilibrium point. The system undergoes Hopf-bifurcating small amplitude periodic solutions around the equilibrium point with respect to the growth rate parameter of micro-organism pool. Numerical simulations are done with respect to different values of the system parameters to understand the qualitative behaviour of our analytical results.
keywords: detritus, micro-organism pool, invertebrate predator, carrying capacity, uptake function, global stability, stable manifold, unstable manifold, Hopf-bifurcation, non-constant periodic orbit.

1. INTRODUCTION

Sundarban is one of the largest mangrove forests in the world, lies in the Ganga-Brahmaputra-Meghna delta particularly at the border of the northern margin of the Bay of Bengal. Mangroves are one special type of trees that grow in brackish wetlands where the land meets with sea. In the Sundarban estuary, mangrove forests

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are very important that support detritus based food chain and play a crucial role to shape the dynamics of the adjacent estuary. A huge amount of detritus is formed from the litter of several species of mangrove forest such as *B. gymnorhiza, Avicennia Sp., Heritiera Sp., Exocaria Sp., Sonneratia Sp.*. Among these species *Avicannia alba, Avicannia marina* and *Avicennia officinalis* are the leading sources of detritus. The plant litter is decomposed by several micro-organisms namely fungi, bacteria and certain protozoa and detritus is formed. The detritus is washed out by tidal flow into the adjacent estuary. This detritus plays a major role to regulate the productivity of the forest and the estuary. Several invertebrate predators such as nematodes, certain insect larvae are present in the ecosystem and feed on the micro-organisms. The invertebrate predators are one of the most important members in the ecosystem to regulate the level of nutrients in the food chain/web. In this way a detritus based prey predator system works in the Sundarban estuary. Lot of theoretical works have been done on detritus based prey predator system in the Sundarban estuarine ecosystem [1–4]

The present study we consider a non-spatial detritus-based prey predator model where detritus is the primary energy/food source for micro-organism pool and invertebrate predator. Here it is proposed that the loss rate of detritus due to micro organism pool follows Holling Type-II functional response [5,6]and the growth rate of micro-organism follows donor-controlled type function [7] and the functional response of invertebrate predator is taken as Ivlev-type response function [8, 9]. It is observed that the growth rate of micro-organism plays the significant role to shape the dynamics of the mangrove ecosystem.

The present study has been carried out sequentially as follows : At first the basic assumptions and formulation of mathematical model have been described and we derive the equilibria and their feasibility conditions. Then we have shown the boundedness of the solutions [4, 10] and the dynamical behaviour of the system. We have also shown the local as well as global stability of the planner and interior equilibria and the existence criteria of Hopf-bifurcation [11, 12], criteria for non-constant large amplitude periodic solutions and the direction of flow of trajectory of the periodic orbit. Finally numerical simulation is carried out with respect to different values of system parameters and we discuss and interpret the results of the given system.

2. MODEL FORMULATION

We consider a deterministic mathematical model of a non-spatial detritus-based micro-organism pool and its invertebrate predator as :

$$\begin{aligned}\frac{dx}{dt} &= x(a_1 - bx) - \frac{hxy}{k+x}\\ \frac{dy}{dt} &= y\left(a_2 - \frac{cy}{bx}\right) - gz\big\{1 - exp(-ey)\big\}\\ \frac{dz}{dt} &= z\big[-m + g\big\{1 - exp(-ey)\big\}\big]\end{aligned} \qquad (1)$$

with the initial conditions,
$x(0) = x_0 > 0$, $y(0) = y_0 > 0$ and $z(0) = z_0 > 0$.
where x, y, z are the biomass of detritus produced from plant litter of mangrove, population density of micro-organism pool and population density of invertebrate predator respectively at time t. Here $a_1, b, h, k, a_2, c, g, e, m$ are positive constants. In the system (1) it is assumed that the detritus grows logistically with carrying capacity $\frac{a_1}{b}$. The loss rate of detritus due to micro-organism pool follows Holling type-II functional response. The growth of micro-organism pool is donor controlled type which is proportional to the available biomass of detritus. The functional response of predator is taken as Ivlev-type response function where e is the hunting success and g is the maximum number of micro-organisms that can be eaten by invertebrate predator and m is the mortality rate of invertebrate predator.
We consider the following substitution to make the system (1) into non dimensional form:

$x = kX$, $y = \frac{mk}{h}Y$, $z = \frac{km^2 Z}{hg}$, $t = \frac{T}{m}$

Then the model system(1) reduces to

$$\begin{aligned}\frac{dX}{dT} &= X\left[(\alpha - \eta X) - \frac{Y}{1+X}\right]\\ \frac{dY}{dT} &= Y\left(\beta - \frac{\gamma Y}{X}\right) - Z\big\{1 - exp(-\phi Y)\big\}\\ \frac{dZ}{dT} &= Z\big[-1 + \sigma\big\{1 - exp(-\phi Y)\big\}\big]\end{aligned} \qquad (2)$$

where $\alpha = \frac{a_1}{m}$, $\eta = \frac{bk}{m}$, $\beta = \frac{a_2}{m}$, $\gamma = \frac{c}{bh}$, $\sigma = \frac{g}{m}$, $\phi = \frac{emk}{h}$.
It is known that the qualitative behaviour of the system (1) and the non-dimensional system (2) are the same.

3. EQUILIBRIA

The system (2) has the following equilibria.
(i) The boundary equilibrium i.e. $E_1(\frac{\alpha}{\eta}, 0, 0)$
(ii) The invertebrate predator free equilibrium $E_2(\overline{X}, \overline{Y}, 0)$
where

$$\overline{X} = \frac{-(\eta\gamma+\beta-\alpha\gamma)+\sqrt{(\eta\gamma+\beta-\alpha\gamma)^2+4\alpha\eta\gamma^2}}{2\eta\gamma}$$
$$\overline{Y} = \frac{\beta\left\{-(\eta\gamma+\beta-\alpha\gamma)+\sqrt{(\eta\gamma+\beta-\alpha\gamma)^2+4\alpha\eta\gamma^2}\right\}}{2\eta\gamma^2}$$

(iii) The interior equilibrium $E_3(X^*, Y^*, Z^*)$ where,

$$X^* = \frac{(\alpha-\eta)+\sqrt{(\alpha+\eta)^2-\frac{4\eta}{\phi}ln\left(\frac{\sigma}{\sigma-1}\right)}}{2\eta}$$
$$Y^* = \frac{1}{\phi}ln\left(\frac{\sigma}{\sigma-1}\right)$$
$$Z^* = \sigma\frac{1}{\phi}ln\left(\frac{\sigma}{\sigma-1}\right)\left[\beta - \frac{\frac{2\gamma\eta}{\phi}ln\left(\frac{\sigma}{\sigma-1}\right)}{(\alpha-\eta)+\sqrt{(\alpha+\eta)^2-\frac{4\eta}{\phi}ln\left(\frac{\sigma}{\sigma-1}\right)}}\right]$$

We see that the interior equilibrium $E_3(X^*, Y^*, Z^*)$ is feasible under the given conditions
(i) $\sigma > 1$
(ii) $Y^* > \alpha$
(iii) $\beta > \frac{2\gamma\eta Y^*}{(\alpha-\eta)+\sqrt{(\alpha+\eta)^2-4\eta Y^*}} = \beta^*$(say).

4. QUALITATIVE ANALYSIS OF THE MODEL

4.1. Boundedness of the solutions

The boundedness of solutions of the system (2) is shown in the following lemma.

Lemma 1

All the solutions of the system (2) originating in $\mathbb{R}^3_+$ are uniformly bounded and are confined into a region $\mathbb{B}$ defined by
$\mathbb{B} = \left\{(X, Y, Z) : 0 \leq X \leq \frac{\alpha}{\eta}, 0 \leq Y \leq \frac{\alpha\beta}{\eta\gamma}, 0 \leq X + Y + Z \leq \frac{L}{m}\right\}$

where, $L = \frac{2\alpha^2}{\eta} + \frac{2\alpha\beta^2}{\eta\gamma}$ and m=min$\left\{\alpha, \beta, 1\right\}$

Proof

To prove this we shall use the standard comparison theorem and follow the approach of Freedman [13].
From (2) we get

$$\frac{dX}{dT} \leq X(\alpha - \eta X), \text{ So } X(T) \leq \frac{\alpha}{\eta} \text{ for } T \to \infty$$
$$\frac{dY}{dT} \leq Y\left(\beta - \frac{\gamma Y}{X}\right), \; Y(T) \leq \frac{\alpha\beta}{\eta\gamma} \text{ for } T \to \infty$$

Let us define a function S=X+Y+Z.
The time derivative of S along the solution of (2) is

$$\begin{aligned}
\dot{S}(T) &= \dot{X} + \dot{Y} + \dot{Z} \\
&\leq X(\alpha - \eta X) + Y\left(\beta - \tfrac{\gamma Y}{X}\right) - Z\{1 - exp(-\phi Y)\} - Z + Z\sigma\{1 - exp(-\phi Y)\} \\
&= \alpha X - \eta X^2 + \beta Y - \tfrac{\gamma Y^2}{X} - Z\{1 - exp(-\phi Y)\}(1-\sigma) - Z \\
&\leq \alpha X + \beta Y - Z\{1 - exp(-0)\}(1-\sigma) - Z \\
&= \alpha X + \beta Y - Z \\
&= -\alpha X + 2\alpha X - \beta Y + 2\beta Y - Z \\
&\leq -m(X+Y+Z) + \max_{0 \leq X \leq \frac{\alpha}{\eta}} 2\alpha X + \max_{0 \leq Y \leq \frac{\alpha\beta}{\eta\gamma}} 2\beta Y
\end{aligned}$$

$$\Rightarrow \dot{S}(T) + mS(T) \leq \tfrac{2\alpha^2}{\eta} + \tfrac{2\alpha\beta^2}{\eta\gamma} = L \text{ (say)}$$

Applying the theorem of differential inequality we obtain
$0 \leq S \leq \frac{L}{m} + e^{-mT} S(X(0), Y(0), Z(0))$.
At $T \to \infty$, $e^{-mT} \to 0$.
So, $0 \leq S \leq \frac{L}{m}$ where m=min $\{\alpha, \beta, 1\}$.
Hence system (2) is dissipative with the asymptotic bound $\frac{L}{m}$.

4.2. Dynamic behaviour of the system

In this subsection we discuss the stability of the equilibria E_1, E_2, E_3 respectively. Let J_k be the Jacobian of the system (2) at the equilibrium points E_k where $k = 1, 2, 3$.
At E_1, the Jacobian matrix is

$$J_1 = \begin{bmatrix} -\alpha & -\frac{\alpha}{\alpha+\eta} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So the eigenvalues of the matrix J_1 is $-\alpha$, β, -1. Therefore, E_1 is stable in X and Z direction and unstable in Y direction.

At E_2, Jacobian matrix is

$$J_2 = \begin{bmatrix} \alpha - 2\eta\overline{X} - \frac{\overline{Y}}{(1+\overline{X})^2} & -\frac{\overline{X}}{1+\overline{X}} & 0 \\ \frac{\gamma\overline{Y}^2}{\overline{X}^2} & \beta - \frac{2\gamma\overline{Y}}{\overline{X}} & -1 + e^{-\phi\overline{Y}} \\ 0 & 0 & -1 + \sigma\left(1 - e^{-\phi\overline{Y}}\right) \end{bmatrix}$$

The characteristic equation of J_2 is

$$\left[-1 + \sigma\left(1 - e^{-\phi\overline{Y}}\right) - \lambda\right]\left[\lambda^2 - \lambda\left\{\alpha - 2\eta\overline{X} - \frac{\overline{Y}}{(1+\overline{X})^2} + \beta - \frac{2\gamma\overline{Y}}{\overline{X}}\right\} + \left(\beta - \frac{2\gamma\overline{Y}}{\overline{X}}\right)\right.$$
$$\left.\left\{\alpha - 2\eta\overline{X} - \frac{\overline{Y}}{(1+\overline{X})^2}\right\} + \frac{\gamma\overline{Y}^2}{\overline{X}\left(1+\overline{X}\right)}\right] = 0$$

$$\Rightarrow \left[-1 + \sigma\left(1 - e^{-\phi\overline{Y}}\right) - \lambda\right]\left[\lambda^2 + P\lambda + Q\right] = 0$$

where,

$$P = -\alpha + 2\eta\overline{X} + \frac{\overline{Y}}{(1+\overline{X})^2} - \beta + \frac{2\gamma\overline{Y}}{\overline{X}}$$
$$Q = \left(\beta - \frac{2\gamma\overline{Y}}{\overline{X}}\right)\left\{\alpha - 2\eta\overline{X} - \frac{\overline{Y}}{(1+\overline{X})^2}\right\} + \frac{\gamma\overline{Y}^2}{\overline{X}\left(1+\overline{X}\right)}$$

If $P > 0$ and $Q > 0$, the equilibrium $E_2(\overline{X}, \overline{Y}, 0)$ is stable in XY plane otherwise unstable.

In the Z direction $E_2(\overline{X}, \overline{Y}, 0)$ is unstable or stable according to the condition $\overline{Y} > Y^*$ or $\overline{Y} < Y^*$.

At E_3, Jacobian matrix is

$$J_3 = \begin{bmatrix} \alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2} & -\frac{X^*}{1+X^*} & 0 \\ \frac{\gamma Y^{*2}}{X^{*2}} & \beta - \frac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*} & -\frac{1}{\sigma} \\ 0 & \phi\sigma Z^* e^{-\phi Y^*} & 0 \end{bmatrix}$$

Then the characteristic equation of J_3 is

$$\lambda^3 - \lambda^2\left[\left\{\alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2}\right\} + \left\{\beta - \frac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*}\right\}\right] + \lambda\left[\left\{\alpha - 2\eta X^*\right.\right.$$
$$\left.\left. - \frac{Y^*}{(1+X^*)^2}\right\}\left\{\beta - \frac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*}\right\} + \phi Z^* e^{-\phi Y^*} + \frac{X^*}{1+X^*}\frac{\gamma Y^{*2}}{X^{*2}}\right] - \phi Z^* e^{-\phi Y^*}$$
$$\left\{\alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2}\right\} = 0$$

$$\Rightarrow \lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{3}$$

$$\Rightarrow f(\lambda) = 0 \text{(say)}$$

where,

$$A = -\left\{\alpha - 2\eta X^* - \tfrac{Y^*}{(1+X^*)^2}\right\} - \left\{\beta - \tfrac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*}\right\}$$
$$B = \left\{\alpha - 2\eta X^* - \tfrac{Y^*}{(1+X^*)^2}\right\}\left\{\beta - \tfrac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*}\right\} + \phi Z^* e^{-\phi Y^*} + \tfrac{X^*}{1+X^*}\tfrac{\gamma Y^{*2}}{X^{*2}}$$
$$C = -\phi Z^* e^{-\phi Y^*}\left\{\alpha - 2\eta X^* - \tfrac{Y^*}{(1+X^*)^2}\right\}$$

From Routh-Hurwitz criterion the system (2) is locally asymptotically stable if $A > 0$, $C > 0$, $AB - C > 0$.
Here $A > 0$ if

$$\beta < \frac{1}{1-\phi Y^*(\sigma-1)}\left\{-\alpha + 2\eta X^* + \frac{Y^*}{(1+X^*)^2} + \frac{2\gamma Y^*}{X^*} - \frac{\phi\gamma Y^{*2}(\sigma-1)}{X^*}\right\} = \beta^{**}(say)$$

and $C > 0$ (always).
Now, $AB - C = \sigma_1\beta^2 + \sigma_2\beta + \sigma_3$
where,

$$\sigma_1 = \left\{\phi Y^*(\sigma-1) - 1\right\}\left\{-\eta X^* + \frac{X^*Y^*}{(1+X^*)^2} + \phi Y^*(\sigma-1)\left(1 + \eta X^* - \frac{X^*Y^*}{(1+X^*)^2}\right)\right\}$$

$$\begin{aligned}\sigma_2 = {} & \left\{-\eta X^* + \frac{X^*Y^*}{(1+X^*)^2} + \phi Y^*(\sigma-1)\left(1+\eta X^* - \frac{X^*Y^*}{(1+X^*)^2}\right)\right\}\left\{-\frac{\phi\gamma(\sigma-1)Y^{*2}}{X^*}\right.\\ & \left. +\eta X^* - \frac{X^*Y^*}{(1+X^*)^2} + \frac{2\gamma Y^*}{X^*}\right\} + \phi Y^*(\sigma-1)\left(-\eta X^* + \frac{X^*Y^*}{(1+X^*)^2}\right) + \left\{\phi Y^*(\sigma-1)\right.\\ & \left. -1\right\}\left\{-\frac{\phi\gamma(\sigma-1)Y^{*2}}{X^*}\left(1+\eta X^* - \frac{X^*Y^*}{(1+X^*)^2}\right) - \frac{2\alpha\gamma Y^*}{X^*} + 4\eta\gamma Y^* + \frac{2\gamma Y^{*2}}{X^*(1+X^*)^2}\right.\\ & \left. + \frac{\gamma Y^{*2}}{X^*(1+X^*)}\right\}\end{aligned}$$

$$\begin{aligned}\sigma_3 = {} & \left\{\frac{-\phi\gamma(\sigma-1)Y^{*2}}{X^*} + \eta X^* - \frac{X^*Y^*}{(1+X^*)^2} + \frac{2\gamma Y^*}{X^*}\right\}\left\{\frac{-\phi\gamma(\sigma-1)Y^{*2}}{X^*}\right.\\ & \left.\left(1+\eta X^* - \frac{X^*Y^*}{(1+X^*)^2}\right) - \frac{2\alpha\gamma Y^*}{X^*} + 4\eta\gamma Y^* + \frac{2\gamma Y^{*2}}{X^*(1+X^*)^2} + \frac{\gamma Y^{*2}}{X^*(1+X^*)}\right\}\\ & -\frac{\phi\gamma(\sigma-1)Y^{*2}}{X^*}\left(-\eta X^* + \frac{X^*Y^*}{(1+X^*)^2}\right)\end{aligned}$$

It is quite difficult to find an explicit parametric condition for which all above conditions are satisfied. But through numerical simulations it is observed that there exists some threshold value of β say $\beta = \beta_1$ such that A, C and $AB - C$ are positive for all $0 < \beta < \beta_1$ and E_3 is locally asymptotically stable. When $\beta = \beta_1$, $AB - C = 0$ and we conclude that there exists Hopf-bifurcating small amplitude periodic solutions near E_3. If $\beta > \beta_1$, then $AB - C < 0$ and E_3 is unstable. Then there exists a large amplitude non-constant periodic orbit near E_3.

Theorem 1

There exists a Hopf-bifurcating small amplitude periodic solution if there exists a $\beta = \beta_1 (> 0)$ such that $A(\beta_1) > 0, C(\beta_1) > 0$ and satisfying the equation
$\sigma_1 \beta^2 + \sigma_2 \beta + \sigma_3 = 0$ where $2\sigma_1 \beta_1 + \sigma_2 \neq 0$

Proof

The characteristic equation (3) has one real and two purely imaginary roots at $\beta = \beta_1$ [14].
Now, $AB - C = 0$
$\Rightarrow \sigma_1 \beta^2 + \sigma_2 \beta + \sigma_3 = 0$ has one positive $\beta = \beta_1$ such that $A(\beta_1)B(\beta_1) - C(\beta_1) = 0$.
Let us consider the real root is $\rho(\beta_1) = u(\beta_1) + iv(\beta_1)$
We know that $u(\beta_1) = 0$, we have to show $\frac{du}{d\beta}|_{\beta=\beta_1} \neq 0$
Since $\rho(\beta_1)$ is a root of equation (3), we get
$(u + iv)^3 + A(u + iv)^2 + B(u + iv) + C = 0$
Separating the real and imaginary parts we get

$$
\begin{aligned}
u^3 - 3uv^2 + Au^2 - Av^2 + Bu + C &= 0 \\
3u^2 v - v^3 + 2Auv + Bv &= 0
\end{aligned}
\tag{4}
$$

When $v(\beta_1) \neq 0$ then (4) has a solution iff $v^2 = 3u^2 + 2Au + B$ which gives

$$
8u^3 + 8Au^2 + 2(A^2 + B)u + AB - C = 0 \tag{5}
$$

Equation (5) has a root $u(\beta_1)$ such that $u(\beta_1) = 0$ iff $A(\beta_1)B(\beta_1) - C(\beta_1) = 0$ implies
$\sigma_1 \beta_1^2 + \sigma_2 \beta_1 + \sigma_3 = 0$
Now at $\beta = \beta_1$, $u(\beta_1) = 0$ is the only root, since $8u^2 + 8Au + 2(A^2 + B) = 0$ has no real root.
Again it is seen that $v(\beta_1) = \sqrt{B(\beta_1)}$ holds.

Our aim is to show $\frac{du}{d\beta}|_{\beta=\beta_1} \neq 0$
Now from (5) we get
$24u^2\frac{du}{d\beta} + 16Au\frac{du}{d\beta} + 8\frac{dA}{d\beta}u^2 + 2(A^2+B)\frac{du}{d\beta} + 2(2A\frac{dA}{d\beta} + \frac{dB}{d\beta})u + 2\sigma_1\beta + \sigma_2 = 0$
At $\beta = \beta_1$ using $u(\beta_1) = 0$ we get
$\frac{du}{d\beta}|_{\beta=\beta_1} = \frac{-(2\sigma_1\beta_1+\sigma_2)}{2[A^2(\beta_1)+B(\beta_1)]} \neq 0$
This completes the proof of the Theorem.

4.3. Criteria for non-constant large amplitude periodic solutions

When $AB-C$ is negative then the characteristic equation (3) has one real negative root and two complex roots with a positive real part. Then E_3 is locally unstable. Now using the stable manifold theorem [15] we can say that there exists a one dimensional stable manifold and a two dimensional unstable manifold at E_3. We have already shown that there exists a bounded invariant domain $\mathbb{B} \subset \mathbb{R}^3_+$. To know the exact location of the stable manifold and to describe the flow of trajectories of the system through octants in the positively invariant bounded set $\mathbb{B} \subset \mathbb{R}^3_+$ we divide $\mathbb{B}$ into eight octants as follows(see figure 1)

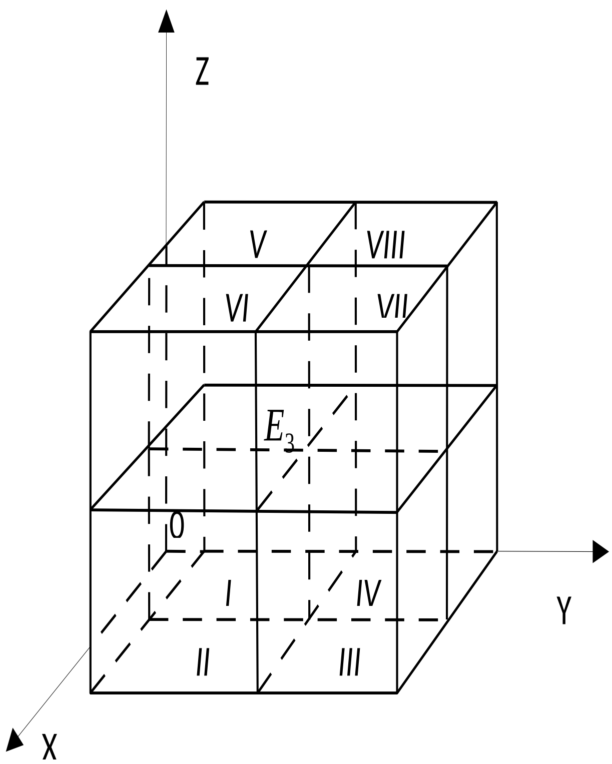

Figure 1: Rectangular region $\mathbb{B}$ is divided into eight sub-boxes as defined in the text.

$$\begin{aligned}
I &= \left\{(X,Y,Z) : \epsilon_1 \le X \le X^*, 0 \le Y \le Y^*, 0 \le Z \le Z^*\right\} \\
II &= \left\{(X,Y,Z) : X^* \le X \le \frac{\alpha}{\eta}, 0 \le Y \le Y^*, 0 \le Z \le Z^*\right\} \\
III &= \left\{(X,Y,Z) : X^* \le X \le \frac{\alpha}{\eta}, Y^* \le Y \le \frac{\alpha\beta}{\eta\gamma}, 0 \le Z \le Z^*\right\} \\
IV &= \left\{(X,Y,Z) : \epsilon_1 \le X \le X^*, Y^* \le Y \le \frac{\alpha\beta}{\eta\gamma}, 0 \le Z \le Z^*\right\} \\
V &= \left\{(X,Y,Z) : \epsilon_1 \le X \le X^*, 0 \le Y \le Y^*, Z^* \le Z \le \frac{L}{m} - \epsilon_1\right\} \\
VI &= \left\{(X,Y,Z) : X^* \le X \le \frac{\alpha}{\eta}, 0 \le Y \le Y^*, Z^* \le Z \le \frac{L}{m} - \epsilon_1\right\} \\
VII &= \left\{(X,Y,Z) : X^* \le X \le \frac{\alpha}{\eta}, Y^* \le Y \le \frac{\alpha\beta}{\eta\gamma}, Z^* \le Z \le \frac{L}{m} - \epsilon_1\right\} \\
VIII &= \left\{(X,Y,Z) : \epsilon_1 \le X \le X^*, Y^* \le Y \le \frac{\alpha\beta}{\eta\gamma}, Z^* \le Z \le \frac{L}{m} - \epsilon_1\right\}
\end{aligned}$$

where $\epsilon_1 < \frac{(\alpha-\eta)-\sqrt{(\alpha-\eta)^2+4\alpha\eta\left(1-\frac{\beta}{\eta\gamma}\right)}}{2\eta}$

Lemma 2

The eigenvector associated with the negative real eigenvalue of the Jacobian matrix at E_3 points into the boxes IV and VI.

Proof

Let λ_1 denote the unique real negetive eigenvalue. Then the eigenvector $(x,y,z)^T$ must satisfy

$$\begin{aligned}
\left[\alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2} - \lambda_1\right] x &= \frac{X^*}{1+X^*} y \\
\frac{\gamma Y^{*2}}{X^{*2}} x + \left\{\beta - \frac{2\gamma Y^*}{X^*} - Z^* \phi e^{-\phi Y^*} - \lambda_1\right\} y &= \frac{1}{\sigma} z \\
\phi\sigma Z^* e^{-\phi Y^*} y &= \lambda_1 z
\end{aligned} \tag{6}$$

From the last equation of (6) we see that y and z are opposite sign.

Now, $\alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2} = X^* \left[\frac{Y^*}{(1+X^*)^2} - \eta\right]$
Let
$Y^* > \eta(1+X^*)^2$
$\Rightarrow 0 > \sqrt{(\alpha+\eta)^2 - 4\eta Y^*}$
$\Rightarrow 0 > (\alpha+\eta)^2 - 4\eta Y^*$
which is a contradiction.
Therefore, $\alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2} < 0$
From (6), we conclude that x and y are opposite sign when $\beta - \frac{2\gamma Y^*}{X^*} - Z^*\phi e^{-\phi Y^*} > 0$ and $\lambda_1 > \alpha - 2\eta X^* - \frac{Y^*}{(1+X^*)^2}$ for $\beta > \beta_1$
Hence from the above discussion we conclude that the eigenvector associated with $\lambda_1 < 0$ points into the boxes IV and VI.

Lemma 3

Any positive solution of the system (2) except those on the stable manifold must eventually start oscillating according to the sequence
$II \to III \to VII \to VIII \to V \to I \to II$

Proof

Let F_{ij} denote the face between the boxes i and j. Form the symmetric properties of boxes, we consider the trajectory which starts on the face F_{23} will intersect the face F_{58} excluding the point E_3. It is seen that on F_{23}, $\dot{Y} > 0$ and so the trajectory will move from II to III. Similarly on F_{37}, $\dot{Z} > 0$ and the trajectory will move from III to VII. Therefore we conclude that the trajectory does not return to II or III.
Since the trajectory can not tend to E_3 from VII, it must intersect F_{78}. On the face F_{78} we get $\dot{X} < 0$. So the trajectory enters into the box $VIII$. On the face F_{58}, $\dot{Y} < 0$ and the trajectory enters into the box V. Similarly on the face F_{15}, $\dot{Z} < 0$ and the trajectory enters into the box I. It is also seen that on the face F_{12}, $\dot{X} > 0$ and the trajectory will move from I to II.
Hence the trajectory will start oscillating according to the sequence $II \to III \to VII \to VIII \to V \to I \to II$. This completes the proof.

5. NUMERICAL SIMULATION AND DISCUSSIONS

In this section we perform numerical simulations to validate our analytical findings. We are using MATLAB-R2017a and the standard MATLAB differential equations integrator for Runge-Kutta method, i.e., MATLAB routine ODE 45.

For this purpose we take $\alpha = 5, \eta = 0.2, \phi = 1, \sigma = 1.005, \gamma = 0.92$ and get the value of $\beta^* = 0.2038$. From the existence criteria we know that when $\beta < \beta^*$ the interior equilibrium E_3 does not exist and the system is locally asymptotically stable around E_2 in the XY plane if $P > 0$ and $Q > 0$ is satisfied. Using the same set of parameter values we get a critical value of β say $\beta = \beta_1 = 0.4229$ such that if $\beta < \beta_1$ then the system is locally asymptotically stable around the interior equilibrium E_3. If $\beta > \beta_1$ then the system is unstable.

Figure 2 and 3 show the solution curve of the system against time and the phase portrait in the XY plane respectively when $\beta < \beta^*, P > 0, Q > 0$.

Figure 4 and 5 show the solution curve of the system (2) against time and the corresponding phase portrait around the interior equilibrium E_3 for $\beta^* < \beta < \beta_1$.

Figure 6 shows the oscillatory behaviour for plant litter, micro-organism and invertebrate predator population against time and figure 7 gives the corresponding phase portrait of the system which shows the periodic behaviour of the system (2) around E_3 for $\beta = 0.5 > \beta_1$.

If $\beta = \beta_1 = 0.4229$, then there exists small amplitude periodic oscillation which leads to Hopf-bifurcating small amplitude periodic solution of the system.

Through the numerical simulations it is shown that the parameter β (which represents the growth rate parameter of micro-organisms) plays an important role to shape the dynamics of the detritus based prey-predator system.

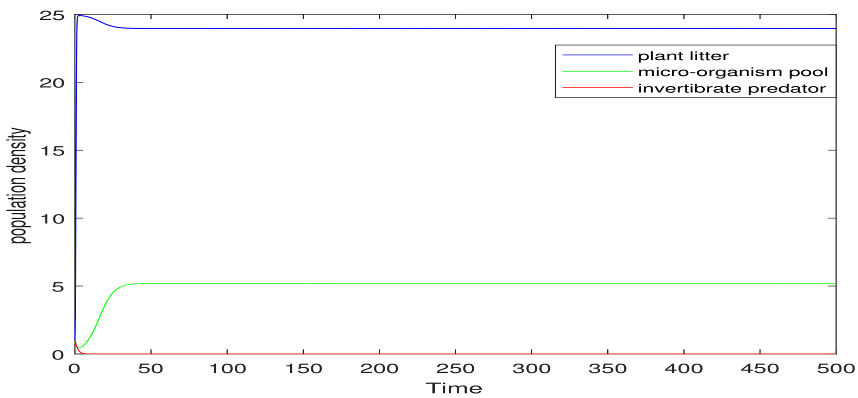

Figure 2: Solution curve for non existence of interior equlibrium for same initial values for $\alpha = 5$, $\beta = 0.2$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

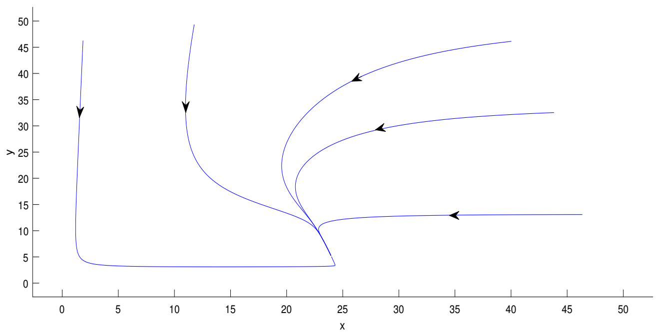

Figure 3: Phase portrait of the system (2) for different initial values for $\alpha = 5$, $\beta = 0.2$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$ in XY plane.

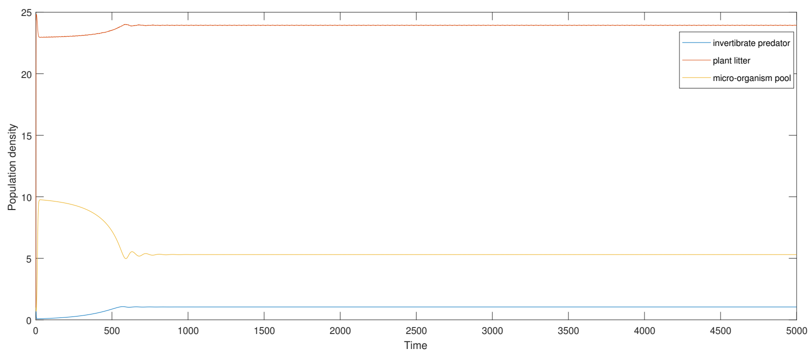

Figure 4: Globally asymptotically stable steady-state of the system (2) for same initial values for $\alpha = 5$, $\beta = 0.4$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$ in XY plane.

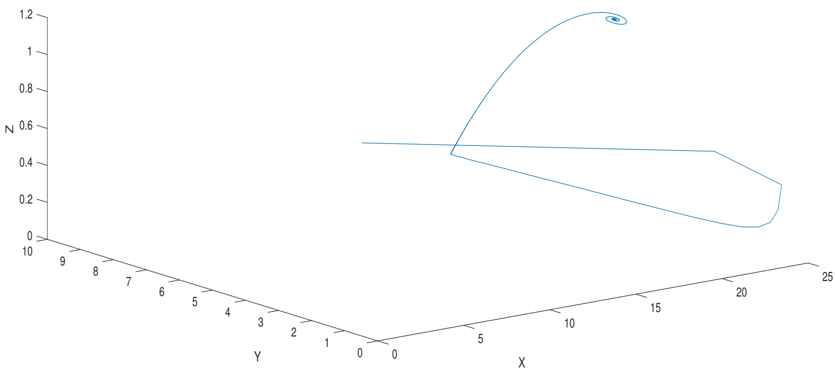

Figure 5: Globally asymptotically stable steady-state around E_3 for different initial values for $\alpha = 5$, $\beta = 0.4$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

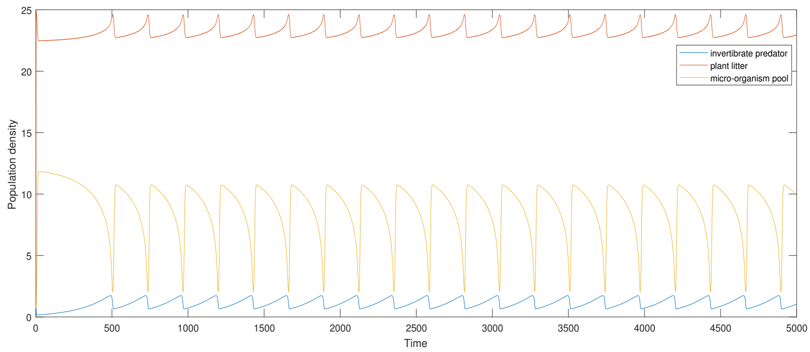

Figure 6: Oscillatory behaviour of the system (2) for same initial values for $\alpha = 5$, $\beta = 0.5$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

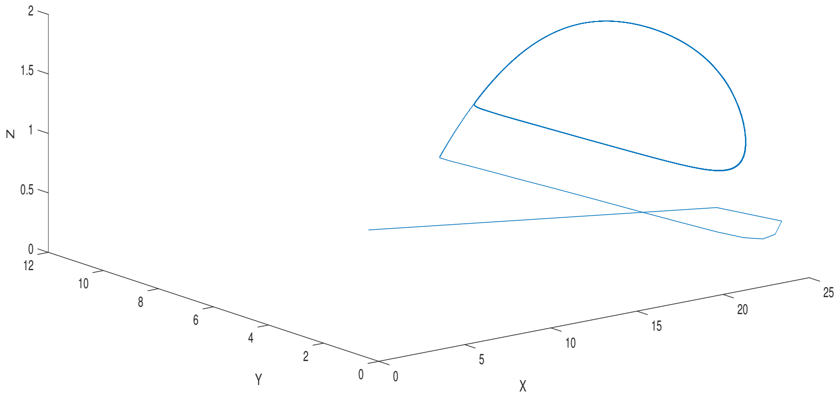

Figure 7: existence of periodic orbit for different initial values for $\alpha = 5$, $\beta = 0.5$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

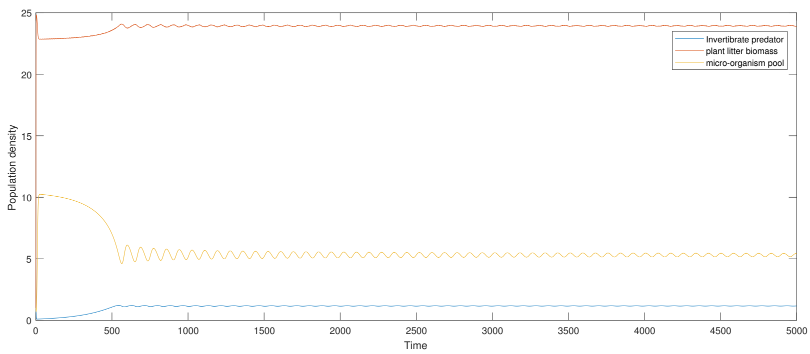

Figure 8: Hopf-bifurcating small-amplitude periodic solution for same initial values for $\alpha = 5$, $\beta = 0.4229$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

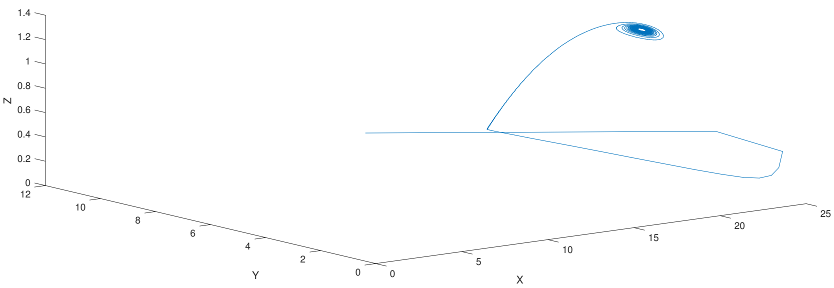

Figure 9: Hopf-bifurcating small-amplitude periodic solution for different initial values for $\alpha = 5$, $\beta = 0.4229$, $\gamma = 0.92$, $\phi = 1$, $\eta = 0.2$ and $\sigma = 1.005$.

6. CONCLUSIONS

We know that mangrove forest has a significant role due to formation of the detritus from the mangrove litter which plays a key role to maintain the nutrient level of the estuary. In this study we consider the mathematical model which is completely based on realistic situation of Sundarban estuary. A general dynamical behaviour of a homogeneous model of detritus based ecosystem comprised of micro-organism pool and invertebrate predator is discussed. Here the loss rate of detritus due to micro organism pool follows Holling Type-II functional response and the growth rate of micro-organism follows donor-controlled type function and the functional response of invertebrate predator is taken as Ivlev-type response function. These functions have a dense ecological interpretation and show a complex nature of the model system which leads to various mathematical results that are quite realistic from the ecological point of view. In this study we derive the conditions of global attractor, periodic orbit and Hopf-bifurcation under different conditions. It is seen that the growth rate of micro-organism pool acts as bifurcation parameter and it is an important factor which affects the stability of the mangrove ecosystem. This study provides an overall understanding of the detritus based prey-predator model of the mangrove ecosystem of Sundarban, India.

REFERENCES

[1] Bandyopadhyay, M., Bhattacharya, R., and Chakrabarti, C.G., 2003, ”A nonlinear two-species oscillatory system: bifurcation and stability analysis,” International Journal of Mathematics and Mathematical Sciences, Vol. 31, pp.1981-1991.

[2] Das, K., and Ray, S., 2008, ”Effect of delay on nutrient cycling in phytoplankton–zooplankton interactions in estuarine system,” ecological modelling, Vol. 215, pp. 1-3:69-76.

[3] Gazi, N.H., and Bandyopadhyay, M., 2006, ”Effect of time delay on a detritus-based ecosystem,” International journal of mathematics and mathematical sciences.

[4] Sarkar, A.K., Mitra, D., Ray, S., and Roy, A.B., 1991, ”Permanence and oscillatory co-existence of a detritus-based prey-predator model,” Ecological modelling, Vol. 53, pp. 147-156.

[5] Chen, L., Chen, F., and Chen, L., 2010, ”Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a constant prey refuge,” Nonlinear Analysis: Real World Applications, 11(1), 246-252.

[6] Ko, W., Ryu, K., 2006, ”Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a prey refuge,” Journal of Differential Equations, 231(2), pp. 534-550.

[7] Pimm, S. L. , 1982, ”Food Webs,” London and New York: Chapman and Hall.

[8] Kooij, R.E., and Zegeling, A., 1996, ”A predator-prey model with Ivlev's functional response,” Journal of Mathematical Analysis and Applications, Vol. 198, No.2, pp. 473-489.

[9] Wang, W., Zhang, L. , Wang, H. , and Li, Z. , 2010, ”Pattern formation of a predator-prey system with Ivlev-type functional response,” Ecological Modelling, Vol. 221, No. 2.

[10] Ghosh, D. , and Sarkar, A. K. , 1997, ”Oscillatory behaviour of an autotroph-herbivore system with a type-III uptake function,” International journal of systems science, Vol. 28, No. 3, pp. 259-264.

[11] Roy, U. , Sarwardi, S. , Majee N. C. , and Ray, S. , 2016, ”Effect of salinity and fish predation on zooplankton dynamics in Hooghly–Matla estuarine system, India,” Ecological informatics, Vol. 35, pp. 19-28.

[12] Sarwardi, S., Mandal, P.K. , and Ray, S. , 2012, ”Analysis of a competitive prey–predator system with a prey refuge,”Biosystems, Vol. 110, No. 3, pp. 133-148.

[13] Freedman, H. I. , 1980, ”Deterministic Mathematical Models in Population Ecology,” Marcel Dekker, New York, NY.

[14] Rai, B. , Freedman, H. I. and Addicott, J. F. ,1983, ”Analysis of three species models of mutualism in predator-pre and competitive systems,” Math. Biosci, Vol. 65, pp. 13-50.

[15] Hale, J. K. , 1969, ”Ordinary Differential Equations,” Wiley-Interscience, New York, NY, pp.242.

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