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Effect of Growth Functions on Jatropha Curcas Plant with Random Attack Pattern of Whitefly: A mathematical study

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Abstract

Jatropha curcas is an important plant which can provide an affordable solution of shrinkage of fossil fuel by producing alternative fuel (biodiesel). The seeds of this plant contain a high amount of oil that can be used to obtain a better quality of biodiesel . So the economic value of the plant is very high , but this plant is affected by the mosaic virus (*Begomovirus*) through whitefly (Bemisia tabaci) which causes mosaic disease. In this paper we consider two mathematical models of different growth functions of the Jatropha curcas plant with random attack pattern of the whitefly. These two nonlinear deterministic models of jatropha curcas plant and whitefly is studied analytically where the distribution of whitefly on plants follows poisson distribution. The result shows that the first system possesses a fragile behavior and the other shows a steady state which is globally asymptotically stable.

Keywords: Jatropha Curcas plant ; Mosaic virus (Begomovirus) ; Whitefly (Bemisia tabaci); Mosaic disease ; Poisson distribution ; Random attack ; Fragile behaviour ; Global stability .

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1. INTRODUCTION

Jatropha curcas plant is one of the wonder plant with economic potentiality and ecological applications in various aspects. The plant produces seeds with oil (biodiesel) that can be combusted as fuel without being refined. Jatropha curcas is a species of flowering plant in the spurge family, Euphorbiaceae and popularly called as physic nut. This plant has been introduced to Africa and Asia and is now cultivated world-wide (Pandey et al., 2012).

Jatropha curcas is a semi-evergreen shrub or small tree with large green to pale green leaves. It grows between (3-5) meter in height but grows upto (8-10) meter under favourable conditions. The tree can be grown in dry and infertile conditions and cultivated also in rough, sandy and salty soils. It has low plantation cost and the first harvesting is made just after 18 months. It grows quickly and lives producing seeds for 50 years. The most successful cultivation occurs in the drier regions of the tropics with annual rainfall of (300-1000) mm. It occurs mainly at lower altitudes (0-500) meter in areas with average annual temperature well above 20 °C but it can also grow at higher altitudes, low nutrient and tolerates slight frost. Fruits are produced in winter or it may produce several crops during the year if soil moisture is good and temperature is sufficiently high. The seeds become mature when the capsule changes from green to yellow, after two to four months. It's life span is around 45-50 years. Seed production ranges from about 0.4 tonnes to 12 tonnes per hectare per year. The seeds of this plant contain 37% oil that can be used to obtain a better quality of biodiesel (Sahoo et al., 2009).

Jatropha curcas is naturally infected by *Begomovirus*. The symptoms of mosaic disease are severe mosaic , mottling, blistering of leaves, yellowing of leaves, reduced leaf size, stunting of diseased plants. It mainly attacks its fruits, considerably reducing the production and quality of seeds. The mosaic virus is carried through infected whiteflies (Gao et al., 2010 ; Holt et al., 1997).

The population of whitefly is controlled by temperature and rainfall. Heavy rainfall creates an obstruction for the growth of whiteflies. In this disease the mosaic virus passes from an infected whitefly to a susceptible plant and vice-versa. The spread of the virus is highly dependent on the plant density. A single whitefly is adequate to infect the host plants but transmission of the disease spread when numerous infected whiteflies feed on the host plants through massive flux of saliva. As a result host plant (Jatropha curcas) faces leaf damage and sap drainage due to such feeding. Whiteflies are tremendously productive, if once they get conventional on any part of the plants they will voluntarily roam and try to attack any other immediate vegetation (Narayana

D.S.A. et al., 2006 ; Venturino et al., 2016). Normally they need 3 hours feeding time to procure the virus and a latent phase of 8 hours. It requires 10 minutes time to contaminate the young leaves. Symptoms seem to be appeared after a latent period of 3-5 weeks. Moreover the infected whiteflies inject the virus to the plant with the infection being more likely if more insects attack the same plant . After acquisition of the mosaic virus adult whiteflies can infect the host plants within 48 hours.The purpose of this study is to show the dynamics due to different growth function of the plant Jatropha Curcas and the effect of random attack pattern of whitefly on the plant (Sarkar and Roy , 1989).
It is observed that if the Jatropha curcas plant grows exponentially then it shows fragile behaviour and it can be stabilize by considering logistic growth of the plant. The result so obtained is verified by numerical simulation.

2. STATEMENT OF THE MODEL

In our model we have considered that *v* whiteflies are distributed over *x* plants in such a way that some plants are whitefly free and others have 1,2,..........,i whiteflies per plant. Thus we have:

$$\sum_{i=1}^{x} i = v$$

We here assumed that the whiteflies are distributed over *x* plants according to a probability distribution so that the proportion of plants with i whiteflies is p(i). So the number of plants with i whiteflies is p(i)*x*. If the intrinsic plant loss-rate per whitefly is f then the loss-rate of plants with i whiteflies will be fip(i)*x*. Therefore the total loss-rate of plants is

$$fx\sum_{i=0}^{\infty} ip(i)$$

Here $\sum ip(i)$ is the mean number of whitefly per plant and *v*/*x* is the expectation of i. So the loss rate due to whitefly consumption is f*v*. The loss of whiteflies occur in the following ways.
e=natural mortality of whitefly.
b=natural mortality of the host plant.
f=by their killing the host plant.
This self induced mortality occurs at a rate $fi^2p(i)x$. So for the whole plant population it is

$$fx\sum_{i=0}^{\infty} i^2p(i)$$

The term $\sum i^2 p(i)$ is the expectation of i^2. We have chosen here the poisson distribution which ecologically reflects random attack pattern. Here whitefly-inflicted losses through the plant death are $fxE(i^2)$. For Poisson distribution we have $E(i^2) = \frac{v}{x} + (\frac{v}{x})^2$ (Sarkar and Roy , 1989).
We have chosen two different types of growth function of the plant (Jatropha curcas) population. In the first model we have chosen the growth of plant population in exponential form and in the second model the growth of of plant population is assumed to be in logistic form. The attack pattern of whitefly on the plant is taken as holling type-1 function in both cases . Here r is growth rate of the whitefly, k is the carrying capacity.
Based on the above assumptions the first model is formulated as:

3. MODEL 1

Assuming that the plant and whitefly follow the exponential growth the model takes the figure:

$$\begin{aligned} \frac{dx}{dt} &= rx - axv - fv \\ \frac{dv}{dt} &= v[cx - (e + b + f) - \frac{fv}{x}] \end{aligned} \tag{1}$$

where $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$.
For mathematical simplicity we consider the following transformation:
$x = \frac{rX}{c}, v = \frac{r^2V}{cf}, t = \frac{\tau}{r}$.

Based on the transformation the model becomes:

$$\begin{aligned} \frac{dX}{d\tau} &= X - \alpha XV - V \\ \frac{dV}{d\tau} &= V[X - \beta - \frac{V}{X}] \end{aligned} \tag{2}$$

Where $\alpha = \frac{ar}{c^2 f}$, $\beta = \frac{b+e+f}{r}$

3.1. Equilibria

The steady state of the system is obtained by setting $\frac{dX}{d\tau} = 0$, $\frac{dV}{d\tau} = 0$ and solving the equations :

$$X - \alpha XV - V = 0$$

$$X - \beta - \frac{V}{X} = 0$$

We have seen that the system has only one equilibrium point i.e. the interior equilibrium point $E(X^*, V^*)$. To solve $E(X^*, V^*)$ we have a quadratic equation which has atleast one positive real root. Therefore $E(X^*, V^*)$ exists.

3.2. Dynamic behavior

At $E(X^*, V^*)$ the characteristic equation is,
$\lambda^2 + \lambda(\frac{V^*}{X^*} - \frac{1}{(1+\alpha X^*)}) + \frac{V^*}{X^*} + X^* - \frac{V^*}{X^*(1+\alpha X^*)} = 0$ which can be written as,

$$\lambda^2 + A\lambda + B = 0$$

Where $A = \frac{V^*}{X^*} - \frac{1}{(1+\alpha X^*)} = 0$

$$B = \frac{V^*}{X^*} + X^* - \frac{V^*}{X^*(1+\alpha X^*)} = \frac{\alpha V^*}{1+\alpha X^*} + X^* > 0$$

Therefore the roots are purely imaginary and $E(X^*, V^*)$ is a centre which shows the fragile behavior of the system (2).

4. MODEL 2

Assuming that the plant follows logistic growth then the model becomes:

$$\begin{aligned} \frac{dx}{dt} &= rx(1 - \frac{x}{k}) - axv - fv \\ \frac{dv}{dt} &= v[cx - (e + b + f) - \frac{fv}{x}] \end{aligned} \quad (3)$$

where $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$.

Here x_0 is the initial plant population density and v_0 is the initial whitefly density. For mathematical simplicity we consider the following transformation.

$x = kX$, $v = \frac{ck^2V}{f}$, $t = \frac{\tau}{ck}$

The transformed equation is,

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X(1-X) - \beta XV - V \\ \frac{dV}{d\tau} &= V[X - \gamma - \frac{V}{X}] \end{aligned} \tag{4}$$

where $\alpha = \frac{r}{ck}$, $\beta = \frac{ak}{f}$, $\gamma = \frac{b+e+f}{ck}$

4.1. Solution Properties

4.1.1 Lemma 1 :

The solutions of (4) are positive.

Proof:
Since $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$ and $V(0) = V_0 > 0$. Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leqslant \tau \leqslant \tau_0$. For $0 \leqslant \tau < \tau_0$
$\frac{\dot{X}(\tau)}{X(\tau)} = \alpha(1-X) - \beta V - \frac{V}{X} > -\alpha X - \beta V - \frac{V}{X}$
$X(\tau_0) > X_0 exp[-\int_0^{\tau_0} V(\eta)/X(\eta)d\eta] > 0$
This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $V(\tau)$ is also positive for all $\tau \geq 0$.

4.2. Equilibria

The equilibrium point is obtained by setting $\frac{dX}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations of (4):

$\alpha X(1-X) - \beta XV - V = 0$

$X - \gamma - \frac{V}{X} = 0$

We have seen that the system has two equilibrium points i.e. $E_1(X, 0) = (1, 0)$ which is the whitefly free equilibrium and $E_2(X^*, V^*)$ which is the interior equilibrium.

Here $V^* = \frac{\alpha X^*(1-X^*)}{\beta X^*+1}$

$$X^* = \frac{-(\alpha+1-\beta\gamma)\pm\sqrt{(\alpha+1-\beta\gamma)^2+4\beta(\alpha+\gamma)}}{2\beta}$$

For feasibility of X^* we have chosen the positive sign.

4.3. Dynamic behavior

From the variational matrix we obtained the behavior of different equilibrium points of the system. E_1 is saddle as $\gamma < 1$. The characteristic equation for $E_2(X^*, V^*)$ is given by,
$\lambda^2 + \alpha X^*\lambda - \frac{V^{*2}}{X^{*2}} + \alpha V^* + \alpha X^*(1 - X^*) + \frac{\alpha V^*(1-X^*)}{X^*} = 0$
which can be written as,
$\lambda^2 + A\lambda + B = 0$
where $A = \alpha X^* > 0$

$$B = \frac{\alpha(1-X^*)[\alpha X^*+\alpha\beta X^{*2}+\alpha X^*+\beta^2 X^{*3}+\beta X^{*2}+X^*+\alpha\beta X^*]}{\beta X^*+1} > 0.$$

Therefore $E_2(X^*, V^*)$ is locally asymptotically stable equilibrium if $\gamma < 1$.

4.4. Global Stability

4.4.1 Lemma 2

The XV subsystem is globally asymptotically stable.
Proof:
$H(X, V) = \frac{1}{XV}$, then $H > 0$ if $X > 0$ and $V > 0$.
$h_1(X, V) = \alpha X(1 - X) - \beta XV - V$
$h_2(X, V) = V(X - \gamma - \frac{V}{X})$
therefore $\nabla(X, V) = \frac{\partial(h_1 H)}{\partial X} + \frac{\partial(h_2 H)}{\partial V}$

$= \frac{\partial[\frac{\alpha(1-X)}{V} - \beta - \frac{1}{X}]}{\partial X} + \frac{\partial[1 - \frac{\gamma}{X} - \frac{V}{X^2}]}{\partial V}$
$= -\frac{\alpha}{V} < 0$

Hence by Bendixon-Dulac criteria $E_2(X^*, V^*)$ is globally asymptotically stable in the positive XV plane (Konar et al., 1999). This completes the proof of the lemma.

5. PERSISTENCE AND PERMANENCE OF THE SYSTEM

From the biological point of view persistence means that all the populations are present and none of them will become extinct. Persistence and permanence is very logical to settle the questions of survival and extinction of n-species whose growth equations are given by the differential equations

$\dot{x_i} = x_i f_i(x_1, x_2, ..., x_n)$

The notion of persistence (weak and strong) came to the light by Freedman and Waltman. The system (4) is said to be weakly persistent if $limsup\ x_i(t) > 0$ for all orbits in $int\mathbb{R}^n_+$ and strongly persistent if $liminf\ \ x_i(t) > 0$.

Again the system is said to be permanent if there exists a compact set $B \subset int\ \mathbb{R}^n_+$ such that all orbits in $int\ \mathbb{R}^n_+$ end up in B. The system is uniformly persistence if there exist $\delta > 0$ such that for each compact set x_i , $liminf\ \ x_i(t) \geq \delta > 0$ for all $(x_1(t), x_2(t), ..., x_n(t)) = X(t) \in int\ \mathbb{R}^n_+$. We now discuss the concept of saturated equilibria. An equilibrium fixed point x^* is said to be saturated equilibrium if $x^*_i = 0$ then $f_i(x^*_1, x^*_2, ..., x^*_n) \leq 0$. With the concept of saturated equilibria and by the method of average Lyapunov function we have the following theorem for permanent coexistence of both the species of the system.

5.1. Theorem

The system (4) is permanent if $\gamma < 1$.

Proof:

The index theorem states that the system with dissipativeness assumption has at least one saturated equilibrium. If all these saturated equilibria are regular, then the sum of their indices is +1. From the lemma 1, the system is dissipative and so there exists atleast one saturated equilibrium and the sum of their indices is +1 if they are regular. The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated. Therefore all the eigenvalues are negative or have negative real parts, which we have shown before.

We now construct the average Lyapunov function . In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.V^{r_2}$ where $r_i > 0$, i=1,2.

Let, $\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)}$

$$= r_1\frac{\dot{X}}{X} + r_2\frac{\dot{V}}{V}$$

$$= r_1(\alpha(1-X) - \beta V - \frac{V}{X}) + r_2(X - \gamma - \frac{V}{X})$$

If $\psi(X) > 0$ for the ω -limit sets of trajectories initiated in $\mathbb{R}^3_+$, then the trajectories more away from the boundary and the system (4) is permanent. It is evident that there is no periodic trajectory. Hence if there exist $r_1 > 0$ such that $\Psi(E_1) > 0$,then (4) is permanent.

Therefore for $E_1(1,0)$, $\psi(X) = r_2(1-\gamma) > 0$ should be satisfied for atleast one positive vector $r = (r_1, r_2, r_3)$ since $\gamma < 1$

Hence the system (4) is uniformly persistent(or permanent) if $\gamma < 1$. This completes the proof of the theorem.

6. NUMERICAL SIMULATIONS OF MODEL 1 AND MODEL 2

We here used ode23 solver for numerical simulations using MATLAB 2017a. Keeping in mind all the feasibility criteria the numerical values are chosen for different parameter values. The equilibrium point corresponding to the parameter values for model 1 is (0.475330834 , 0.159393085) and the equilibrium point corresponding to the parameter values for the model 2 is (0.390263193,0.103522461). The numerical results also supports the theoretical findings of the model 1 and model 2.

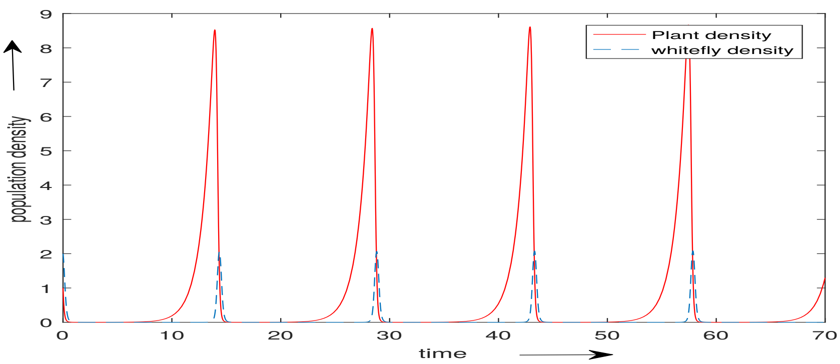

Figure 1: Variation of plant-herbivore densities with time in model 1 for $\alpha = 4.17$ and $\beta = 0.14$.

Here we observed large amplitude oscillations that indicates the unstable condition for both the populations.

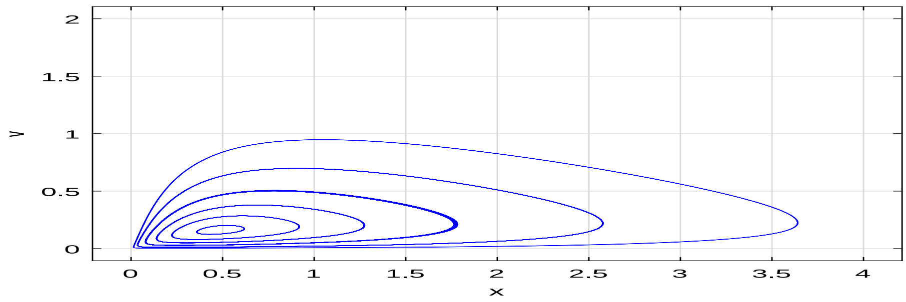

Figure 2: Variation of plant-herbivore densities in model 1 for $\alpha = 4.17$, $\beta = 0.14$.
This shows the phase portrait in the XV-plane which shows the fragile behavior.

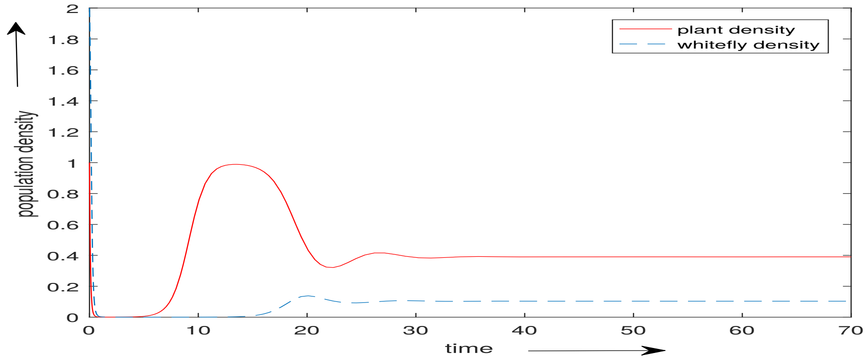

Figure 3: Globally asymptotically stable steady-state in model 2 for $\alpha = 1.25$, $\beta = 4.8$ and $\gamma = 0.125$.
This shows the stable behavior as time increases for both the populations.

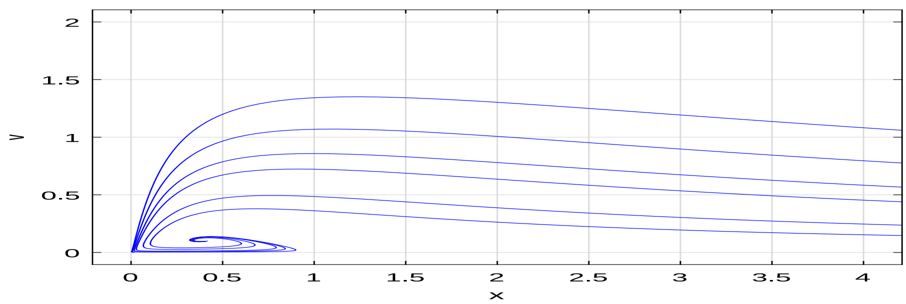

Figure 4: Globally asymptotically stable steady-state in model 2 for $\alpha = 1.25$, $\beta = 4.8$ and $\gamma = 0.125$.
This shows the phase portrait for global asymptotic behavior in the XV-plane.

7. CONCLUSIONS

These two models represents the description of interaction between Jatropha curcas plant and the whitefly. Here we made a comparative study between two different growth functions of the Jatropha curcas plant population with random attack pattern of whitefly using poisson distribution. From the study it is revealed that if the plant grows logistically then the effect of whitefly can not destabilize the system but if the plant growth is exponential, then it shows a fragile behavior. Therefore growth function of the plant (Jatropha curcas) plays an important role for the stability of plant-herbivore system. Our numerical results also reflects the same phenomena.

REFERENCES

[1] Gao , S.Q. , Qu J., Chua N.H., Ye J. (2010) 'A new strain of Indian cassava mosaic virus causes a mosaic disease in the biodiesel crop Jatropha curcas', Archives of virology ; 155, 607-612.

[2] Holt , J. Jeger M.J., Thresh J.M., Otim-Nape G.W. (1997) 'An epidemiological model incorporating vector population dynamics applied to African cassava mosaic virus disease', J Appl Ecol 34(3) : 793-806

[3] Konar, Debajyoti and Das, Kalyan and Sarkar, A.K. (1999). 'Global stability and persistence of a three species donar-controlled model with extensive omnivory', The Bulletin of the Hong Kong Mathematical Society. 2.

[4] Narayana , D.S.A., Shankarappa K.S., Govindappa M.R., Prameela H.A., Rao M.R.G., Rangaswamy K.T. (2006) 'Natural occurence of Jatropha mosaic virus disease in India', Current Science ; 91(5) , 584-586.

[5] Pandey , V.C., Singh K., Singh J.S., Kumar A., Singh B., Singh R.P. (2012) 'Jatropha curcas: A potential biofuel plant for sustainable environmental development', Renew sustain energy Rev 16, 2870-2883.

[6] Sahoo , N.K., Kumar A., Sharma S., Naik S.N. (2009) 'Interaction of Jatropha Curcas plantation with ecosystem', Proceedings of international conference on Energy and environment, chandigarh, India, 19-21.

[7] Sarkar , A.K. and Roy, A.B., (1989) 'Role of herbivore attack pattern in growth of plant populations', Ecol. Modelling, 45, 307-316 .

[8] Venturino , E., Roy, P.K., Al Basir, F. et al. (2016) 'A model for the control of the mosaic virus disease in Jatropha curcas plantations', Energ. Ecol. Environ. 1, 360.

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