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The Black Scholes Option Pricing Model for Insurance Derivative

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ABSTRACT

An insurance derivative is a financial instrument that derives its value from an underlying insurance index or the characteristics of an event related to insurance. Insurance derivatives are useful for insurance companies that want to hedge their exposure to catastrophic losses due to exceptional events, such as earthquakes or hurricanes. Unlike financial derivatives, which typically use marketable securities as their underlying assets, insurance derivatives base their value on a predetermined insurance-related statistic. For example, an insurance derivative could offer a cash payout to its owner if a specific index of hurricane losses reached a target level. This would protect an insurance company from catastrophic losses if an exceptional hurricane caused unforeseen amounts of damage. Our aim of this paper is to construct the Black Scholes option pricing model for Insurance Derivative. The main goal of this study is three fold: 1) First, we begin our approach to explain the stochastic process for the catastrophe derivative. 2) Then we explore, how describe the insurance contract in a financial option context. 3) Finally, we extent our approach to construct and derive mathematical for the insurance option pricing model from Black Scholes Equation. In addition, this paper ends with conclusion.

Keywords: Insurance Derivatives, Option Pricing, Insurance Option Contract (Call and Put), Fair Risk and Black Scholes Equation.

INTRODUCTION

Derivatives are important risk management tools that have made it possible for financial and non – financial institutions to buy and sell exposure, thereby diversity their risk portfolio and reducing earnings volatility. Today, derivatives are being extended beyond the mainstream interest rate, currency, commodity, equity, and credit markets to manage new underlying risks, such as catastrophe, pollution, electricity, weather, solar and inflation. Insurance derivatives are now beginning to be an alternative to traditional reinsurance as a way for insurance companies to manage the risks of a catastrophic event such as a hurricane or an earthquake.

Insurers can use derivatives to effectively manage their risks. A life insurer with a large portfolio of Guaranteed Minimum Death Benefit annuities can hedge against a steep decline in equity markets. Life insurers offering interest rate guarantee on their life savings products can use derivatives to hedge against low interest rate. Property and casualty insurer can transfer can transfer some of their catastrophe risk to the capital markets via swap transaction. So, insurer can use derivatives to manage their assets and liabilities and to enhance their capital adequacy. For example, they can use the derivatives to redress any assets – liability mismatches by adjusting the duration of their assets in line with that of their liabilities. Additionally, they can purchase the options to sell their equity to a counterparty at a pre – negotiated price should they face a liquidity crisis.

Insurance futures and options on insurance futures were the first type of contracts traded at the CBOT. Due to the very low trading activity, they were replaced in 1995 by catastrophe spread options based on underlying loss indexes which are provided by Property Claim Services (PCS), an independent statistical agency. We refer to D'Arcy and France (1992) for a detailed description of insurance futures and to O'Brien (1997) for catastrophe spread options.

1. STOCHASTIC MODEL FOR CATASTROPHE INSURANCE DERIVATIVES

The pricing of catastrophe derivatives is challenging as the market is incomplete. Natural catastrophes cause jumps in the underlying indexes of random size at random points in time. It is thus not possible to determine a unique price process for catastrophe derivatives purely based on the exclusion of arbitrage opportunities. The literature on pricing of catastrophe derivatives either assumes that jump sizes of the underlying loss index are constant or specifies investors' preferences.

In this section, we define the stochastic process for the underlying loss index, the specifications of the catastrophe derivative and reinsurance portfolio, and their price processes.

Uncertainty in the market is described by a probability space (Ω, F, P) on which random variables will be defined. We assume that the economy is of finite horizon $T < \infty$ and the flow of information is modelled by a non-decreasing family of σ - algebras $(F_t)_{0 \le t \le T}$, a filtration.

We assume that $F_T = F$, each F_t contains the events in F that are of P - measure zero, and the filtration is right-continuous, i.e. $F_t = F_{t+}$ where $F_{t+} = \bigcap_{s>t} F_s$.

The risk faced by the insurance and reinsurance industry is inherent in their exposure to accumulated insured property losses. As natural catastrophes cause claims of extreme magnitude, we follow the approach of Aase (1999) and assume that the process of accumulated insured property losses is a compound Poisson process $X = (X_t)_{0 \le t \le T}$.

The random variables X_t thus represents the sum of insured property losses to the industry in $[0, t]$, that is

$$X_t = \sum_{\{k \mid T_k \le t\}} Y_k = \sum_{k=1}^{N_t} Y_k \tag{1}$$

Where T_k is the random time point of occurrence of the k^{th} catastrophe that cause a corresponding insured property loss of size Y_k, and N_t is a random variable counting catastrophe up to and including time t.

We assume that the past evolution and current state of the risk process X is observable by every agent in the economy, i.e. X is assumed to be adapted to the filtration $(F_t)_{0 \le t \le T}$.

We thus exclude any effects that asymmetric information may have on the market through moral hazard or adverse selection problems. For simplicity, it is assumed that X generates the flow of information, i.e. $F_t = \sigma(\sigma(X_s, s \le t) \cup N)$ where N denotes the events of P - measure zero.

Changing the probability measure plays a central role in the context of no-arbitrage valuation of contracts as their discounted price processes are martingales under the appropriate probability measure.

For compound Poisson processes, Delbaen and Haezendonck (1989) characterized the set of probability measures Q on (Ω, F) that are equivalent to the "reference" measure P and that preserve the structure of the underlying risk process X, i.e. such that X is a compound Poisson process under the new probability measure Q.

Aase (1992) showed that this set can be parameterized by a pair $(k, v(\cdot))$ where k is a strictly positive constant k and $R_+ \to R$ is a measurable function that is strictly positive on the support of G with $E^P[v(Y_1)] = 1$.

The density process $\xi_t = E^P[\xi_T \mid F_t]$ of the Radon – Nikodym derivatives $\xi_T = \frac{dQ}{dP}$ is then given by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} In(kv(Y_k)) + (\lambda - k)t \right) \tag{2}$$

for any $0 \le t \le T$ under the assumption that $E^P\left[\exp\left(\sum_{k=1}^{N_t} In\left(kv\left(Y_k\right)\right)\right)\right] < 0$. Let $P^{k,v}$ denote the equivalent probability measure that corresponds to the constant k and the function $v(\cdot)$. Under the new measure $P^{k,v}$ the process X is then a compound Poisson process with characteristics $\left(\lambda^*, dG^*(y)\right)$ where $\lambda^* = \lambda k$ and $dG^*(y) = v(y)dG(y)$.

Where, k the market price of frequency risk and $v(\cdot)$ the market price of claim size risk.

Further, here we assume that three assets are traded in the capital market, a risk-free bond, a reinsurance portfolio, and a catastrophe derivative.

- The risk-free bond accumulates interest at a deterministic rate r, continuously compounded.
- The reinsurance portfolio specifies a premium process for the industry's overall insured losses $p = (p_t)_{0 \le t \le T}$ for industry´s overall insured losses $X = (X_t)_{0 \le t \le T}$.

  The premium p_t defines the price at time t for the remaining risk, $X_T - X_t$.
- The catastrophe derivative is a European-style derivative with maturity T which is written on the same underlying risk process X.

The payoff of the catastrophe derivative thus depends on the realization of X_T only and specifies a price process $\pi = (\pi_t)_{0 \le t \le T}$.

Let $\phi : R_+ \to R$ be a measurable function that specifies the payoff at maturity to the buyer of the catastrophe derivative, i.e. at T the buyer receives $\phi(X_T)$. The price π_t defines the price at time t the buyer has to pay to enter into the financial contract.

In the absence of arbitrage strategies the fundamental theorem of asset pricing implies that both discounted price processes are martingales under some but the same equivalent probability measure $P^{k,v}$.

The two price processes can therefore be represented as

$$p_t^{k,v} = e^{-r(T-t)} E^{P^{k,v}}\left[E_T - X_t \middle| F_t\right] \text{ and } \tag{3}$$

$$\pi_t^{k,v} = e^{-r(T-t)} E^{P^{k,v}}\left[\phi(X_t) \middle| F_t\right] \tag{4}$$

2. INSURANCE CONTRACT IN A FINANCIAL OPTION CONTEXT

An option is a contract giving the holder a right to buy or sell an underlying object at a predefined price during a predefined period of time. An option condition is that the underlying object has an uncertain stochastic future value. Financial stocks are the most common option object. However, the claims risk of an insurance customer may be interpreted as another option object. Hence we may define a non-life insurance contract as an option.

In financial option contact, we can define, the holder of a call option gives the right, but not the obligation, to buy a stock at a predetermined date (maturity time) and price (strike price). Alternatively, the holder of an insurance contract gives the right to get covered all incurred insurance claims within a predetermined date (maturity time) and at a predetermined price (the deductible or excess point).

Assume two different underlying risk processes $S(t)$ and $X(t)$, where $S(t)$ is the stock price process up to time t and $X(t)$ is the accumulated insurance claim process up to time t.

That is, assume $X(t)=\sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is the number of incurred claims up to time t and the Y_i - are the claims severities.

Let C be a European call option contract on the stock price process $S(t)$ and Z and Z^* two different insurance contracts on the claims risk process $X(t)$. Assume time $t=0$ as the start time of all contracts and time $t=T$ as the maturity time.

The payoff from a long position in a European call option is

$$C(T)=Max\left[(S(T)-K)^+,0\right] \qquad (5)$$

Where $C(T)$ is the call option payment at time T, $S(T)$ is the stock price on T, K is the strike price.

For insurance contract, we rewrite the payoff from a long position in European call option is

$$Z(T)=\sum_{i=1}^{N(T)} Max\left[(Y_i-D)^+,0\right] \qquad (6)$$

Where $Z(T)$ is the sum insurance payment value at time T, $N(T)$ is the number of claims amount up to time T, Y_i is incurred claim amount of claim number i up to time T and D deductible for each claim occurrence.

$$Z^*(T)=Max\left[\left(X(T)-D^*\right)^+,0\right] \qquad (7)$$

Where $Z^*(T)$ is the sum insurance payment value at time T , $X(T)$ is accumulated claim sizes at time T, and D^* is deductible or excess point of $X(T)$.

Hence Z is an excess-of-loss insurance contract and Z^* is a stop-loss insurance contract. We also observe that Z may be interpreted as a stochastic sum of N single European call options and Z^* as an ordinary European call option. Hence within this context we may name these insurance contracts as insurance call contracts, and the

pricing of them as insurance option pricing.
Based on dynamic hedging of the underlying stochastic portfolios and the Black-Scholes formula of financial option values is to create risk free synthetic portfolios by continuously (dynamic) purchasing (hedging) a certain share of the underlying asset and a certain amount of a risk less bond.

The hedging strategy has two key properties:

- It replicates the payoff of the option, and
- It has a fixed and known total cost.

Hence the value of the option at any time t may be explicitly expressed by the combination of the shares of the asset and the bond.

The most important assumption behind this dynamic hedging based pricing theory is the no-arbitrage assumption. That is, we assume no opportunities to make risk less profits through buying and selling financial security contracts. This assumption generates the existence of a put-call parity, that is, a fundamental relationship between the values of a call option and a put option (the right to sell a security to a strike price). Let P be an European put option with the same strike K and maturity T as the call option C:

The payoff to the holder of a long position in a European put call option is

$$P(T) = Min\left[(S(T) - K)^{-}, 0\right] \quad (5)$$

Where $P(T)$ is the put option value maturity time T, $S(T)$ is the stock price on T, K is the strike price.

The important relationship between call and put is known as put-call parity. It shows that the value of a European call with a certain strike price and exercise date can be deducted from the value of a European put option with the same strike price and exercise price date, vice versa.

$$C(t) - P(t) = S(t) - e^{-rT} K \quad (6)$$

Where r is the free rate of interest.

Turning to the insurance contracts, we may also construct put-call parities for the excess-of-loss and stop-loss contract. This depends however on the existence of so-called insurance put contracts. Hence let $\tilde{Z}(T)$ and $\tilde{Z}^{*}(T)$ respectively be the values at maturity time T of an excess-of-loss put contract and a stop-loss put contract.

We then have:

$$\tilde{Z}(T) = \sum_{i=1}^{N(T)} Max\left[D - Y_i, 0\right] \quad (7)$$

$$\tilde{Z}^*(T) = Max[D^* - X(T), 0] \quad (8)$$

The practical interpretation of the insurance put contract is that the holder of the contract gives the right to a payment of D or D^* against a self-financing of the claims $X(T)$ during the period $(0,T)$.

This is obviously a right the holder of the contract only will use if the incurred claims are less than D or D^*.

Hence the owner of a claims risk could by buying an insurance call contract and at the same time sell an insurance put contract on the same underlying risk gain a risk-free cash flow equal to the deductible D or D^*. The claims occurrence during the contract period will not influence this cash flow.

By use of simple algebra, we find straightforward the general put-call parity expressions of the insurance contracts:

$$Z(T) - \tilde{Z}(T) = X(T) - D(T), \qquad Where \quad D(T) = N(T)D \quad (9)$$

$$Z^*(T) - \tilde{Z}^*(T) = X(T) - D^*(T), \qquad Where \quad D^*(T) = D^* \quad (10)$$

That is, the value of the call minus the value of the put is equal to the value of the accumulated claim amounts minus the (present) value of $D(T)$ each contract. Hence the structure of the parities is identical to the financial option parity.

any deviation from the put - call parity does constitute an arbitrage opportunity in the insurance market because simultaneously buying and selling the insurance put-call portfolio (left hand side of (9) and (10)) and the claims-bond portfolio (right hand side of (9) and (10)) yield a risk less profit equal the difference between the values of the portfolios. It is however important that these relationships depend on the existence of an insurance put contract $\tilde{Z}$ or $\tilde{Z}^*$ and the existence of an efficient market place of buying and selling insurance calls and puts.

Now, discuss what is happened in dynamic hedging of insurance contracts. The existence of insurance put-call parities and no-arbitrage in an insurance market are important elements, but not sufficient to replicate a risk free payoff of an insurance risk by a hedging strategy.

It is well known that the ordinary insurance claims risk process $X(t)$ is difficult to directly hedge away by dynamic trading, and by this follow the classical financial pricing theory. Delbaen and Haezendonck (1989), however, assumed that an insurer at any time t can sell the remaining claim payments $X(T) - X(t)$ over the period (t,T) for some premium $p(t-T)$.

Hence the value of the portfolio of claims risks at time t is

$$I(t) = X(t) + p(T - t) \tag{11}$$

Where $I(0) = p(T)$ *and* $I(T) = X(T)$

This assumption of dynamic buying and selling insurance risks during the insurance period makes the crucial possibility for dynamic hedging also in insurance markets.

3. INSURANCE OPTION PRICING MODEL

It is often said that derivatives are used either for speculation or hedging. In particular, it seems that from the derivatives, options were originally designed as insurance contracts for the case of loss or damage in positions on other assets, indexes or securities but especially of futures. By buying contracts of put options we can insure long positions in futures while by buying contracts of call options we can insure short positions on futures.

Here, I introduce a new model for option fair pricing, which is based on principles of actuarial mathematics and insurance, and which includes the Black-Scholes as a special case. The main advantage of the present suggested model, is that the final option's fair price, does depend and includes, the trend of the underlying asset. The present option fair pricing does suggest nevertheless a better system of choices for the market maker or the investor, based on two parameters rather than one, the trend and the volatility of the underlying. When the trend of the underlying is put equal to the risk-free rate, then the present formulae for the option's fair price coincide with those of Black-Scholes model.

The present suggested option fair pricing, based on principles of insurance, is not included of course in any treatise or publication (as far as we know) of actuarial mathematics, as it is standard that in the applications of actuarial mathematics is not included the Financial Derivatives, and options, that is a topic of finance rather than insurance.

The Black-Scholes formula (also called Black-Scholes-Merton) was the first widely used model for option pricing. It's used to calculate the theoretical value of European-style options using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration and expected volatility.

Before we proceed we have to outline the model for the movements of underlying assets that is assumed in the Black-Scholes Model as well.

A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift (Karlin and Taylor 1975). It is an important example of stochastic processes satisfying a stochastic differential equation (SDE); in particular, it is used in mathematical finance to model stock prices in the Black–Scholes model.

The technical definition as follows.

A stochastic process S_t is said to spot price of an assets and it is assumed an (Ito) stochastic process (Oksendal, 1995) to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{12}$$

where μ and σ are constants representing the long term drift and the noisiness (diffusion) or coefficient of volatility respectively in the stock price and W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and correlation in time.

The drift parameter μ and the coefficient of volatility σ in discrete time and a sufficient fine grid or resolution of time step δt, can be estimated by the formulae:

$$\mu = \frac{1}{N\delta t}\sum_{i=1}^{N} R_i \tag{13}$$

Where $R_i = \dfrac{S_{i+1} - S_i}{S_i}$

$$\sigma = \sqrt{\frac{1}{(N-1)\delta t}\sum_{i=1}^{N}\left(R_i - \overline{R}\right)^2} \tag{14}$$

Where $\overline{R} = \mu\delta t$

Obviously if the time unit is the pixel δt, then μ and σ are the average and standard deviation of the rates of return per time step which is also the time unit.

As a result of the assumption to derive the Black-Scholes-Merton equation and it can be proved that an option price f has to satisfy the next Partial Differential equation.

$$\frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{15}$$

The solution of this equation with boundary final condition

$$f(S_T, T) = Max\left[(S_T - K)^+, 0\right] \tag{16}$$

when $t = T$, which is the payoff for a standard European call option at expiry, and at time

$t = 0$ is:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{17}$$

Where

$$d_1 = \frac{In\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{18}$$

$$d_2 = \frac{In\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{19}$$

And $N(x)$is the distribution of a normal random variables.

The options fair price P at a present time moment t is the discount with the risk-free rate ρ at this moment t, of the average paid value at expiration T, given that the value of the price random variable, S_T, of the underlying asset at expiration is calculated by an assumed model M and by the spot price S_t at the present time t.

Now revolving in to prove that the Black-Scholes option fair price formula is a special case of the above general insurance principle of option fair pricing, the following lemma is required (Hull 2000).

If V is lognormal distributed with average value m and the standard deviation of $\ln N$ is s, then

$$E\left[Max\left[(V-K)^+, 0\right]\right] = mN(d_1) - KN(d_2) \tag{20}$$

Where

$$d_1 = \frac{\ln\left(\frac{m}{K}\right) + \frac{s^2}{2}}{s} \qquad d_2 = \frac{\ln\left(\frac{m}{K}\right) - \frac{s^2}{2}}{s} \tag{21}$$

$E(A)$ denotes the average value of the random variable A and $N(x)$ is the distribution function of a normal variable x.

Using this result, we may derive the general form of the option fair price, and the exact formula in the case we assume (as is also assumed in the B-S model) that the underlying follows a geometric Brownian motion (lognormal distribution).

If we interpret the average value m as the average value of the price of one unit of the underlying (one item of the security, if the underlying is a stock) at expiration, and the standard deviation s as the standard deviation of the price of the underlying at expiration, then the average value of payoff of one contract of a call option at expiration of exercise price X (and assuming that one contract insures one item of the underlying) is

$$C = E\left[Max\left((V-K)^{+},0\right)\right] \tag{22}$$

The general principle of pricing requires of course to have this average payoff value discounted at present values, if the pricing is not at expiration.

So if the current time is t , and expiration is time T , while the risk-free (continuous time) rate by which we discount is ρ , then the present fair price of the option is,

$$C = e^{-\rho(T-t)}E\left[Max\left((V-K)^{+},0\right)\right] \tag{23}$$

This is the general formula of the option fair price, which is model-free in the sense that any model may be assumed for the changes of the prices of the underlying.

If we assume that the price of the underlying at expiration is also log normally distributed with average value m and standard deviation s, then we may apply the above lemma to transform the general formula into:

$$C = e^{-\rho(T-t)}\left[mN(d_1) - KN(d_2)\right] \tag{24}$$

$$d_1 = \frac{\ln\left(\frac{m}{K}\right) + \frac{s^2}{2}}{s} \qquad d_2 = \frac{\ln\left(\frac{m}{K}\right) - \frac{s^2}{2}}{s} \tag{25}$$

This is again a general formula of the option fair price, which is also model-free in the sense that, we may assume any model for the changes of the prices of the underlying, provided it is log normally distributed at expiration, with average value m and standard deviation s .

Let us now choose a particular model for the changes of the prices of the underlying, which is the same as the one assumed by the Black-Scholes model, namely a geometric Brownian motion of drift r and volatility s .

Then the average value of the price of the underlying at expiration T is

$$m = Se^{\left(r-\frac{1}{2}s^2\right)(T-t)} \tag{26}$$

where t is the present time and S is the present value of the underlying.

This yields the following formula for the fair option's premium:

$$C = Se^{-\rho(T-t)}\left(e^{\left(r-\frac{1}{2}s^2\right)(T-t)}N(d_1)-KN(d_2)\right) \qquad (27)$$

$$d_1 = \frac{\left(\frac{Se^{\left(r-\frac{1}{2}s^2\right)(T-t)}}{K}+\frac{s^2}{2}\right)}{s} \qquad d_2 = \frac{\left(\frac{Se^{\left(r-\frac{1}{2}s^2\right)(T-t)}}{K}-\frac{s^2}{2}\right)}{s} \qquad (28)$$

The reader should notice that both the drift r and the risk-free rate ρ enter the formula; in general, these two figures are different.

If we now assume that the drift of the underlying r is equal to the risk-free rate ρ, then the above formula reduces to the familiar formula of option fair price (for call options) of the Black-Scholes Model (Wilmot 1999):

$$C = SN(d_1)-Ke^{-\rho(T-t)}N(d_2) \qquad (29)$$

$$d_1 = \frac{In\left(\frac{S_0}{K}\right)+\left(\rho+\frac{s^2}{2}\right)}{s\sqrt{T-t}} \qquad (30)$$

$$d_2 = \frac{In\left(\frac{S_0}{K}\right)+\left(\rho-\frac{s^2}{2}\right)}{\sigma\sqrt{T-t}} \qquad (31)$$

The insurance model of option fair pricing has as special case the Black-Scholes model is proved in equation (29), when in the insurance model we assume that the drift of the underlying is equal to the risk-free rate.

CONCLUSION

In recent years there has been an ongoing economic and political debate on whether financial markets should be used to insure risk that has been traditionally hedged through other channels. Famous examples include the discussion about the change to a funded pension scheme, equity-linked life insurance contracts, and insurance derivatives. This need for an alternative way of insurance resulted in a growing number of insurance products coming onto the market and containing a financial component of some sort. In order to tailor these new financial products optimally to the needs of the different markets, both finance experts as well as actuaries will have to get to know the other expert's field better. This overlap suggests that combining the methods used in both areas, insurance mathematics and mathematical finance should prove indispensable.

The above approach gives an alternative formulation and derivation of the Black - Scholes option fair pricing formula that completely avoids the ironic, ambiguous and controversial Black-Scholes assumptions about risk less arbitrage opportunities, continuous delta hedging trading, zero transaction costs and infinite continuous divisibility of invested size. To derive exactly the Black-Scholes option fair pricing formula we only require the assumption that in the average the underlying has a drift, equal to the risk-free rate. To account for other drifts, and with the same natural assumptions we should resort to the insurance option fair pricing.

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