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# Weiner Index of a Vertex with Respect to a CDPU Set

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**Abstract**

A graph $G = (V, E)$ is complementary distance pattern uniform (CDPU), if there exists $M \subset V(G)$ such that $f_M(u) = \{d(u, v) : v \in M\}$, for every $u \in V(G) - M$, is independent of the choice of $u \in V(G) - M$ and the set $M$ is called the CDPU set. In this paper, we define the Weiner index of a vertex with respect to a CDPU set.

**Keywords:** Complementary distance pattern uniform set, Weiner index

**AMS Subject Classification:** 05C22

## 1. INTRODUCTION

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to Harary [5]. Unless mentioned otherwise, all the graphs considered in this note are simple, self-loop-free and finite.

B.D.Acharya [6] define the $M-$ distance pattern of a vertex as follows :

**Definition 1.1.** [6] Let $G = (V, E)$ be a $(p, q)$ graph and $M$ be any non-empty subset of $V(G)$. Then, the $M-$distance pattern of $u$ is the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between $u$ and $v$ in $G$. If for a subset $M$ of vertices in a graph $G = (V, E)$, $f_M$ is injective, then the set $M$ is called the distance pattern distinguishing set (DPD-set in short).

Germina and Beena [3] defined *Complementary Distance Pattern Uniform (CDPU) Graph* as follows:

**Definition 1.2.** [3] If $f_M(u)$ is independent of the choice of $u \in V - M$, then $G$ is called a Complementary Distance Pattern Uniform (CDPU) Graph and the set $M$ is called the CDPU set. The least cardinality of CDPU set in $G$ is called the *CDPU number* of $G$, denoted $\sigma(G)$.

**Theorem 1.3.** *[3] Every connected graph has a CDPU set.*

**Theorem 1.4.** *[3] Every self-centered graph of order $p$ has a CDPU set $M$ with $|M| \leq p - 2$.*

**Theorem 1.5.** *[3] A graph $G$ has $\sigma(G) = 1$ if and only if $G$ has atleast one vertex of full degree.*

**Theorem 1.6.** *[3] Let $G$ be a non self-centered graph having no full degree vertex. Then, $\sigma(G) = 2$ if and only if the vertices of $G$ have exactly two different eccentricities such that, the number of vertices corresponding to atleast one of the eccentricities should be two.*

**Theorem 1.7.** *[3] For any integer $n \geq 4$, $\sigma(P_n) = n - 2$.*

**Corollary 1.8.** *[3] $\sigma(K_{a_1,a_2}) = 2$.*

**Theorem 1.9.** *[3]* $\sigma(C_n) = \begin{cases} n-2, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \geq 8 \text{ is even} \end{cases}$

## 2. WEINER INDEX OF A VERTEX WITH RESPECT TO A CDPU SET

In chemical graph theory, the Wiener index (also Wiener number) is a topological index of a molecule, defined as the sum of the numbers of edges in the shortest paths in a chemical graph, between all pairs of non-hydrogen atoms in a molecule. It was introduced by H. Wiener in 1947 [9]. Wiener index is the oldest topological index related to molecular branching [7]. A tentative explanation of the relevance of the Wiener index in research of QSPR and QSAR is that it correlates with the van der Waals surface area of the molecule [4]. Natarajan and Basak [2] define *Weiner index* of a vertex as follows: Weiner index of a vertex $u$, denoted by $W(u, G)$, is the sum of the distances of the vertex $u$ to each of the vertices of $G$.

$$W(u, G) = \sum_{v \in V(G)} d(u, v).$$

In this section, we define the Weiner index of graphs with respect to the CDPU set $M$ and establish some of the properties of this index.

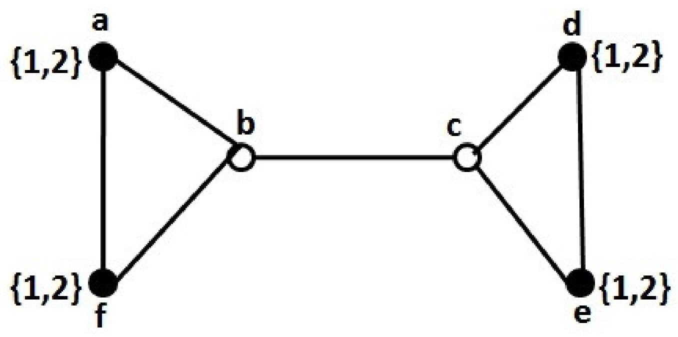


Figure 1:

**Definition 2.1.** Let $G = (V, E)$ be a graph with CDPU set $M \subset V(G)$. Let $u \in V - M$. Then the Weiner index of u with respect to $M$, denoted by $W_M^u(G)$, is the sum of the distances from $u$ to each of the vertices of $M$.

$W_M^u(G) = \sum_{v \in M} d(u, v)$

For any $u \in V - M$, $W(u, G) > W_M^u(G)$. Also, $\sum_{f_M(u)} \leq W_M^u(G)$, where $\sum_{f_M(u)}$ denotes the sum of the elements in the set $f_M(u)$.

In Figure 1, $M = \{c, d\}$ which gives $W_M^a(G) = 3, W_M^b(G) = 3, W_M^e(G) = 3$ $and$ $W_M^f(G) = 3$. Since every vertices in $V - M$ have the same distance pattern, there arise a question whether $W_M^u(G), u \in V - M$ is same for every graph?

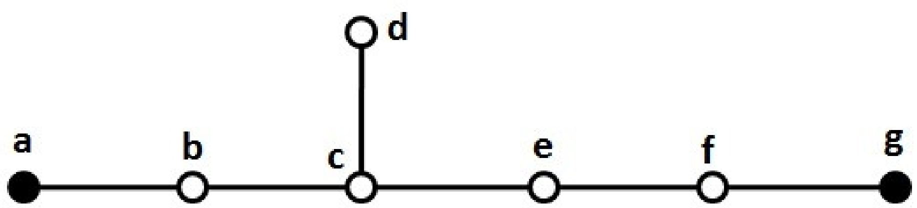


Figure 2:

In Figure 2, $W_M^a(G) = 1 + 2 + 3 + 3 + 4 = 13$ and $W_M^g(G) = 1 + 2 + 3 + 4 + 4 = 14$. Hence, $W_M^u(G)$ is not same for every graph.

For a path $G \cong P_n$, $W_M^u(G) = 1 + 2 + 3 + \cdots + (n - 2) = \frac{(n-2)(n-1)}{2}$ for every

$u \in V - M$ and for a star $K_{1,n}$ and complete graph $K_n$, Weiner index of every vertex with respect to $M$ is one.

For a complete bipartite graph $G \cong K_{m,n}$ with partitions $U$ and $V$, $M = \{u, v\}$, where $u \in U$ and $v \in V$. Thus, $W_M^u(G) = 1 + 2 = 3$, for every $u \in U$ and $W_M^v(G) = 3$.

For an even cycle $G \cong C_{2n},\ n \geq 2$ and $n \neq 3$ with $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ and $M = \{v_2, v_4, \ldots, v_{2n}\}$. When $n$ is even, $W_M^{v_i}(G) = 1 + 1 + 3 + 3 + 5 + 5 + \cdots + (n-1) + (n-1)$, for every $i = 1, 3, \ldots, 2n-1$. When $n$ is odd, $W_M^{v_i}(G) = 1 + 1 + 3 + 3 + 5 + 5 + \cdots + n$, for every $i = 1, 3, \ldots, 2n-1$. When $G \cong C_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$ and $M = \{v_1, v_4\}$, $W_M^{v_i}(G) = 1 + 2 = 3$, for every $i = 2, 3, 5, 6$.

For an odd cycle $G \cong C_n,\ n \geq 5$ with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $M = \{v_3, v_4, \ldots, v_n\}$, $W_M^{v_i}(G) = 1 + 3 + 3 + 5 + 5 + \cdots + \frac{n-1}{2} + \frac{n-1}{2}$, for $i = 1, 2$.

Now, we establish a characterization for $W_M^u(G) = 1$, for every $u \in V - M$.

**Theorem 2.2.** *Let $G$ be a graph with CDPU set $M$ and $u \in V - M$. Then $W_M^u(G) = 1$, for every $u \in V - M$, if and only if $G$ has atleast one full degree vertex.*

*Proof.* Suppose that $G$ has a full degree vertex $v$. Then, $M = \{v\}$ and $W_M^u(G) = 1$, for every $u \in V - M$.

Conversely, assume that $W_M^u(G) = 1$ for every $u \in V - M$. Then, $f_M(u) = \{1\}$ for every $u \in V - M$ which implies $v$ is a full degree vertex. □

Theorem 2.3 is a necessary condition for a graph to have $W_M^u(G) = 2$, connecting with degree of each vertex.

**Theorem 2.3.** *Let $G$ be a graph with $n$ vertices and CDPU set $M$. Let $u \in V - M$. If $W_M^u(G) = 2$, for every $u \in V - M$, then there exist exactly two vertices $v_1$ and $v_2 \in M$ such that for $i = 1, 2;\ d(v_i) = n - 1$ or $n - 2$, where $d(v_i)$ represents the degree of the vertex $v_i$. Also, there exist $n - 2$ vertices in $V - M$ such that $2 \leq d(u) \leq n - 1$ for every $u \in V - M$.*

*Proof.* Suppose that $W_M^u(G) = 2$, for every $u \in V - M$. Then, either $f_M(u) = \{1, 1\}$ *or* $f_M(u) = \{2\}$.

$f_M(u) = \{2\}$ is not possible, since $M$ has only one vertex and the distance from all other vertices to $M$ is two, which is not possible for any graph.

If $f_M(u) = \{1,1\}$ then there exists two vertices $x, y \in M$. Hence, every $u \in V - M$ is adjacent to the two vertices $x$ and $y$. Hence, $d(u) = 2$, for every $u \in V - M$. Also, $d(u) \geq 2$, since attaching any edge to the vertex $u$ does not change the distance pattern, $f_M(u) = \{1,1\}$.

Since every $u \in V - M$ is adjacent to the two vertices $x$ *and* $y \in M$, $d(x) = d(y) = n - 2$. If there is an edge connecting $x$ and $y$, then also $W_M^u(G)$ remains the same. Hence, $d(x) = d(y) = n - 1$ *or* $n - 2$. The proof of the second part follows. □

**Theorem 2.4.** *Let $G$ be a non self-centered graph with $n$ vertices and CDPU set $M$. Let $u \in V - M$. Then, $W_M^u(G) = 3$ if and only if there exist exactly two eccentricities $e_i < e_j$ for the vertices of $G$, with $e_i$ corresponds to two vertices.*

*Proof.* Let the vertices corresponds to $e_i$ be $u$ *and* $v$ and the vertices corresponds to $e_j$ be $v_1, v_2, \ldots, v_{n-2}$. Choose $M = \{u, v\}$ which implies $f_M(v_i) = \{1,2\}$, for every $v_i \in V - M$. Hence, $W_M^u(G) = 3$.

Conversely, assume that $W_M^u(G) = 3$. Then, either $f_M(u) = \{3\}, f_M(u) = \{1,2\}$ *or* $f_M(u) = \{1,1,1\}$.

**Case 1:** $f_M(u) = \{3\}$ is not possible, since $M$ has only one vertex and the distance from all other vertices to $M$ is three, which is not possible for any graph.

**Case 2:** $f_M(u) = \{1,1,1\}$ is not possible, since $M$ has three vertices and the distance from all other vertices to $M$ is one, which results a self-centered graph, which is a contradiction to the assumption that $G$ is not self centered.

**Case 3:** If $f_M(u) = \{1,2\}$, then there exists two vertices in $M$, say, $u$ *and* $v$ such that $d(v_i, u) = 1$ and $d(v_i, v) = 2$, for $v_i \in V - M$. Also, $d(v_j, v) = 1$ and $d(v_j, u) = 2$, for $v_j \in V - M$. Thus, the eccentricity of $v_i$ and $v_j$ should necessarily be three and eccentricity of $u$ *and* $v$ should be two. □

**Remark 2.5.** Let $G$ be a self-centered graph with CDPU set $M$. Then, $W_M^u(G) = 1$ if and only if $G \cong K_n$.

Let $G$ be a self-centered graph with CDPU set $M$. If $W_M^u(G) = 2$, then either $f_M(u) = \{1,1\}$ or $f_M(u) = \{2\}$. $f_M(u) = \{2\}$ is not possible for any graph. If $f_M(u) = \{1,1\}$, $M$ should contain two vertices and from Theorem 2.3, $G \cong C_4$ *or* $K_4$.

Hence, we have

**Theorem 2.6.** *Let $G$ be a self-centered graph with CDPU set $M$. Then, $W_M^u(G) = 2$ if and only if $G \cong K_4$ or $C_4$.*

**Remark 2.7.** In a non self-centered graph $G$, if both $M$ and $M^c$ are CDPU sets, then $W_M^u(G) = 3$.

**Remark 2.8.** Let $G$ be a non self-centered graph with CDPU set $M$. Then, $W_M^u(G) = 1$ if and only if $G$ has atleast one full degree vertex. Also, $W_M^u(G) = 3$ if and only if the vertices of $G$ have exactly two different eccentricities. However, a characterization for $W_M^u(G) = 2$ is yet to be established.

**Problem 1.** Let $G$ be any graph of order $n$ with CDPU set $M$. Then, characterization for $W_M^u(G) = i$, for $i = 2, \ldots, n$ is an interesting problem for further investigation.

## 3. CONCLUSION

Weiner Index of a vertex with respect to a CDPU set is defined in this paper.$W_M^u(G)$ is not same for every graph. This paper gives the Weiner Index of a path, complete bipartite graph and cycles of odd and even length with respect to a CDPU set. Also establish a characterisation for a graph having $W_M^u(G) = 1$, for every $u \in V - M$, necessary condition for a graph having $W_M^u(G) = 2$ and characterisation for a non self-centered graph having $W_M^u(G) = 3$.

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