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Existence of a Weak Solution to the Maxwell-Stokes Type Equation Associated with a Slip-Navier Boundary Condition

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Abstract

In this paper, we show the existence of a weak solution to the Maxwell-Stokes type equation associated with a slip-Navier boundary condition by the penalty method introduced by Témam. Our approximate equation is nonlinear and contains so called p-curlcurl system. Furthermore, we obtain the continuous dependence of the weak solution on the data.

Keywords: Maxwell-Stokes type equation, weak solution, penalty method, minimization problem.

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1. INTRODUCTION

In this paper, we show the existence of a weak solution to the Maxwell-Stokes type equation by the penalty method introduced by Temam [11] (cf. Dautray and Lions [8, vol. 7] or Girault and Raviart [10]).

More precisely, we consider the following Stokes problem in a bounded, Lipschitz-continuous domain $\Omega \subset \mathbb{R}^d$ with the Dirichlet boundary condition on a boundary Γ.

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}$ denotes a velocity, π a pressure, and $\boldsymbol{f}$ is the external force.

The penalty method replaces the Stokes problem (1.1) by

$$\begin{cases} -\Delta \boldsymbol{u}_\varepsilon - \frac{1}{\varepsilon}\nabla \mathrm{div}\, \boldsymbol{u}_\varepsilon = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where ε is a positive parameter which will tend to zero. The pressure π is approximated by $\pi_\varepsilon = -\frac{1}{\varepsilon}\mathrm{div}\, \boldsymbol{u}_\varepsilon$, and $\boldsymbol{u}_\varepsilon$ approximates $\boldsymbol{u}$. Since the problem (1.2) is the Dirichlet problem for the elliptic equation, (1.2) has a unique solution $\boldsymbol{u}_\varepsilon$. Temam established the convergence of $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon)$ to a solution $(\boldsymbol{u}, \pi)$ of (1.1), and more precisely, he proved the following theorem.

Theorem 1.1 (Temam)**.** *Let Ω be a bounded, Lipschitz-continuous domain in $\mathbb{R}^d$, and $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$. Then, as $\varepsilon \to 0$, we have*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \textit{ in } \boldsymbol{H}_0^1(\Omega) \textit{ and } \pi_\varepsilon \to \pi \textit{ in } L^2(\Omega),$$

where $(\boldsymbol{u}, \pi)$ is the solution of the homogeneous Stokes problem (1.1).

Amrouche and Girault [1] established the regularity of a weak solution of (1.1) belongs to $\boldsymbol{W}^{m,p}(\Omega) \times W^{m-2,p}(\Omega)$ if $\boldsymbol{f} \in \boldsymbol{W}^{m-2,p}(\Omega)$ $(m \geq 2)$ to the Stokes problem with the Dirichlet boundary condition. However, in the physical applications, we are often encounter situation where this Dirichlet condition does not quite feasible.

In the present paper, we shall show that the penalty method can be applicable to the Maxwell-Stokes problem in the case $d = 3$. We impose the boundary condition associated with the slip-Navier boundary condition (cf. Amrouche and Rejaiba [2]). In order to do so, we replace $-\Delta \boldsymbol{u}$ in the first equation of (1.1) with a nonlinear term as in the following system.

$$\begin{cases} \mathrm{curl}\,[S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u}] + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma \end{cases} \tag{1.3}$$

where $\boldsymbol{n}$ is the outer unit normal vector to Γ and the functions $S(x, t)$ and $F(x, \boldsymbol{z})$ are Carathéodory functions in $\Omega \times [0, \infty)$ and $\Omega \times \mathbb{R}^3$, respectively, and $\nabla_{\boldsymbol{z}} F(x, \boldsymbol{z})$ denotes the gradient of $F(x, \boldsymbol{z})$ in $\boldsymbol{z}$-variable. They satisfy some structural conditions. The first equation of (1.3) contains so called p-curlcurl system

$$\mathrm{curl}\,[|\mathrm{curl}\, \boldsymbol{u}|^{p-2}\mathrm{curl}\, \boldsymbol{u}] + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \tag{1.4}$$

as a special case. The equation (1.4) is nonlinear, and when $p > 2$, it is degenerate and when $1 < p < 2$, it is singular. In a special case of $S(x, t)$ with $p = 2$ and $F(x, \boldsymbol{z}) = 0$, (1.4) satisfying divergence free condition reduces to the first equation of (1.1). Since the system (1.3) is not quasi-linear elliptic, it is necessary to investigate this problem from a different point of view. Our approximate system by the penalty method is as follows.

$$\begin{cases} \operatorname{curl}\left[S_t(x, |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2)\operatorname{curl} \boldsymbol{u}_\varepsilon\right] + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_\varepsilon) & \\ \quad -\frac{1}{\varepsilon}\nabla[S_t(x, (\operatorname{div} \boldsymbol{u}_\varepsilon)^2)\operatorname{div} \boldsymbol{u}_\varepsilon] = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x, |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2)\operatorname{curl} \boldsymbol{u}_\varepsilon \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma. \end{cases} \tag{1.5}$$

We show that (1.5) has a unique weak solution $\boldsymbol{u}_\varepsilon$ by solving a variational problem, and then we obtain that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } \mathbb{X} \text{ and } \pi_\varepsilon := -\frac{1}{\varepsilon} S_t(x, (\operatorname{div} \boldsymbol{u}_\varepsilon)^2)\operatorname{div} \boldsymbol{u}_\varepsilon \to \pi \text{ in } L^{p'}(\Omega),$$

where $(\boldsymbol{u}, \pi)$ is a weak solution of (1.3). Here the space $\mathbb{X}$ is defined in section 2.

The paper is organized as follows. In section 2, we give some preliminaries and the main theorem that states the existence of the weak solution to the problem (1.3). Section 3 is devoted the proof of the main theorem (Theorem 2.8). In section 4, we show the continuous dependence of the solution on the data.

2. PRELIMINARIES AND THE MAIN THEOREM

This section consists of two subsections. In subsection 2.1, we give some preliminaries that are necessary later. In subsection 2.2, we give the notion of a weak solution for the Maxwell-Stokes problem and state the main theorem.

2.1. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ and let $1 < p < \infty$. We denote the conjugate exponent of p by p', i.e., $(1/p) + (1/p') = 1$. From now on we use $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ for the standard L^p and Sobolev spaces of functions. For any Banach space B, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we write $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$ for the duality bracket.

We assume that a Carathéodory function $S(x,t)$ in $\Omega \times [0, \infty)$ satisfies the following structural conditions. For a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C^0([0,\infty))$, and there exist $1 < p < \infty$ and positive constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x \in \Omega$,

$$S(x,0) = 0 \text{ and } \quad \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, S_{tt}(x,t) \ge 0 \text{ for } t > 0, \tag{2.1c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.1a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0. \tag{2.2}$$

Example 2.1. *If $S(x,t) = \nu(x)g(t)t^{p/2}$, where ν is a measurable function in Ω and satisfies $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. $x \in \Omega$ for some constants ν_* and ν^* and $g \in C^\infty([0,\infty))$,*

When $g(t) \equiv 1$, it follows from elementary calculations that (2.1a)-(2.1c) *hold.*

As an another example, we can take

$$g(t) = \begin{cases} c(e^{-1/t}+1) & \text{if } t > 0, \\ c & \text{if } t = 0 \end{cases}$$

with a constant $c > 0$. Then $S(x,t) = \nu(x)g(t)t^{p/2}$ satisfies (2.1a)-(2.1c) *(cf. Aramaki [7, Example 3.2]).*

We give a monotonic property of S_t.

Lemma 2.2. *There exists a constant $c > 0$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}) \cdot (\boldsymbol{a}-\boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p \ge 2, \\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

For the proof, see Aramaki [6, Lemma 3.6].

Lemma 2.3. *There exists a constants $C_1 > 0$ depending only on Λ and p such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$|S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C_1|\boldsymbol{a}-\boldsymbol{b}|^{p-1} & \text{if } 1 < p < 2, \\ C_1(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}| & \text{if } p \ge 2. \end{cases}$$

For the proof, see Aramaki [4].

Next, let $F(x,\boldsymbol{z})$ be a Carathédory function in $\Omega \times \mathbb{R}^3$ satisfying that for a.e. $x \in \Omega$, $F(x,\boldsymbol{z}) \in C^1(\mathbb{R}^3)$ in $\boldsymbol{z}$ and the following structural conditions. There exist positive constants c_1, c_2, c_3, b_1 and b_2 such that

$$c_1|\boldsymbol{z}|^p - c_2 \le F(x,\boldsymbol{z}) \le b_1(|\boldsymbol{z}|^p+1), \tag{2.3a}$$

$$\nabla_{\boldsymbol{z}} F(x,\boldsymbol{0}) = \boldsymbol{0} \text{ and} \tag{2.3b}$$

$$|\nabla_{\boldsymbol{z}} F(x,\boldsymbol{z}_1) - \nabla_{\boldsymbol{z}} F(x,\boldsymbol{z}_2)| \le \begin{cases} b_2|\boldsymbol{z}_1-\boldsymbol{z}_2|^{p-1} & \text{if } 1 < p < 2, \\ b_2(|\boldsymbol{z}_1|+|\boldsymbol{z}_2|)^{p-2}|\boldsymbol{z}_1-\boldsymbol{z}_2| & \text{if } p \ge 2, \end{cases}$$

$$\begin{aligned} &\big(\nabla_{\boldsymbol{z}} F(x,\boldsymbol{z}_1) - \nabla_{\boldsymbol{z}} F(x,\boldsymbol{z}_2)\big) \cdot (\boldsymbol{z}_1-\boldsymbol{z}_2) \\ &\quad \ge \begin{cases} c_3|\boldsymbol{z}_1-\boldsymbol{z}_2|^p & \text{if } p \ge 2, \\ c_3(|\boldsymbol{z}_1|+|\boldsymbol{z}_2|)^{p-2}|\boldsymbol{z}_1-\boldsymbol{z}_2|^2 & \text{if } 1 < p < 2, \end{cases} \end{aligned} \tag{2.3c}$$

Lemma 2.4. *If* $F(x, \boldsymbol{z})$ *satisfies* (2.3a)-(2.3c), *then* $F(x, \boldsymbol{z})$ *is a strictly convex function.*

Proof. Let $\boldsymbol{z}_1 \neq \boldsymbol{z}_2$ and $0 < \lambda < 1$. It suffices to prove

$$I = F(x, \boldsymbol{z}_1) - F(x, \boldsymbol{z}_2) - \frac{F(x, \boldsymbol{z}_2 + \lambda(\boldsymbol{z}_1 - \boldsymbol{z}_2)) - F(x, \boldsymbol{z}_2)}{\lambda} > 0.$$

Since

$$\begin{aligned} F(x, \boldsymbol{z}_1) - F(x, \boldsymbol{z}_2) &= \int_0^1 \frac{d}{ds} F(x, \boldsymbol{z}_2 + s(\boldsymbol{z}_1 - \boldsymbol{z}_2)) ds \\ &= \int_0^1 \nabla_{\boldsymbol{z}} F(x, \boldsymbol{z}_2 + s(\boldsymbol{z}_1 - \boldsymbol{z}_2)) \cdot (\boldsymbol{z}_1 - \boldsymbol{z}_2) ds, \end{aligned}$$

$$\begin{aligned} &\frac{F(x, \boldsymbol{z}_2 + \lambda(\boldsymbol{z}_1 - \boldsymbol{z}_2)) - F(x, \boldsymbol{z}_2)}{\lambda} \\ &= \frac{1}{\lambda} \int_0^1 \frac{d}{ds} F(x, \boldsymbol{z}_2 + s\lambda(\boldsymbol{z}_1 - \boldsymbol{z}_2)) ds = \int_0^1 \nabla_{\boldsymbol{z}} F(x, \boldsymbol{z}_2 + s\lambda(\boldsymbol{z}_1 - \boldsymbol{z}_2)) \cdot (\boldsymbol{z}_1 - \boldsymbol{z}_2) ds, \end{aligned}$$

and

$$\boldsymbol{z}_2 + s(\boldsymbol{z}_1 - \boldsymbol{z}_2) - (\boldsymbol{z}_2 + s\lambda(\boldsymbol{z}_1 - \boldsymbol{z}_2) = (1 - \lambda)s(\boldsymbol{z}_1 - \boldsymbol{z}_2),$$

using (2.3c), when $p \geq 2$, we have

$$\begin{aligned} I &\geq \frac{1}{1 - \lambda} \int_0^1 c_3 \big((1 - \lambda)s|\boldsymbol{z}_1 - \boldsymbol{z}_2|\big)^p \frac{1}{s} ds \\ &\geq c_3 (1 - \lambda)^{p-1} |\boldsymbol{z}_1 - \boldsymbol{z}_2|^p \int_0^1 s^{p-1} ds \\ &= \frac{c_3}{p} (1 - \lambda)^{p-1} |\boldsymbol{z}_1 - \boldsymbol{z}_2| > 0, \end{aligned}$$

and when $1 < p < 2$,

$$\begin{aligned} I &\geq \frac{1}{1 - \lambda} \int_0^1 c_3 (|\boldsymbol{z}_2 + s(\boldsymbol{z}_1 - \boldsymbol{z}_2)| + |\boldsymbol{z}_2 + \lambda s(\boldsymbol{z}_1 - \boldsymbol{z}_2)|)^{p-2} \big((1 - \lambda)s|\boldsymbol{z}_1 - \boldsymbol{z}_2|\big)^2 \frac{1}{s} ds \\ &\geq c_3 (1 - \lambda) \big(2(|\boldsymbol{z}_2| + |\boldsymbol{z}_1 - \boldsymbol{z}_2|)\big)^{p-2} |\boldsymbol{z}_1 - \boldsymbol{z}_2|^2 \int_0^1 s ds \\ &\geq \frac{c_3}{2} (1 - \lambda) 2^{p-2} (|\boldsymbol{z}_1| + |\boldsymbol{z}_1 - \boldsymbol{z}_2|))^{p-2} |\boldsymbol{z}_1 - \boldsymbol{z}_2|^2 > 0. \end{aligned}$$

□

Example 2.5. *If* $F(x, \boldsymbol{z}) = c(x)|\boldsymbol{z}|^p$, *where* $c(x)$ *is a measurable function such that* $0 < c_* \leq c(x) \leq c^* < \infty$ *a.e.* $x \in \Omega$ *for some constants* c_* *and* c^*, *then* $F(x, \boldsymbol{z})$ *satisfies* (2.3a)-(2.3c).

The following inequality is used frequently (cf. Amrouche and Seloula [3]). If Ω is a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ, and if $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfies $\mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \mathrm{div}\, \boldsymbol{u} \in L^p(\Omega)$ and $\boldsymbol{u} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\mathrm{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{u}\|_{L^p(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)}). \quad (2.4)$$

Define a Banach space

$$\mathbb{X} = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \mathrm{div}\, \boldsymbol{v} \in L^p(\Omega), \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{X}} = (\|\mathrm{curl}\, \boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^p_{L^p(\Omega)} + \|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)})^{1/p}.$$

We note that from (2.4), $\|\boldsymbol{v}\|_{\mathbb{X}}$ is equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ for $\boldsymbol{v} \in \mathbb{X}$. Since $\mathbb{X}$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, we can see that $\mathbb{X}$ is a reflexive Banach space. Furthermore, we define a closed subspace $\mathbb{X}_0$ of $\mathbb{X}$ by

$$\mathbb{X}_0 = \{\boldsymbol{v} \in \mathbb{X}; \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

Furthermore, we define a Banach space

$$\mathbb{H} = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\, \boldsymbol{v} \in L^p(\Omega), \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{H}} = (\|\mathrm{div}\, \boldsymbol{v}\|^p_{L^p(\Omega)} + \|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)})^{1/p}.$$

Since $\mathbb{X} \subset \mathbb{H}$ and the inclusion mapping is linear and continuous, we can regard $\mathbb{H}' \subset \mathbb{X}'$ and the inclusion mapping is linear and continuous.

2.2. The main theorem

We rewrite the Maxwell-Stokes type system.

$$\begin{cases} \mathrm{curl}\, [S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u}] + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \end{cases} \quad (2.5)$$

where $\boldsymbol{f} \in \mathbb{H}'$ and $\boldsymbol{g} \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$ are given functions, where

$$\boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma) = \{\boldsymbol{v} \in W^{-1/p',p'}(\Gamma); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

We give the notion of a weak solution of the system (2.5).

Definition 2.6. *We say* $(\boldsymbol{u}, \pi) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ *is a weak solution of* (2.5)*, if* $(\boldsymbol{u}, \pi)$ *satisfies*

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} dx + \int_\Omega \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}) \cdot \boldsymbol{v} dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v} dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle + \langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma \quad (2.6)$$

for all $\boldsymbol{v} \in \mathbb{X}$*, where* $\langle \boldsymbol{f}, \boldsymbol{v}\rangle$ *denotes the duality bracket* $\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbb{H}',\mathbb{H}}$ *and* $\langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma$ *denotes the duality bracket* $\langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p',p'}(\Gamma),\boldsymbol{W}^{1-1/p,p}(\Gamma)}$.

Remark 2.7. *If* $(\boldsymbol{u}, \pi) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ *satisfies* (2.6)*, then* (2.5) *holds.*

Proof. If $(\boldsymbol{u}, \pi) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ satisfies (2.6), then taking $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)$ as a test function of (2.6), the first equation of (2.5) holds in the distribution sense. Moreover, since $\nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}), \nabla\pi, \boldsymbol{f} \in \mathbb{H}'$, from the first equation of (2.5) we can see that

$$\operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u}] \in \mathbb{H}'.$$

If we define a space

$$\boldsymbol{M}^{p'}(\Omega) = \{\boldsymbol{S} \in \boldsymbol{L}^{p'}(\Omega); \operatorname{curl} \boldsymbol{S} \in \mathbb{H}'\}$$

then we can see that $\boldsymbol{C}^\infty(\overline{\Omega})$ is dense in $\boldsymbol{M}^{p'}(\Omega)$. The mapping $\boldsymbol{S} \mapsto \boldsymbol{S}\big|_\Gamma \times \boldsymbol{n}$ on $\boldsymbol{C}^\infty(\overline{\Omega})$ can be extended by continuity to a linear, continuous mapping from $\boldsymbol{M}^{p'}(\Omega)$ into $\boldsymbol{W}_{no}^{-1/p',p'}(\Gamma)$ as follows. For any $\boldsymbol{\phi} \in \boldsymbol{W}_{n0}^{1-1/p,p}(\Gamma)$, there exists $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on Γ and $\|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{\phi}\|_{W^{1-1/p,p}(\Omega)}$ with some constant $C > 0$. If $\boldsymbol{S} \in \boldsymbol{C}^\infty(\overline{\Omega})$, then we can define

$$\langle \boldsymbol{S} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle = \langle \operatorname{curl} \boldsymbol{S}, \widetilde{\boldsymbol{\phi}}\rangle_{\mathbb{H}',\mathbb{H}} - \int_\Omega \boldsymbol{S} \cdot \operatorname{curl} \widetilde{\boldsymbol{\phi}} dx.$$

Since

$$|\langle \boldsymbol{S} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle| \le C(\|\operatorname{curl} \boldsymbol{S}\|_{\mathbb{H}'} + \|\boldsymbol{S}\|_{\boldsymbol{L}^{p'}(\Omega)})\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)},$$

we have

$$\|\boldsymbol{S} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} \le C(\|\operatorname{curl} \boldsymbol{S}\|_{\mathbb{H}'} + \|\boldsymbol{S}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

By denseness, we can extend the mapping to a linear, continuous mapping from $\boldsymbol{M}^{p'}(\Omega)$ into $\boldsymbol{W}^{-1/p'.p'}(\Gamma)$.

From this, we see that $S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ is well defined. From (2.6), we have

$$\langle \operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u}], \boldsymbol{v}\rangle_{\mathbb{H}',\mathbb{H}} + \langle S_t(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma + \int_\Omega \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}) \cdot \boldsymbol{v} dx + \langle \nabla\pi, \boldsymbol{v}\rangle_{\mathbb{H}',\mathbb{H}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbb{H}',\mathbb{H}} + \langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v} \in \mathbb{X}.$$

Therefore, taking the first equation of (2.5), we have

$$\langle S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v} \in \mathbb{X}.$$

Since the trace operator $\mathbb{X} \to \boldsymbol{W}^{1-1/p,p}(\Gamma)$ is surjective, we have

$$S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma.$$

□

We are in a position to state the main theorem.

Theorem 2.8. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ, and assume that Carathéodory functions $S(x,t)$ and $F(x, \boldsymbol{z})$ satisfy the structure conditions* (2.1a)-(2.1c) *and* (2.3a)-(2.3c)*, respectively. Then for any $\boldsymbol{f} \in \mathbb{H}'$ and $\boldsymbol{g} \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$, the Maxwell-Stokes system* (2.5) *has a unique weak solution $(\boldsymbol{u}, \pi) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$ depending only on p, Ω and the constants in* (2.1a)-(2.1c) *and* (2.3a)-(2.3c) *such that*

$$\|\boldsymbol{u}\|_{\mathbb{X}}^p + \|\pi\|^{p'}_{L^{p'}(\Omega)/\mathbb{R}} \le C(\|\boldsymbol{f}\|^{p'}_{\mathbb{H}'} + \|\boldsymbol{g}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}). \tag{2.7}$$

3. PROOF OF THEOREM 2.8

In this section, we prove Theorem 2.8 by the penalty method introduced by [11]. In order to do so, let $0 < \varepsilon \le 1$. We rewrite the approximate system: to find $\boldsymbol{u}_\varepsilon \in \mathbb{X}$ such that

$$\begin{cases} \mathrm{curl}\,[S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon] \\ \qquad -\frac{1}{\varepsilon}\nabla[S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)\mathrm{div}\, \boldsymbol{u}_\varepsilon] + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_\varepsilon) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma \end{cases} \tag{3.1}$$

where $\boldsymbol{f} \in \mathbb{H}'$ and $\boldsymbol{g} \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$ are given functions.

Here we give the notion of a weak solution of (3.1).

Definition 3.1. *We say that $\boldsymbol{u}_\varepsilon \in \mathbb{X}$ is a weak solution of* (3.1)*, if $\boldsymbol{u}_\varepsilon$ satisfies*

$$\int_\Omega \big\{S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon \cdot \mathrm{curl}\, \boldsymbol{v} + \frac{1}{\varepsilon} S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon)(\mathrm{div}\, \boldsymbol{v})\big\}dx \\ + \int_\Omega \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_\varepsilon) \cdot \boldsymbol{v} dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle + \langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma \textit{ for all } \boldsymbol{v} \in \mathbb{X}. \tag{3.2}$$

We obtain a weak solution of (3.1) by solving a variational problem. For any fixed $0 < \varepsilon \le 1$, define a functional

$$E_\varepsilon[\boldsymbol{v}] = \frac{1}{2}\int_\Omega \big\{S(x, |\mathrm{curl}\, \boldsymbol{v}|^2) + \frac{1}{\varepsilon} S(x, (\mathrm{div}\, \boldsymbol{v})^2)\big\}dx \\ + \int_\Omega F(x, \boldsymbol{v})dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle - \langle \boldsymbol{g}, \boldsymbol{v}\rangle_\Gamma. \tag{3.3}$$

We consider the following minimization problem: to find $\boldsymbol{u}_\varepsilon \in \mathbb{X}$ such that

$$E_\varepsilon[\boldsymbol{u}_\varepsilon] = \alpha := \inf_{\boldsymbol{v}\in\mathbb{X}} E_\varepsilon[\boldsymbol{v}]. \tag{3.4}$$

We call such a $\boldsymbol{u}_\varepsilon$ a minimizer of α. Then we have the following proposition.

Proposition 3.2. *Let* $0 < \varepsilon \leq 1$ *and* $\boldsymbol{f} \in \mathbb{H}'$ *and* $\boldsymbol{g} \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$. *Then the minimization problem* (3.4) *has a unique minimizer* $\boldsymbol{u}_\varepsilon \in \mathbb{X}$.

Proof. Step 1. E_ε is a proper convex functional.

In fact, from (2.1b), $S(x,t^2)$ is convex in t and from Lemma 2.4, $F(x,\boldsymbol{z})$ is also convex in $\boldsymbol{z}$

Step 2. E_ε is coercive on $\mathbb{X}$.

In fact, from (2.2) and (2.3a) and the Young inequality, for any $\delta > 0$,

$$\begin{aligned} E_\varepsilon[\boldsymbol{v}] \;&\geq\; \frac{\lambda}{p}\int_\Omega (|\mathrm{curl}\,\boldsymbol{v}|^p + |\mathrm{div}\,\boldsymbol{v}|^p)dx + c_1\int_\Omega |\boldsymbol{v}|^p dx - c_2|\Omega| \\ &\quad -\|\boldsymbol{f}\|_{\mathbb{H}'}\|\boldsymbol{v}\|_{\mathbb{H}} - \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \\ &\geq\; \min\{\frac{\lambda}{p}, c_1\}\|\boldsymbol{v}\|_{\mathbb{X}}^p - c_2|\Omega| - \delta\|\boldsymbol{v}\|_{\mathbb{X}}^p \\ &\quad -C(\delta)(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}). \end{aligned}$$

If we choose $\delta = c := \frac{1}{2}\min\{\frac{\lambda}{p}, c_1\}$, then we have

$$E_\varepsilon[\boldsymbol{v}] \geq c\|\boldsymbol{v}\|_{\mathbb{X}}^p - c_2|\Omega| - C(c)(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}). \tag{3.5}$$

Thus E_ε is coercive on $\mathbb{X}$.

Step 3. E_ε is weakly lower semi-continuous on $\mathbb{X}$.

In fact, since $S(x,t^2)$ is continuous and convex in t and $F(x,\boldsymbol{z})$ is also continuous and convex in $\boldsymbol{z}$, E_ε is a lower semi-continuous and convex functional. Therefore, it is weakly lower semi-continuous on $\mathbb{X}$. For the direct proof, see Aramaki [5].

Hence a minimizer $\boldsymbol{u}_\varepsilon \in \mathbb{X}$ exists. (cf. Ekeland and Temam [9, Proposition 2.1, p. 35]). Then the uniqueness follows from the strictly convexity of F (Lemma 2.4). □

If $\boldsymbol{u}_\varepsilon$ is a minimizer of α, then the Euler-Lagrange equation becomes

$$\begin{aligned} &\int_\Omega \big\{S_t(x,|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon\cdot\mathrm{curl}\,\boldsymbol{v} + \frac{1}{\varepsilon}S_t(x,(\mathrm{div}\,\boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\,\boldsymbol{u}_\varepsilon)(\mathrm{div}\,\boldsymbol{v})\big\}dx \\ &\qquad + \int_\Omega \nabla_{\boldsymbol{z}}F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}dx = \langle\boldsymbol{f},\boldsymbol{v}\rangle + \langle\boldsymbol{g},\boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v}\in\mathbb{X}. \end{aligned} \tag{3.6}$$

Furthermore, we obtain the following proposition.

Proposition 3.3. *The minimizer* $\boldsymbol{u}_\varepsilon$ *in Proposition* 3.2 *is a unique weak solution of* (3.1) *in the sense of Definition* 3.1.

Proof. It suffices to prove the uniqueness. Let $\boldsymbol{u}_\varepsilon^1, \boldsymbol{u}_\varepsilon^2 \in \mathbb{X}$ be two weak solutions of (3.1). Taking $\boldsymbol{v} = \boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2$ as a test function of (3.2), we have

$$\int_\Omega \Big\{ \big(S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1 - S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2)\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) \\ + \frac{1}{\varepsilon}\big(S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^1)^2)\mathrm{div}\, \boldsymbol{u}_\varepsilon^1 - S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^2)^2)\mathrm{div}\, \boldsymbol{u}_\varepsilon^2\big)\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)\Big\} dx \\ + \int_\Omega \big(\nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_1) - \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_2)\big) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) dx = 0.$$

It follows from Lemma 2.2 and (2.3c) that

$$\int_\Omega \Big\{ c\big(|\mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^p + |\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^p\big) + c_3 |\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2|^p \Big\} dx \le 0$$

if $p \ge 2$, and

$$\int_\Omega \Big\{ c(|\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1| + |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2|)^{p-2} |\mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^2 \\ + c(|\mathrm{div}\, \boldsymbol{u}_\varepsilon^1| + |\mathrm{div}\, \boldsymbol{u}_\varepsilon^2|)^{p-2} |\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^2 \\ + c_3(|\boldsymbol{u}_\varepsilon^1| + |\boldsymbol{u}_\varepsilon^2|)^{p-2} |\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2|^2 \Big\} dx \le 0$$

if $1 < p < 2$. Therefore, we have $\boldsymbol{u}_\varepsilon^1 = \boldsymbol{u}_\varepsilon^2$. □

We obtain the following estimates.

Proposition 3.4. *The minimizer* $\boldsymbol{u}_\varepsilon \in \mathbb{X}$ *of* (3.4) *satisfies the following estimates. There exist constants* $C_2, C_3 > 0$ *depending only on* p, Ω, λ, C_1 *and* c_3*, but independent of* $0 < \varepsilon \le 1$ *such that*

$$\|\boldsymbol{u}_\varepsilon\|_{\mathbb{X}}^p \le C_2(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}), \tag{3.7}$$

and

$$\|\mathrm{div}\, \boldsymbol{u}_\varepsilon\|_{L^p(\Omega)}^p \le C_3 \varepsilon(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}). \tag{3.8}$$

Proof. Taking $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ as a test function of (3.2) and using (2.1a) and (2.3c) with $\boldsymbol{z}_2 = \mathbf{0}$, for any $\delta > 0$, we have

$$\min\{\lambda, c_3\} \|\boldsymbol{u}_\varepsilon\|_{\mathbb{X}}^p \le \delta \|\boldsymbol{u}_\varepsilon\|_{\mathbb{X}}^p + C(\delta)(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}).$$

If we choose $\delta > 0$ so that $\delta < \min\{\lambda, c_3\}$, we get (3.7). If we use (2.1a), (3.2) with $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ and (3.7), we can see that (3.8) holds. □

Proof of Theorem 2.8

By (3.8), we see that $\operatorname{div} \boldsymbol{u}_\varepsilon \to 0$ strongly in $L^p(\Omega)$. Define

$$\pi_\varepsilon = -\frac{1}{\varepsilon} S_t(x, (\operatorname{div} \boldsymbol{u}_\varepsilon)^2) \operatorname{div} \boldsymbol{u}_\varepsilon.$$

By (3.7), $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $\mathbb{X}$, so bounded in $\boldsymbol{W}^{1,p}(\Omega)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in $\mathbb{X}$, strongly in $\boldsymbol{L}^p(\Omega)$ and a.e. in Ω. Thus it follows from (3.7) that

$$\|\boldsymbol{u}\|_{\mathbb{X}}^p \le \liminf_{\varepsilon \to 0} \|\boldsymbol{u}_\varepsilon\|_{\mathbb{X}}^p \le C_2(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}). \tag{3.9}$$

We show that $\{\pi_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^{p'}(\Omega)/\mathbb{R}$. To show this, for any $\phi \in L_0^p(\Omega) = \{\phi \in L^p(\Omega); \int_\Omega \phi dx = 0\}$ we note that the following Neumann problem to the Poisson equation

$$\begin{cases} \Delta\psi = \phi & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma, \\ \int_\Omega \psi dx = 0 \end{cases}$$

has a unique solution $\psi \in W^{2,p}(\Omega)$. If we define $\boldsymbol{v}_\phi = \nabla\psi \in \boldsymbol{W}^{1,p}(\Omega)$, then we see that $\operatorname{curl} \boldsymbol{v}_\phi = \boldsymbol{0}, \operatorname{div} \boldsymbol{v}_\phi = \phi$ in Ω, $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, and $\|\boldsymbol{v}_\phi\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\phi\|_{L^p(\Omega)}$. Taking $\boldsymbol{v} = \boldsymbol{v}_\phi$ as a test function in (3.6) and using (3.7) and (2.3c), we have

$$\begin{aligned} \left| \int_\Omega \pi_\varepsilon \phi dx \right| &= |\langle \boldsymbol{f}, \boldsymbol{v}_\phi\rangle| + |\langle \boldsymbol{g}, \boldsymbol{v}_\phi\rangle_\Gamma| + \left| \int_\Omega \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_\varepsilon) \cdot \boldsymbol{v}_\phi dx \right| \\ &\le C(\|\boldsymbol{f}\|_{\mathbb{H}'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\|\phi\|_{L^p(\Omega)} \text{ for all } \phi \in L_0^p(\Omega), \end{aligned}$$

where C is a constant independent of ϕ and ε. Since $L_0^p(\Omega)$ is isomorphic onto $(L^{p'}(\Omega)/\mathbb{R})'$, the above inequality implies that

$$\|\pi_\varepsilon\|_{L^{p'}(\Omega)/\mathbb{R}} \le C(\|\boldsymbol{f}\|_{\mathbb{H}'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}).$$

Thus $\{\pi_\varepsilon\}$ is bounded in $L^{p'}(\Omega)/\mathbb{R}$. Hence passing to a subsequence, we may assume that $\pi_\varepsilon \to \pi$ weakly in $L^{p'}(\Omega)/\mathbb{R}$, and

$$\|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}^{p'} \le \liminf_{\varepsilon\to 0} \|\pi_\varepsilon\|_{L^{p'}(\Omega)/\mathbb{R}}^{p'} \le \Lambda C_3(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}). \tag{3.10}$$

On the other hand, it follows from (2.1a) and (3.2) with $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ that

$$\int_\Omega |S_t(x, \operatorname{curl} \boldsymbol{u}_\varepsilon|^2)\operatorname{curl} \boldsymbol{u}_\varepsilon|^{p'} dx \le \Lambda^{p'} \|\operatorname{curl} \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^p(\Omega)}^p \le \Lambda^{p'} C_2(\|\boldsymbol{f}\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'}).$$

Therefore, we may assume that $S_t(x, |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2)\operatorname{curl} \boldsymbol{u}_\varepsilon \to \boldsymbol{w}$ weakly in $\boldsymbol{L}^{p'}(\Omega)$ for some $\boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)$.

Lemma 3.5. *We have the following.*

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot \boldsymbol{v}dx = \int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot \boldsymbol{v}dx \textit{ for any } \boldsymbol{v}\in\mathbb{X}.$$

Proof. We use the Vitali convergence theorem. From (2.3c) and the Hölder inequality,

$$\begin{aligned}\int_\Omega |\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}|dx &\le b_2\int_\Omega (|\boldsymbol{u}_\varepsilon|^{p-1}+1)|\boldsymbol{v}|dx\\ &\le b_2(\|\boldsymbol{u}_\varepsilon\|^{p-1}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+|\Omega|^{1/p'}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\\ &\le C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)},\end{aligned}$$

where the constant C depends on $b_2,\Omega,C_1,\|\boldsymbol{f}\|^{p'}_{\mathbb{H}'}$ and $\|\boldsymbol{g}\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}$. Thus the function $\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}\in L^1(\Omega)$. Since $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ strongly in $\boldsymbol{L}^p(\Omega)$, for any $\delta>0$, there exists $\varepsilon_0>0$ such that for any $0<\varepsilon\le\varepsilon_0$, $\|\boldsymbol{u}_\varepsilon-\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}<\delta$. There exists $\gamma\in(0,\delta)$ such that if $\omega\subset\Omega$ is a measurable subset of Ω and $|\omega|<\gamma$, then $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\omega)}<\delta/C$. Hence

$$\int_\omega |\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}|dx\le C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\omega)}<\delta.$$

Thus $\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}$ is uniformly integrable. Since $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ a.e. in Ω, we have $\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}\to\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot\boldsymbol{v}$ a.e. in Ω. Therefore, we can apply the Vitali convergence theorem. □

Letting $\varepsilon\to 0$ in (3.2), we have

$$\begin{aligned}\int_\Omega \boldsymbol{w}\cdot\operatorname{curl}\boldsymbol{v}dx-\int_\Omega \pi\operatorname{div}\boldsymbol{v}dx+\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot\boldsymbol{v}dx&\\ =\langle\boldsymbol{f},\boldsymbol{v}\rangle+\langle\boldsymbol{g},\boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v}\in\mathbb{X}.&\quad(3.11)\end{aligned}$$

In particular, since $\operatorname{div}\boldsymbol{u}=0$ in Ω, we have

$$\int_\Omega \boldsymbol{w}\cdot\operatorname{curl}\boldsymbol{u}dx+\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot\boldsymbol{u}dx=\langle\boldsymbol{f},\boldsymbol{u}\rangle+\langle\boldsymbol{g},\boldsymbol{u}\rangle_\Gamma. \qquad (3.12)$$

Lemma 3.6. *We have the following.*

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{u}_\varepsilon dx=\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot\boldsymbol{u}dx.$$

Proof. We have

$$\begin{aligned}\left|\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{u}_\varepsilon dx-\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\cdot\boldsymbol{u}dx\right| &\le \left|\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot(\boldsymbol{u}_\varepsilon-\boldsymbol{u})dx\right|\\ &+\left|\int_\Omega \big(\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)-\nabla_{\boldsymbol{z}} F(x,\boldsymbol{u})\big)\cdot\boldsymbol{u}dx\right|.\end{aligned}$$

Here we have

$$\begin{aligned}\left|\int_\Omega \nabla_{\boldsymbol{z}} F(x,\boldsymbol{u}_\varepsilon)\cdot(\boldsymbol{u}_\varepsilon-\boldsymbol{u})dx\right| &\le b_2\int_\Omega (|\boldsymbol{u}_\varepsilon|^{p-1}+1)|\boldsymbol{u}_\varepsilon-\boldsymbol{u}|dx\\ &\le b_2\left(\int_\Omega (|\boldsymbol{u}_\varepsilon|^{p-1}+1)^{p'}dx\right)^{1/p'}\|\boldsymbol{u}_\varepsilon-\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\\ &\le C\|\boldsymbol{u}_\varepsilon-\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},\end{aligned}$$

where C is a constant independent of ε. From the fact that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $\boldsymbol{L}^p(\Omega)$ and Lemma 3.5, the conclusion holds. □

From (3.2) with $\boldsymbol{v}=\boldsymbol{u}_\varepsilon$,

$$\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2dx+\int_\Omega \nabla_{\boldsymbol{z}}F(x,\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{u}_\varepsilon dx\le\langle\boldsymbol{f},\boldsymbol{u}_\varepsilon\rangle+\langle\boldsymbol{g},\boldsymbol{u}_\varepsilon\rangle_\Gamma. \quad (3.13)$$

Since $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ weakly in $\mathbb{X}$. it follows from Lemma 3.6 and (3.12) that we have

$$\limsup_{\varepsilon\to 0}\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2dx\le\int_\Omega \boldsymbol{w}\cdot\mathrm{curl}\,\boldsymbol{u}dx. \quad (3.14)$$

By the monotonicity lemma (Lemma 2.2) for S_t, we have

$$\begin{aligned}&\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon\cdot\mathrm{curl}\,(\boldsymbol{u}_\varepsilon-\boldsymbol{v})dx\\ &\qquad -\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}\cdot\mathrm{curl}\,(\boldsymbol{u}_\varepsilon-\boldsymbol{v})dx\ge 0.\end{aligned}$$

Since $\mathrm{curl}\,\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ weakly in $\boldsymbol{L}^p(\Omega)$ and $S_t(x,|\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon\to\boldsymbol{w}$ weakly in $\boldsymbol{L}^{p'}(\Omega)$, taking upper limit of this inequality as $\varepsilon\to 0$ and using (3.14), we have

$$\int_\Omega(\boldsymbol{w}-S_t(x,|\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v})\cdot\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{v})dx\ge 0 \text{ for all } \boldsymbol{v}\in\mathbb{X}.$$

For any $\boldsymbol{\phi}\in\mathbb{X}$, put $\boldsymbol{v}=\boldsymbol{u}-\alpha\boldsymbol{\phi}$ $(\alpha>0)$. Then we have

$$\int_\Omega\big(\boldsymbol{w}-S_t(x,|\mathrm{curl}\,\boldsymbol{u}-\alpha\mathrm{curl}\,\boldsymbol{\phi}|^2)\big)(\mathrm{curl}\,\boldsymbol{u}-\alpha\mathrm{curl}\,\boldsymbol{\phi})\cdot\alpha\mathrm{curl}\,\boldsymbol{\phi}dx\ge 0.$$

If we divide this inequality by α, and then let $\alpha\to 0$, we have

$$\int_\Omega\big(\boldsymbol{w}-S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\big)\cdot\mathrm{curl}\,\boldsymbol{\phi}dx\ge 0$$

for all $\boldsymbol{\phi}\in\mathbb{X}$. This implies

$$\int_\Omega(\boldsymbol{w}-S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u})\cdot\mathrm{curl}\,\boldsymbol{\phi}dx=0$$

for all $\boldsymbol{\phi} \in \mathbb{X}$. Therefore, according to (3.11), we see that (2.6) holds, so $(\boldsymbol{u}, \pi) \in \mathbb{X}_0 \times L^p(\Omega)$ is a weak solution of (2.5).

Next we show the uniqueness of solution. Let $(\boldsymbol{u}_1, \pi_1), (\boldsymbol{u}_2, \pi_2) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ be two weak solutions of (2.5). Taking $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ as a test function of (2.6), since $\mathrm{div}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = 0$ in Ω, we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_i|^2)\mathrm{curl}\,\boldsymbol{u}_i \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_i) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx$$
$$= \langle \boldsymbol{f}, \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle + \langle \boldsymbol{g}, \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle_\Gamma \text{ for } i = 1, 2.$$

Thus we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx$$
$$+ \int_\Omega \big(\nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_1) - \nabla_{\boldsymbol{z}} F(x, \boldsymbol{u}_2)\big) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx = 0.$$

By the monotonicity (Lemma 2.2) and (2.3c), we can see that $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Furthermore, it follows from (2.6) that

$$\int_\Omega (\pi_1 - \pi_2)\mathrm{div}\,\boldsymbol{v} dx = 0 \text{ for all } \boldsymbol{v} \in \mathbb{X}.$$

This implies that $\nabla(\pi_1 - \pi_2) = \boldsymbol{0}$ in the distribution sense, so $\pi_1 - \pi_2$ is a constant, i.e., $\pi_1 = \pi_2$ in $L^{p'}(\Omega)/\mathbb{R}$.

Finally we show the estimate (2.7). Since the weak solution $(\boldsymbol{u}, \pi)$ is a weak limit of $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon)$ in $\mathbb{X} \times L^{p'}(\Omega)$, it follows from (3.9) and (3.10) that there exists a constant $C > 0$ depending only on $\lambda, \Lambda, C_1, C_3, p$ and Ω such that (2.7) holds. This completes the proof of Theorem 2.8.

4. CONTINUOUS DEPENDENCE OF A WEAK SOLUTION ON THE DATA

In this section, we consult the continuous dependence of a weak solution of (2.5) on the data. In order to do so, for every $n = 0, 1, \ldots$, let $S^{(n)}(x, t)$ and $F^{(n)}(x, \boldsymbol{z})$ satisfy (2.1a)-(2.1c) with the same constants λ and Λ and (2.3a)-(2.3c) with the same constants c_1, c_2, c_3, b_1 and b_2, respectively. Let $\boldsymbol{f}_n \in \mathbb{H}'$ and $\boldsymbol{g}_n \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$ for $n = 0, 1, \ldots$. Assume that $(\boldsymbol{u}_n, \pi_n) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ is a weak solution of (2.5), i.e.,

$$\begin{cases} \mathrm{curl}\,[S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n] + \nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_n) + \nabla \pi_n = \boldsymbol{f}_n & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u}_n = 0 & \text{in } \Omega, \\ \boldsymbol{u}_n \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n \times \boldsymbol{n} = \boldsymbol{g}_n & \text{on } \Gamma \end{cases} \tag{4.1}$$

for every $n = 0, 1, \ldots$. Thus $(\boldsymbol{u}_n, \pi_n)$ satisfies

$$\int_\Omega S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n \cdot \mathrm{curl}\,\boldsymbol{v}dx + \int_\Omega \nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_n) \cdot \boldsymbol{v}dx \\ - \int_\Omega \pi_n \mathrm{div}\,\boldsymbol{v}dx = \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle + \langle \boldsymbol{g}_n, \boldsymbol{v}\rangle_\Gamma \quad (4.2)$$

for all $\boldsymbol{v} \in \mathbb{X}$.

Then we have the following theorem on the continuous dependence on the data.

Theorem 4.1. *We assume that for every* $n = 0, 1, \ldots$, *Carathéodory functions* $S^{(n)}(x,t)$ *and* $F^{(n)}(x, \boldsymbol{z})$ *satisfy* (2.1a)-(2.1c) *with the same constants and* (2.3a)-(2.3c) *with the same constants, respectively, and* $\boldsymbol{f}_n \in \mathbb{H}'$ *and* $\boldsymbol{g}_n \in \boldsymbol{W}_{n0}^{-1/p',p'}(\Gamma)$. *Let* $(\boldsymbol{u}_n, \pi_n) \in \mathbb{X}_0 \times L^{p'}(\Omega)/\mathbb{R}$ *be a unique weak solution of* (4.1). *If* $S_t^{(n)}(x,t) \to S_t^{(0)}(x,t)$ *a.e. in* $\Omega \times [0, \infty)$, $\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{z}) \to \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{z})$ *a.e. in* $\Omega \times \mathbb{R}^3$, *and* $\boldsymbol{f}_n \to \boldsymbol{f}_0$ *in* $\mathbb{H}'$ *and* $\boldsymbol{g}_n \to \boldsymbol{g}_0$ *in* $\boldsymbol{W}^{-1/p',p'}(\Gamma)$ *as* $n \to \infty$, *then* $\boldsymbol{u}_n \to \boldsymbol{u}_0$ *in* $\mathbb{X}$ *and* $\pi_n \to \pi_0$ *in* $L^{p'}(\Omega)/\mathbb{R}$ *as* $n \to \infty$.

In the particular case where $S^{(n)}(x,t) = S^{(0)}(x,t)$ *and* $F^{(n)}(x, \boldsymbol{z}) = F^{(0)}(x, \boldsymbol{z})$ *for all* $n = 1.2, \ldots$, *there exists a constant* $C > 0$ *depending only on* $p, \lambda, \Lambda, c_1, c_2, c_3, b_1, b_2, \Omega$, $\|\boldsymbol{f}_0\|_{\mathbb{H}'}$ *and* $\|\boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}$ *such that for large* n,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}^{p\vee p'} + \|\pi_n - \pi_0\|_{L^{p'}(\Omega)/\mathbb{R}}^{p\vee p'} \\ \leq C(\|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'}^{p'} + \|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'}^{p} + \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p'} + \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}^{p}),$$

where $p \vee p' = \max\{p, p'\}$.

Proof. Taking $\boldsymbol{v} = \boldsymbol{u}_n - \boldsymbol{u}_0$ as a test function of (4.2), since $\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_0) = 0$ in Ω, we have

$$\int_\Omega \big(S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)dx \\ + \int_\Omega \big(\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_n) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\big) \cdot (\boldsymbol{u}_n - \boldsymbol{u}_0)dx \\ = \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{u}_n - \boldsymbol{u}_0\rangle + \langle \boldsymbol{g}_n - \boldsymbol{g}_0, \boldsymbol{u}_n - \boldsymbol{u}_0\rangle_\Gamma.$$

We write this equality into the form

$$\int_\Omega \big(S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n - S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)dx \\ + \int_\Omega \big(\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_n) - \nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0)\big) \cdot (\boldsymbol{u}_n - \boldsymbol{u}_0)dx = I_1 + I_2 + I_3, \quad (4.3)$$

where

$$\begin{aligned} I_1 &= \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{u}_n - \boldsymbol{u}_0\rangle + \langle \boldsymbol{g}_n - \boldsymbol{g}_0, \boldsymbol{u}_n - \boldsymbol{u}_0\rangle_\Gamma \\ I_2 &= -\int_\Omega \big(S_t^{(n)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0\big) \cdot \mathrm{curl}\, (\boldsymbol{u}_n - \boldsymbol{u}_0)dx \\ I_3 &= \int_\Omega \big(\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\big) \cdot (\boldsymbol{u}_n - \boldsymbol{u}_0)dx. \end{aligned}$$

First we estimate I_1, I_2 and I_3 from above. By the Hölder inequality, we have

$$|I_1| \le (\|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'} + \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}},$$

$$|I_2| \le \|S_t^{(n)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}},$$

and

$$|I_3| \le \|\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}.$$

For the brevity of notations, we put

$$\begin{aligned} G_n = \|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'} &+ \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)} \\ &+ \|S_t^{(n)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\, \boldsymbol{u}_0|^2)\mathrm{curl}\, \boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \\ &\qquad + \|\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\|_{\boldsymbol{L}^{p'}(\Omega)}. \end{aligned}$$

Thus we have

$$|I_1| + |I_2| + |I_3| \le G_n\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}.$$

We estimate the left-hand side of (4.3) from below. When $p \ge 2$, using Lemma 2.2 and (2.3c), we have

$$\min\{c, c_3\} \int_\Omega \big(|\mathrm{curl}\, (\boldsymbol{u}_n - \boldsymbol{u}_0)|^p + |\boldsymbol{u}_n - \boldsymbol{u}_0|^p\big)dx \le G_n\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}.$$

Using the Young inequality, we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}^p \le CG_n^{p'}. \tag{4.4}$$

When $1 < p < 2$, by Lemma 2.2, (2.3c), we have

$$\begin{aligned} \min\{c, c_3\} \int_\Omega \big(&(|\mathrm{curl}\, \boldsymbol{u}_n| + |\mathrm{curl}\, \boldsymbol{u}_0|)^{p-2}|\mathrm{curl}\, (\boldsymbol{u}_n - \boldsymbol{u}_0)|^2 \\ &+ (|\boldsymbol{u}_n| + |\boldsymbol{u}_0|)^{p-2}|\boldsymbol{u}_n - \boldsymbol{u}_0|^2\big)dx \le G_n\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}. \end{aligned}$$

If we use the reverse Hölder inequality (cf. Sobolev [12, p. 8]) with $0 < s = p/2 < 1$ and $s' = p/(p-2)$, then we have

$$\begin{aligned} \int_\Omega \big(&(|\mathrm{curl}\, \boldsymbol{u}_n| + |\mathrm{curl}\, \boldsymbol{u}_0|)^{p-2}|\mathrm{curl}\, (\boldsymbol{u}_n - \boldsymbol{u}_0)|^2 + (|\boldsymbol{u}_n| + |\boldsymbol{u}_0|)^{p-2}|\boldsymbol{u}_n - \boldsymbol{u}_0|^2\big)dx \\ &\ge 2^{p-1}(\|\mathrm{curl}\, \boldsymbol{u}_n\|_{\boldsymbol{L}^p(\Omega)}^p + \|\mathrm{curl}\, \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}^p)^{(p-2)/p}\|\mathrm{curl}\, (\boldsymbol{u}_n - \boldsymbol{u}_0)\|_{\boldsymbol{L}^p(\Omega)}^2 \\ &\qquad + 2^{p-1}(\|\boldsymbol{u}_n\|_{\boldsymbol{L}^p(\Omega)}^p + \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}^p)^{(p-2)/p}\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}^2. \end{aligned}$$

Thus from (2.7), we have

$$\begin{aligned}\|\mathrm{curl}\,(\boldsymbol{u}_n-\boldsymbol{u}_0)\|^2_{\boldsymbol{L}^p(\Omega)} &\le C(\|\mathrm{curl}\,\boldsymbol{u}_n\|^p_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{curl}\,\boldsymbol{u}_0\|^p_{\boldsymbol{L}^p(\Omega)})^{(2-p)/p}G_n\|\boldsymbol{u}_n-\boldsymbol{u}_0\|_{\mathbb{X}}\\ &\le C_1(\|\boldsymbol{f}_n\|^{p'}_{\mathbb{H}'}+\|\boldsymbol{f}_0\|^{p'}_{\mathbb{H}'}+\|\boldsymbol{g}_n\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}\\ &\qquad +\|\boldsymbol{g}_0\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})^{(2-p)/p}G_n\|\boldsymbol{u}_n-\boldsymbol{u}_0\|_{\mathbb{X}}.\end{aligned}$$

Thus we have

$$\|\mathrm{curl}\,(\boldsymbol{u}_n-\boldsymbol{u}_0)\|^2_{\boldsymbol{L}^p(\Omega)}\le C(\|\boldsymbol{f}_0\|^{p'}_{\mathbb{H}'}+\|\boldsymbol{g}_0\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}+1)^{(2-p)/p}G_n\|\boldsymbol{u}_n-\boldsymbol{u}_0\|_{\mathbb{X}}$$

for large n. Similarly, we have

$$\|\boldsymbol{u}_n-\boldsymbol{u}_0\|^2_{\boldsymbol{L}^p(\Omega)}\le C(\|\boldsymbol{f}_0\|^{p'}_{\mathbb{H}'}+\|\boldsymbol{g}_0\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}+1)^{(2-p)/p}G_n\|\boldsymbol{u}_n-\boldsymbol{u}_0\|_{\mathbb{X}}.$$

Thus we have

$$\|\boldsymbol{u}_n-\boldsymbol{u}_0\|_{\mathbb{X}}\le C(\|\boldsymbol{f}_0\|^{p'}_{\mathbb{H}'}+\|\boldsymbol{g}_0\|^{p'}_{\boldsymbol{W}^{-1/p',p'}(\Gamma)}+1)^{(2-p)/p}G_n \tag{4.5}$$

for large n.

On the other hand, we have

$$\begin{aligned}&\int_\Omega\big(S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n-S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\big)\cdot\mathrm{curl}\,\boldsymbol{v}dx\\ &\qquad+\int_\Omega\big(\nabla_{\boldsymbol{z}}F^{(n)}(x,\boldsymbol{u}_n)-\nabla_{\boldsymbol{z}}F^{(0)}(x,\boldsymbol{u}_0)\big)\cdot\boldsymbol{v}dx-\int_\Omega(\pi_n-\pi_0)\mathrm{div}\,\boldsymbol{v}dx\\ &\qquad\qquad=\langle\boldsymbol{f}_n-\boldsymbol{f},\boldsymbol{v}\rangle+\langle\boldsymbol{g}_n-\boldsymbol{g}_0,\boldsymbol{v}\rangle_\Gamma\end{aligned}\tag{4.6}$$

for any $\boldsymbol{v}\in\mathbb{X}$. We write the mean value of a function φ by c_φ, i.e.,

$$c_\varphi=\frac{1}{|\Omega|}\int_\Omega\varphi dx.$$

Since $\boldsymbol{v}\in\mathbb{X}$, we can see that

$$\begin{aligned}\int_\Omega(\pi_n-\pi_0-c_{\pi_n-\pi_0})\mathrm{div}\,\boldsymbol{v}dx&=\int_\Omega(\pi_n-\pi_0)\mathrm{div}\,\boldsymbol{v}dx-c_{\pi_n-\pi_0}\int_\Omega\mathrm{div}\,\boldsymbol{v}dx\\ &=\int_\Omega(\pi_n-\pi_0)\mathrm{div}\,\boldsymbol{v}dx.\end{aligned}$$

Thus we may assume that $\pi_n-\pi_0\in L_0^{p'}(\Omega)$, where

$$L_0^{p'}(\Omega):=\left\{\varphi\in L^{p'}(\Omega);\int_\Omega\varphi dx=0\right\}.$$

For any $\phi \in L^p(\Omega)$, we see that

$$\int_\Omega (\pi_n - \pi_0)\phi dx = \int_\Omega (\pi_n - \pi_0)(\phi - c_\phi)dx.$$

By [1, Corollary 3.1], there exists $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\operatorname{div} \boldsymbol{w} = \phi - c_\phi$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \le C\|\phi\|_{L^p(\Omega)}.$$

Taking $\boldsymbol{v} = \boldsymbol{w}$ as a test function of (4.6),

$$\int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_n|^2)\operatorname{curl} \boldsymbol{u}_n - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx$$
$$- \int_\Omega (\pi_n - \pi_0)\phi dx = \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{w}\rangle + \langle \boldsymbol{g}_n - \boldsymbol{g}, \boldsymbol{w}\rangle_\Gamma.$$

We write this equality in the following form.

$$\int_\Omega (\pi_n - \pi_0)\phi dx = J_1 + J_2 + J_3. \tag{4.7}$$

where

$$\begin{aligned}
J_1 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_n|^2)\operatorname{curl} \boldsymbol{u}_n - S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx \\
&\quad + \int_\Omega (\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_n) - \nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0)) \cdot \boldsymbol{w} dx \\
J_2 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0 - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx, \\
&\quad + \int_\Omega (\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)) \cdot \boldsymbol{w} dx \\
J_3 &= -\langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{w}\rangle - \langle \boldsymbol{g}_n - \boldsymbol{g}, \boldsymbol{w}\rangle_\Gamma.
\end{aligned}$$

We have

$$\begin{aligned}
|J_2| &\le C\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0 \\
&\qquad - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{w}\|_{W_0^{1,p}(\Omega)} \\
&\qquad + C\|\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\
&\le C_1\big(\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0 - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \\
&\qquad + \|\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\|_{\boldsymbol{L}^{p'}(\Omega)}\big)\|\phi\|_{L^p(\Omega)} \\
&\le C_1 G_n \|\phi\|_{L^p(\Omega)}.
\end{aligned}$$

Clearly we have

$$|J_3| \le C(\|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'} + \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p',p'}(\Gamma)})\|\phi\|_{L^p(\Omega)} \le CG_n\|\phi\|_{L^p(\Omega)}.$$

When $1 < p < 2$, using Lemma 2.3, (2.3c), (4.3) and the Hölder inequality, we have

$$\begin{aligned} |J_1| &\le C_2\Big\{\int_\Omega |\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)|^{p-1}|\mathrm{curl}\,\boldsymbol{w}|dx + \int_\Omega |\boldsymbol{u}_n - \boldsymbol{u}_0|^{p-1}|\boldsymbol{w}|dx\Big\} \\ &\le C_3\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}^{p-1}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ &\le C_4 G_n^{p-1}\|\phi\|_{L^p(\Omega)}. \end{aligned}$$

When $p \ge 2$, similarly using Lemma 2.3, (2.3c), the Hölder inequality, (2.7), and (4.4), we have

$$\begin{aligned} |J_1| &\le C_1\Big\{\int_\Omega (|\mathrm{curl}\,\boldsymbol{u}_n| + |\mathrm{curl}\,\boldsymbol{u}_0|)^{p-2}|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)||\mathrm{curl}\,\boldsymbol{w}|dx \\ &\quad + \int_\Omega (|\boldsymbol{u}_n| + |\boldsymbol{u}_0|)^{p-2}|\boldsymbol{u}_n - \boldsymbol{u}_0||\boldsymbol{w}|dx\Big\} \\ &\le C_2(\|\boldsymbol{u}_n\|_{\mathbb{X}} + \|\boldsymbol{u}_0\|_{\mathbb{X}})^{p-2}\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ &\le C_3 G_n^{p'/p}\|\phi\|_{L^p(\Omega)}. \end{aligned}$$

Therefore, we have

$$\|\pi_n - \pi\|_{L^{p'}(\Omega)} \le \begin{cases} C_4 G_n + C_5 G_n^{p-1} & \text{if } 1 < p < 2, \\ C_4 G_n + C_5 G_n^{p'-1} & \text{if } p \ge 2. \end{cases} \tag{4.8}$$

Taking (4.5) into consideration, if $p \ge 2$, we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}^p + \|\pi_n - \pi\|_{L^{p'}(\Omega)}^p \le C_6(G_n^{p'} + G_n^p),$$

and if $1 < p < 2$,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathbb{X}}^{p'} + \|\pi_n - \pi\|_{L^{p'}(\Omega)}^{p'} \le C_6(G_n^{p'} + G_n^p),$$

Finally, we show $G_n \to 0$ as $n \to \infty$ if $\boldsymbol{f}_n \to \boldsymbol{f}_0$ in $\mathbb{X}'$ and $\boldsymbol{g}_n \to \boldsymbol{g}_0$ in $\boldsymbol{W}^{-1/p',p'}(\Gamma)$ and $S_t^{(n)}(x,t) \to S_t^{(0)}(x,t)$, $\nabla_{\boldsymbol{z}} F^{(n)}(x,\boldsymbol{z}) \to \nabla_{\boldsymbol{z}} F^{(0)}(x,\boldsymbol{z})$ a.e..

Since

$$|S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0|^{p'} \le (2\Lambda)^{p'}|\mathrm{curl}\,\boldsymbol{u}_0|^p \in L^1(\Omega)$$

and $S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 \to S_t^0(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0$ a.e. in Ω, it follows from the Lebesgue dominated theorem that

$$\int_\Omega |S_t^{(n)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S_t^{(0)}(x,|\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0|^{p'}dx \to 0$$

as $n \to \infty$. Similarly, since

$$|\nabla_{\boldsymbol{z}} F^{(n)}(x,\boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x,\boldsymbol{u}_0)|^{p'} \le (2b_3)^{p'}|\boldsymbol{u}_0|^p \in L^1(\Omega)$$

and $\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) \to \nabla_{\boldsymbol{z}} F^0(x, \boldsymbol{u}_0)$ a.e. in Ω, we have

$$\|\nabla_{\boldsymbol{z}} F^{(n)}(x, \boldsymbol{u}_0) - \nabla_{\boldsymbol{z}} F^{(0)}(x, \boldsymbol{u}_0)\|_{\boldsymbol{L}^{p'}(\Omega)} \to 0$$

as $n \to \infty$. Hence we have $G_n \to 0$ as $n \to \infty$. In the particular case where $S^{(n)}(x,t) = S^{(0)}(x,t)$ and $F^{(n)}(x, \boldsymbol{z}) = F^{(0)}(x, \boldsymbol{z})$, since $G_n = \|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\mathbb{H}'} + \|\boldsymbol{g}_n - \boldsymbol{g}_0\|_{\boldsymbol{W}^{-1/p'.p'}(\Gamma)}$, the estimate in theorem is clear. □

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