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Improved Estimates on Initial Coefficients of Certain Subclasses of Bi-univalent Functions

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Abstract

In the present paper, we derive estimates on initial coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for functions belong to the two well known subclasses $\mathcal{T}_\Sigma(\beta)$ and $\mathcal{T}_\Sigma^\alpha$ of the bi-univalent function class Σ defined in the open unit disk $\mathbb{U}$. These estimates shows improvements in the earlier known estimates for the two subclasses.
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1. INTRODUCTION

Let $\mathcal{S}$ denote the class described as $\mathcal{S} = \{f : \mathbb{U} \to \mathbb{C}$ such that f is analytic and univalent in $\mathbb{U}; f(0) = 0; f'(0) = 1\}$ such that every function $f \in \mathcal{S}$ has a Taylor-Maclaurin expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

Let g be the analytic extension of inverse function f^{-1} to $\mathbb{U}$ given by (see [6, 15])

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

Then $\Sigma = \{f \in \mathcal{S} : f$ and g both are univalent in the unit disk $\mathbb{U}\}$ defines the bi-univalent function class.

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In 1967, Lewin [10] has extended the theory of univalent functions to bi-univalent functions and proved that $|a_2| < 1.51$ for $f \in \Sigma$. Further in 1979, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$ and subsequently, Goodman [7] beleived that $|a_n| \leq 1$ may be true for every $n \in \mathbb{N}$ and $f \in \Sigma$. But then in 1981, Styer and Wright [18] proved that there exist functions in Σ for which $|a_2| > 4/3$ and in 1984, Tan [19] showed that $|a_2| \leq 1.485$ for $f \in \Sigma$.

In fact, after the work of Brannan and Taha [3], the pioneering research article of Srivastava et al. [17] revived the concept of coefficient inequalities of the functions $f \in \Sigma$. Motivated by them, many researchers (viz. [1, 4, 5, 8, 9, 13, 14, 16, 20]) have been studied and investigated various subclasses of Σ and obtained estimates on initial coefficients for the functions in these subclasses. But still the coefficient estimate problem for $|a_n|,\ (n = 3, 4, 5, \cdots)$ remain open.

We have used the following Lemma to prove our main results (see [6]).

Lemma 1.1. ***(Carathéodory Lemma)*** *Let $\mathcal{P}$ denotes the class of all analytic functions in $\mathbb{U}$ with positive real part. If $p(z) \in \mathcal{P}$ have the form* $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n,\ (z \in \mathbb{U})$, *then $|p_n| \leq 2$ for each $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$.*

Using an univalence criterion given by Ozaki and Nunokawa [12], Naik and Patil [11] introduced the following two subclasses.

Definition 1.2. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{T}_\Sigma(\beta)$ if we have:

$$\Re\left(\frac{z^2 f'(z)}{(f(z))^2}\right) > \beta, \quad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{w^2 g'(w)}{(g(w))^2}\right) > \beta, \quad (w \in \mathbb{U}),$$

where $0 \leq \beta < 1$ and the function $g \equiv f^{-1}$ is given by (1.2).

Definition 1.3. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{T}_\Sigma^\alpha$ if we have:

$$\left| arg\left(\frac{z^2 f'(z)}{(f(z))^2}\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{U})$$

and

$$\left| arg\left(\frac{w^2 g'(w)}{(g(w))^2}\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in \mathbb{U}),$$

where $0 < \alpha \leq 1$ and the function $g \equiv f^{-1}$ is given by (1.2).

In this paper, we obtain estimates on the initial coefficient $|a_2|$, $|a_3|$ and $|a_4|$ for functions belong to the subclasses $\mathcal{T}_\Sigma(\beta)$ and $\mathcal{T}_\Sigma^\alpha$. These results improves the corresponding results of Naik and Patil [11].

2. MAIN RESULTS

Theorem 2.1. *Let the function $f(z) \in \mathcal{T}_\Sigma(\beta)$ be of the form (1.1). Then,*

$$|a_2| \leq \begin{cases} 1, & \left(0 \leq \beta \leq \frac{1}{2}\right) \\ \sqrt{2(1-\beta)}, & \left(\frac{1}{2} \leq \beta < 1\right), \end{cases}$$

$|a_3| \leq 2(1-\beta)$ *and* $|a_4| \leq 3(1-\beta)$, *where* $0 \leq \beta < 1$.

Proof. Using Definition 1.2 we can write

$$\frac{z^2 f'(z)}{(f(z))^2} = \beta + (1-\beta)\, u(z) \tag{2.1}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = \beta + (1-\beta)\, v(w), \tag{2.2}$$

where $u(z), v(w) \in \mathcal{P}$ such that

$$u(z) = 1 + u_1 z + u_2 z^2 + u_3 z^3 + \cdots, \ (z \in \mathbb{U}) \tag{2.3}$$

and

$$v(w) = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \cdots, \ (w \in \mathbb{U}). \tag{2.4}$$

Hence we have

$$\beta + (1-\beta)\, u(z) = 1 + (1-\beta)u_1 z + (1-\beta)u_2 z^2 + (1-\beta)u_3 z^3 + \cdots$$

and

$$\beta + (1-\beta)\, v(w) = 1 + (1-\beta)v_1 w + (1-\beta)v_2 w^2 + (1-\beta)v_3 w^3 + \cdots.$$

Also, some simple calculations gives

$$\frac{z^2 f'(z)}{(f(z))^2} = 1 + \left(a_3 - a_2^2\right) z^2 + 2\left(a_2^3 + a_4 - 2a_2 a_3\right) z^3 + \cdots \tag{2.5}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = 1 - \left(a_3 - a_2^2\right) w^2 - 2\left(2a_2^3 + a_4 - 3a_2 a_3\right) w^3 + \cdots. \tag{2.6}$$

Now, equating the coefficients in (2.1) and (2.2) we get $u_1 = v_1 = 0$ and also,

$$\left(a_3 - a_2^2\right) = (1-\beta)\, u_2, \tag{2.7}$$

$$2\left(a_2^3 + a_4 - 2a_2a_3\right) = (1-\beta)\, u_3, \tag{2.8}$$

$$-\left(a_3 - a_2^2\right) = (1-\beta)\, v_2, \tag{2.9}$$

$$-2\left(2a_2^3 + a_4 - 3a_2a_3\right) = (1-\beta)\, v_3. \tag{2.10}$$

Equation (2.7) and (2.9) along with Lemma 1.1 gives

$$\left|a_3 - a_2^2\right| = \left|(1-\beta)\, u_2\right| = (1-\beta)\ |u_2| \leq 2\,(1-\beta) \tag{2.11}$$

and

$$\left|a_2^2 - a_3\right| = \left|(1-\beta)\, v_2\right| = (1-\beta)\ |v_2| \leq 2\,(1-\beta), \tag{2.12}$$

respectively. Further, adding (2.8) in (2.10), we obtain

$$2\, a_2 \left(a_3 - a_2^2\right) = (1-\beta)\ (u_3 + v_3) \tag{2.13}$$

which, on using Lemma 1.1 gives

$$\left|a_2\ \left(a_3 - a_2^2\right)\right| = |a_2| \left|a_3 - a_2^2\right| \leq 2\,(1-\beta). \tag{2.14}$$

Equations (2.11) and (2.14) together yields

$$|a_2| \leq 1. \tag{2.15}$$

Also, applying the triangle inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$ in (2.12), we obtain

$$\left|a_2^2\right| - |a_3| \leq \left|a_2^2 - a_3\right| \leq 2\,(1-\beta). \tag{2.16}$$

Clearly, it gives

$$\left|a_2^2\right| \leq 2\,(1-\beta). \tag{2.17}$$

Equation (2.15) and (2.17) together gives $|a_2| \leq min\left\{1, \sqrt{2\,(1-\beta)}\right\}$, which proves the required estimate on $|a_2|$ according to the range of β.

Now, subtracting (2.10) from (2.8), we get

$$2\left(3a_2^3 + 2a_4 - 5a_2a_3\right) = (1-\beta)\ (u_3 - v_3)\,. \tag{2.18}$$

Eliminating a_2^3 using (2.13) and (2.18) gives

$$4\,(a_4 - a_2a_3) = (1-\beta)\ (4u_3 + 2v_3) \tag{2.19}$$

which, by using Lemma 1.1 gives

$$|a_4 - a_2a_3| \leq 3\,(1-\beta). \tag{2.20}$$

Now, using the triangle inequality in equation (2.11) and (2.20), we get

$$|a_3| - \left|a_2^2\right| \leq \left|a_3 - a_2^2\right| \leq 2\,(1-\beta) \Longrightarrow |a_3| \leq 2\,(1-\beta)$$

and

$$|a_4| - |a_2a_3| \leq |a_4 - a_2a_3| \leq 3\,(1-\beta) \Longrightarrow |a_4| \leq 3\,(1-\beta),$$

respectively. This completes the proof. □

Theorem 2.2. *Let the function $f(z) \in \mathcal{T}_\Sigma^\alpha$ be of the form (1.1). Then,*

$$|a_2| \leq \begin{cases} \sqrt{2\alpha}, & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1, & \left(\frac{1}{2} \leq \alpha \leq 1\right), \end{cases}$$

$|a_3| \leq 2\alpha$ *and* $|a_4| \leq 3\alpha$, *where* $0 < \alpha \leq 1$.

Proof. Definition 1.3 implies that there exist functions $u(z), v(w) \in \mathcal{P}$ given by (2.3) and (2.4) respectively, such that

$$\frac{z^2 f'(z)}{(f(z))^2} = [u(z)]^\alpha \tag{2.21}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = [v(w)]^\alpha. \tag{2.22}$$

Clearly, we have

$$\begin{aligned}[u(z)]^\alpha =& 1 + \alpha u_1 z + \left[\alpha u_2 + \frac{\alpha(\alpha-1)}{2}u_1^2\right] z^2 + \\ & \left[\alpha u_3 + \alpha(\alpha-1)\,u_1u_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}u_1^3\right] z^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}[v(w)]^\alpha =& 1 + \alpha v_1 w + \left[\alpha v_2 + \frac{\alpha(\alpha-1)}{2}v_1^2\right] w^2 + \\ & \left[\alpha v_3 + \alpha(\alpha-1)\,v_1v_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}v_1^3\right] w^3 + \cdots .\end{aligned}$$

Observe that, by equating the coefficients in (2.21) and (2.22) we get $u_1 = v_1 = 0$ and also the following equalities

$$\left(a_3 - a_2^2\right) = \alpha u_2,$$

$$2\left(a_2^3 + a_4 - 2a_2a_3\right) = \alpha u_3,$$
$$-\left(a_3 - a_2^2\right) = \alpha v_2,$$
$$-2\left(2a_2^3 + a_4 - 3a_2a_3\right) = \alpha v_3.$$

Now, continuing as in Theorem 2.1, we can complete the further proof. □

Above two theorems shows improvements in the corresponding theorems stated and proved by Naik and Patil [11].

Remark 2.3. All the three coefficient inequalities obtained for functions belong to the subclasses $\mathcal{T}_\Sigma(\beta)$ and $\mathcal{T}_\Sigma^\alpha$ are similar along with the following relations:

(1) $(1-\beta) = \alpha \quad (0 < \alpha \le 1,\ 0 \le \beta < 1)$,

(2) $\left(0 \le \beta \le \frac{1}{2}\right) \equiv \left(\frac{1}{2} \le \alpha \le 1\right)$,

(3) $\left(\frac{1}{2} \le \beta < 1\right) \equiv \left(0 < \alpha \le \frac{1}{2}\right)$.

The above two theorems implies to the following conjecture, open to the researchers in this field.

Conjecture 2.4. *For the function $f(z) \in \mathcal{T}_\Sigma(\beta)$, $|a_n| \le (n-1)(1-\beta)$ and for the function $f(z) \in \mathcal{T}_\Sigma^\alpha$, $|a_n| \le (n-1)\,\alpha$ where $n = 3, 4, 5, \cdots$.*

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