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A Companion of Weighted Ostrowski's type Inequality and Applications to Numerical Integration

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Abstract

In the present article, we would establish generalisation of companion of Ostrowski's type integral inequality involving weights. This article recaptures the results of M. W. Alomari's article. Moreover, application is given for numerical integration.

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1. INTRODUCTION

In the development of mathematics, inequalities are one of the most powerful tools. Among these inequalities, the Ostrowski inequality is a remarkable inequality which is introduced by A. M. Ostrowski [12] in 1938 and this result had obtained by applying the Montgomery identity.

Here, we present an inequality from article [5] that is given below. Throughout the article $K \subset \mathbb{R}$ and K^o is the interior of the interval K.

Proposition 1.1. *Suppose $\rho : K \to \mathbb{R}$ is a differentiable function in the interval K^o such that $\rho' \in L[j,k]$, where $j,k \in K$ and $j < k$. If $|\rho'(\theta)| \leq \mathfrak{M} \; \forall \; \theta \in (j,k)$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left|\rho(\theta) - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it can not be replaced by the smaller one.

For the results related to Ostrowski's inequality (see [1, 2]). Also, the reader may be refer to the monograph [5] where various inequalities of Ostrowski type are discussed.

In [9], Guessab $et.al.$ have derived the following companion of Ostrowski's inequality like others.

Proposition 1.2. *Let $\rho : [j,k] \to \mathbb{R}$ be satisfies the Lipschitz condition, i.e., $|\rho(\dagger) - \rho(s)| \leq \mathfrak{M}|\dagger - s|$. Then $\forall \; \theta \in [j, \frac{j+k}{2}]$, then*

$$\left|\frac{\rho(\theta)+\rho(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right]\mathfrak{M}(k-j), \tag{1.2}$$

The constant $\frac{1}{8}$ is the best possible constant that it can not be replaced by the smaller one.

Note that the above inequality is the best due to it gives the trapezoid type inequality for $\theta = \frac{3j+k}{4}$, i.e.,

$$\left|\frac{\rho(\frac{3j+k}{4}) + \rho(\frac{j+3k}{4})}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \leq \frac{\mathfrak{M}(k-j)}{8}. \tag{1.3}$$

The constant $\frac{1}{8}$ is the sharp in above the inequality.

In [7], S. S. Dragomir has derived the following companion of the Ostrowski inequality.

Proposition 1.3. *Let $\rho : K \to \mathbb{R}$ be an absolutely continuous function on $[j,k]$. Then we have the inequalities*

$$\left|\frac{\rho(\theta)+\rho(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right|$$
$$\leq \begin{cases} \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right](k-j)\|\rho'\|_\infty, \quad \rho' \in L_\infty[j,k], \\ \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\theta-j}{k-j}\right)^{q+1} + \left(\frac{\frac{j+k}{2}-\theta}{k-j}\right)^{q+1}\right]^{\frac{1}{q}}(k-j)^{\frac{1}{q}}\|\rho'\|_{[j,k],p}\ , \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ and\ \rho' \in L_p[j,k], \\ \\ \left[\frac{1}{4} + \left|\frac{\theta - \frac{3j+k}{4}}{k-j}\right|\right]\|\rho'\|_{[j,k],1}\ , \end{cases} \tag{1.4}$$

$\forall\, \theta \in [j, \frac{j+k}{2}]$.

In 2002, S. S. Dragomir [6] established some inequalities for this companion for mappings of bounded variation. In 2009, Z. Liu [10] introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. In 2009, Barnett *et. al* [4] have derived some companions for Ostrowski inequality and the generalised trapezoid inequality. In 2011, M. W. Alomari [3] obtained the companion of Ostrowski inequality (1.3) for differentiable bounded functions and also gave the applications.

In the present article we would prove a companion of weighted Ostrowski's type inequality for differentiable bounded functions and then we would give its applications.

2. GENERALISATION OF COMPANION OF OSTROWSKI'S TYPE INEQUALITY

Under present section we would give our results about companion of Ostrowski's type inequality which are as follow:

Theorem 2.1. *Let $\rho : [j,k] \to \mathbb{R}$ be a differentiable function in the interval (j,k) and $j < k$ and $w : [j,k] \to \mathbb{R}$ is an integrable function. If $\rho' \in L^1[j,k]$ and*

$m_1 \leq \rho'(\dagger) \leq M_1$, *for all* $\dagger \in [j,k]$, *then*

$$\left|\rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k \rho(\dagger)w(\dagger)d\dagger\right|$$
$$\leq \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\dagger} w(u)du\right)d\dagger + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\dagger} w(u)du\right)d\dagger - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right.$$
$$+\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\dagger} w(u)du\right)d\dagger$$
$$\left. - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2} \tag{2.1}$$

holds $\forall\ \theta \in [j, \frac{j+k}{2}]$.

Proof. For the sake of proof we state the weighted kernel as;

$$P(\theta,\dagger) = \begin{cases} \displaystyle\int_j^{\dagger} w(u)du, & \text{if} \quad \dagger \in [j,\theta], \\ \displaystyle\int_{\frac{j+k}{2}}^{\dagger} w(u)du, & \text{if} \quad \dagger \in (\theta, j+k-\theta], \\ \displaystyle\int_k^{\dagger} w(u)du, & \text{if} \quad \dagger \in (j+k-\theta, k], \end{cases}$$

$\forall\ \theta \in [j, \frac{j+k}{2}]$.

Applying by parts formula of integration, obtain

$$\int_j^k P(\theta,\dagger)\rho'(\dagger)d\dagger = \rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k \rho(\dagger)w(\dagger)d\dagger. \tag{2.2}$$

We know that

$$\int_j^k P(\theta,\dagger)d\dagger = 0. \tag{2.3}$$

Let $C = \frac{M_1+m_1}{2}$. From (2.2) and (2.3) it follows

$$\int_j^k P(\theta,\dagger)[\rho'(\dagger)-C]d\dagger = \rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k \rho(\dagger)w(\dagger)d\dagger. \tag{2.4}$$

Another way we have

$$\left|\int_j^k P(\theta,\dagger)[\rho'(\dagger)-C]d\dagger\right| \le \max_{\dagger\in[j,k]}|\rho'(\dagger)-C|\cdot\int_j^k |P(\theta,\dagger)|d\dagger. \tag{2.5}$$

Since

$$\max_{\dagger\in[j,k]}|\rho'(\dagger)-C| \le \frac{M_1+m_1}{2} \tag{2.6}$$

and

$$\begin{aligned}\int_j^k |P(\theta,\dagger)|d\dagger &= \int_j^{\frac{j+\theta}{2}}\left(\int_j^\dagger w(u)du\right)d\dagger + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\dagger w(u)du\right)d\dagger \\ &- \int_\theta^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^\dagger w(u)du\right)d\dagger + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^\dagger w(u)du\right)d\dagger \\ &- \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^\dagger w(u)du\right)d\dagger - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^\dagger w(u)du\right)d\dagger. \end{aligned} \tag{2.7}$$

Now from (2.5) to (2.7), it follows that

$$\begin{aligned}&\left|\rho(\theta)\int_j^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(j+k-\theta)\int_{\frac{j+k}{2}}^k w(\dagger)d\dagger - \int_j^k \rho(\dagger)w(\dagger)d\dagger\right| \\ &\le \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^\dagger w(u)du\right)d\dagger + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\dagger w(u)du\right)d\dagger - \int_\theta^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^\dagger w(u)du\right)d\dagger\right. \\ &+ \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^\dagger w(u)du\right)d\dagger - \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^\dagger w(u)du\right)d\dagger \\ &\left.- \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^\dagger w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2},\end{aligned}$$

$\forall\, \theta \in [j, \frac{j+k}{2}]$. □

Remark 2.2. *If put* $w = \frac{1}{k-j}$ *in Theorem* 2.1, *then we recapture the Theorem 4 of [3].*

Corollary 2.3. *In the inequality* (2.1), *select*

(i) $\theta = \frac{3j+k}{4}$, *obtain*

$$\left|\rho(\frac{3j+k}{4})\int_{j}^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(\frac{j+3k}{4})\int_{\frac{j+k}{2}}^{k} w(\dagger)d\dagger - \int_{j}^{k} w(\dagger)\rho(\dagger)d\dagger\right|$$
$$\leq \left[\int_{j}^{\frac{7j+k}{8}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger + \int_{\frac{7j+k}{8}}^{\frac{3j+k}{4}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right.$$
$$+\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{j+3k}{4}}^{\frac{j+7k}{8}}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger$$
$$\left.-\int_{\frac{j+7k}{8}}^{k}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2}, \tag{2.8}$$

(ii) $\theta = \frac{2j+k}{3}$, *obtain*

$$\left|\rho(\frac{2j+k}{3})\int_{j}^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(\frac{j+2k}{3})\int_{\frac{j+k}{2}}^{k} w(\dagger)d\dagger - \int_{j}^{k} w(\dagger)\rho(\dagger)d\dagger\right|$$
$$\leq \left[\int_{j}^{\frac{5j+k}{6}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger + \int_{\frac{5j+k}{6}}^{\frac{2j+k}{3}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{2j+k}{3}}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right.$$
$$+\int_{\frac{j+k}{2}}^{\frac{j+2k}{3}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{j+2k}{3}}^{\frac{j+5k}{6}}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger$$
$$\left.-\int_{\frac{j+5k}{6}}^{k}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2}, \tag{2.9}$$

(iii) $\theta = \frac{j+k}{2}$, *obtain*

$$\left|\rho(\frac{j+k}{2})\int_{j}^{k} w(\dagger)d\dagger - \int_{j}^{k} w(\dagger)\rho(\dagger)d\dagger\right|$$
$$\leq \left[\int_{j}^{\frac{3j+k}{4}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger + \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\left(\int_{j}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger\right.$$
$$\left.-\int_{\frac{j+3k}{4}}^{k}\left(\int_{k}^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2}. \tag{2.10}$$

(iv) $\theta = j$, *obtain*

$$\left|\rho(j)\int_{j}^{\frac{j+k}{2}} w(\dagger)d\dagger + \rho(k)\int_{\frac{j+k}{2}}^{k} w(\dagger)d\dagger - \int_{j}^{k} \rho(\dagger)w(\dagger)d\dagger\right|$$
$$\leq \left[\int_{\frac{j+k}{2}}^{k}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_{j}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2}. \tag{2.11}$$

In the following we present special cases of above corollary.

Special Case 1: If put $w = \frac{1}{k-j}$ in (ii) of Corollary 2.3, then we get

$$\left|\frac{\rho(\frac{2j+k}{3}) + \rho(\frac{j+2k}{3})}{2} - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \le \frac{5(k-j)}{72}(M_1 + m_1).$$

Special Case 2: If put $w = \frac{1}{k-j}$ in (iii) of Corollary 2.3, then we get midpoint inequality

$$\left|\rho(\frac{j+k}{2}) - \frac{1}{k-j}\int_j^k \rho(\dagger)d\dagger\right| \le \frac{(k-j)}{4}(M_1 + m_1).$$

Remark 2.4. *(i) By putting $w = \frac{1}{k-j}$ in (i) of Corollary 2.3, we recapture the Corollary 1 of [3].*

(ii) By putting $w = \frac{1}{k-j}$ in (iv) of Corollary 2.3, we recapture the Corollary 2 of [3].

Ostrowski's type inequality can be defined in the form of following corollary.

Corollary 2.5. *Let the assumptions of Theorem 2.1 be valid. Further, if ρ is symmetric about the θ-axis, i.e., $\rho(j+k-\theta) = \rho(\theta)$, then*

$$\begin{aligned}
&\left|\rho(\theta)\int_j^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \\
&\le \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\dagger} w(u)du\right)d\dagger + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\dagger} w(u)du\right)d\dagger - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right. \\
&+ \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\dagger} w(u)du\right)d\dagger \\
&\left.- \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2} \qquad (2.12)
\end{aligned}$$

holds $\forall\, \theta \in [j, \frac{j+k}{2}]$.

Remark 2.6. *In Corollary 2.5, select $\theta = j$, then obtain*

$$\begin{aligned}
&\left|\rho(j)\int_j^k w(\dagger)d\dagger - \int_j^k w(\dagger)\rho(\dagger)d\dagger\right| \\
&\le \left[\int_{\frac{j+k}{2}}^{k}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger - \int_j^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\dagger} w(u)du\right)d\dagger\right]\frac{(M_1+m_1)}{2}
\end{aligned}$$

Remark 2.7. *By putting $w = \frac{1}{k-j}$ in Corollary 2.5, we recapture the Corollary 3 of [3].*

3. APPLICATION TO NUMERICAL INTEGRATION

Let $K_n : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ be a division of the interval $[j, k]$ and $h_i = \theta_{i+1} - \theta_i$, $(i = 0, 1, 2, \cdots, n-1)$.

Consider the quadrature formula

$$Q_n(K_n, \rho) := \sum_{i=0}^{n-1} \left[\rho(\frac{3\theta_i + \theta_{i+1}}{4}) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger + \rho(\frac{\theta_i + 3\theta_{i+1}}{4}) \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger \right]. \tag{3.1}$$

We give following result.

Theorem 3.1. *Let $\rho : K \to \mathbb{R}$ be a differentiable function in the interval K^o and $w : [j, k] \to \mathbb{R}$ is an integrable function, where $j, k \in K$ with $j < k$. If $\rho' \in L^1[j, k]$ and $m_1 \leq \rho'(\theta) \leq M_1$, for all $\theta \in [j, k]$, then the following holds*

$$\int_j^k w(\dagger)\rho(\dagger)d\dagger = Q_n(K_n, \rho) + R_n(K_n, \rho), \tag{3.2}$$

where $Q_n(K_n, \rho)$ is stated as above and the following remainder $R_n(K_n, \rho)$ satisfies the estimates

$$\begin{aligned} |R_n(K_n, \rho)| \leq \frac{(M_1 + m_1)}{2} \sum_{i=0}^{n-1} \Bigg[& \int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}} \left(\int_{\theta_i}^{\dagger} w(u)du \right) d\dagger \\ & + \int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}} \left(\int_{\theta_i}^{\dagger} w(u)du \right) d\dagger - \int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}} \left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\dagger} w(u)du \right) d\dagger \\ & + \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}} \left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\dagger} w(u)du \right) d\dagger - \int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}} \left(\int_{\theta_{i+1}}^{\dagger} w(u)du \right) d\dagger \\ & - \int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}} \left(\int_{\theta_{i+1}}^{\dagger} w(u)du \right) d\dagger \Bigg]. \end{aligned} \tag{3.3}$$

Proof. Applying inequality (2.8) on the intervals $[\theta_i, \theta_{i+1}]$, we get

$$\begin{aligned} R_i(K_i, \rho) &= \int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \left[\rho(\frac{3\theta_i + \theta_{i+1}}{4}) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger \right. \\ &+ \left. \rho(\frac{\theta_i + 3\theta_{i+1}}{4}) \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger \right]. \end{aligned}$$

Summing (3.4) over i from 0 to $n-1$, then

$$\begin{aligned} R_n(K_n,\rho) &= \sum_{i=0}^{n-1}\int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \sum_{i=0}^{n-1}\left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger\right. \\ &+ \left.\rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right], \end{aligned}$$

which follows the form of (2.8), i.e.

$$\begin{aligned} |R_n(K_n,\rho)| &= \left|\sum_{i=0}^{n-1}\int_{\theta_i}^{\theta_{i+1}} w(\dagger)\rho(\dagger)d\dagger - \sum_{i=0}^{n-1}\left[\rho(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\dagger)d\dagger\right.\right. \\ &+ \left.\left.\rho(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\dagger)d\dagger\right]\right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{(M_1+m_1)}{2}\sum_{i=0}^{n-1}\left[\int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}}\left(\int_{\theta_i}^{\dagger} w(u)du\right)d\dagger + \int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}}\left(\int_{\theta_i}^{\dagger} w(u)du\right)d\dagger\right. \\ &- \int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\dagger} w(u)du\right)d\dagger + \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\dagger} w(u)du\right)d\dagger \\ &- \left.\int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}}\left(\int_{\theta_{i+1}}^{\dagger} w(u)du\right)d\dagger - \int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}}\left(\int_{\theta_{i+1}}^{\dagger} w(u)du\right)d\dagger\right]. \end{aligned}$$

This completes the required proof. □

Remark 3.2. *By putting* $w = \frac{1}{k-j}$ *in Theorem* 3.1*, we recapture the result of Theorem 5 of [3].*

4. CONCLUSION

In this article, our aim was to generalise the results of [3]. We have obtained generalisation of companion of Ostrowski's type integral inequality involving weights. By using suitable substitutions we have recaptured the results of M. W. Alomari's article and given some special cases. Further, we have deduced application to numerical integration.

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