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ON $(N,p,q)(C,\alpha)$ Summability of Product Series

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Abstract

Summability is a branch of mathematical analysis in which an infinite series which is usually divergent can converge to a finite sum s (say) by ordinary summation techniques and become summable with the help of diff erent summation means or methods.C method was given by Ernesto Cesaro such that ordinary Cesaro summation was written as (C,1) summation whereas generalised Cesaro summation was given as (C,α). In 1913, Hardy [1] proved a theorem on (C, a), a > 0 summability of the series. Prasad and Siddiqi [2], J. M. Hyslop [3] studied (N, p. n) summability of derived series of a Fourier series. In 1959, Varshney [4] for the first time studied the sequence {n Bn(x)} by product summability means of form (H, 1) (C, 1). Here (N, p, q) C, summability of derived series of a Fourier series is considered. (N, p, q) C, summability reduces to (N, p,n) Cj if qn=l for every n and to (N, qn) Cj if pn=l for every n. But till now no work seems to have been done so far in the direction of study of derived series of a Fourier series by product summability means of the form (N, p, q) C,. In an attempt to make an advance study in this direction, in present chapter, a theorem on (N, p, q) (C,α) summability of derived series of a Fourier series has been obtained under a very general condition.

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1. INTRODUCTION

Hardy [1], introduced the summability method (C,α) for functions and investigated some of its properties . Prasad and Siddiqi [2], J.M. Hyslop [3] studied (N, p,n) summability of derived series of a Fourier series. Pathak [10], defined the product summability method $(D,k)(C,l)$ for functions and investigated some of its properties . Likewise product summability (N, p,q) (C,α) for functions has been defined for ($p > 0$, $\alpha > 0$, $q > -1$) and investigated some of its properties.

2. SOME RELATIONS AND DEFINITIONS

We would like to first introduce Summability method. Summability method is more general than that of ordinary convergence. If we are given a sequence (s_n), we can construct a generalized sequence (σ_n), the arithmetic mean of (s_n) by this sequence (s_n). If (σ_n) is convergent in ordinary sense for all $n > 0$, then we say that (s_n) is summable $(C,1)$ to the sum s . This $(C,1)$ is called Cesaro mean of first order.

If $s_n \to s \Rightarrow \sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots\ldots\ldots+s_n}{n+1} \to s$, ie if a sequence is convergent, it is summable by method of arithmetic mean. Also a series $1-1+1+1+\cdots\ldots\ldots$ is not convergent , but is summable to the sum $\frac{1}{2}$. The space of summable sequences is larger than space of convergent sequences. If $\sigma_n \to s$ as $n \to \infty$, then we say that sequence (s_n) is summable by method of arithmetic mean.

For example : Consider the series $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \cdots\ldots\ldots..$ (1)

And let $\sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots\ldots\ldots+s_n}{n+1}$, It may happen that whereas (1) diverges , the quantities (the arithmetic mean of partial sum of series) converges to a definite limit as $n \to \infty$. For example $1-1+1-1+\cdots\ldots\ldots$ diverges, but in this case $s_0 = 1$, $s_1 = 1-1 = 0$, $s_2 = 1-1+1 = 1$, $s_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$(s_n) = (1,0,1,0,1\ldots\ldots..)$. Since $s_n = \frac{1+(-1)^n}{2}$,

$$\sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots\ldots\ldots+s_n}{n+1}$$

$$= \frac{1+(-1)^0}{2} + \frac{1+(-1)^1}{2} + \frac{1+(-1)^2}{2} + \cdots\ldots\ldots\ldots\ldots.. + \frac{1+(-1)^n}{2} / (n+1)$$

$$= \frac{(n+1)}{2} + \frac{1}{2}\{1-1+1-\cdots\ldots\ldots\ldots(n+1)terms\}/(n+1)$$

$= \frac{1}{2} + \frac{1+(-1)^n}{4(n+1)}$, If n is even then $\sigma_n = \frac{1}{2} + \frac{1}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$ and if n is odd then $\sigma_n = \frac{1}{2}$. So in either case $\lim_{n\to\infty} \sigma_n = \frac{1}{2}$, $\therefore$ $s_n \notin C$ but $s_n \epsilon S$. Therefore space of summable sequences is larger than thar of space of convergent sequences .

In mathematical analysis, Cesàro summation assigns values to some infinite sums that are not convergent in the usual sense, while coinciding with the standard sum if they are convergent. The Cesàro sum is defined as the limit of the arithmetic mean of the partial sums of the series.

Regularity of a Method: A summability method *M* is regular if it agrees with the actual limit on all convergent series .

Cesàro method is always regular.

Cesàro summation is named for the Italian analyst Ernesto Cesàro (1859–1906).

On the other hand, now let $a_n = n$ for $n \geq 1$. That is, $\{a_n\}$ is the sequence

$1, 2, 3, 4, \ldots.$ and let *G* now denote the series

$$\sum_{n=1}^{\infty} a_n = 1 + 2 + 3 + 4 + 5 + \cdots$$

Then the sequence of partial sums $\{s_n\}$ is

$$1, 3, 6, 10, \ldots,$$

and the evaluation of *G* diverges to infinity. The terms of the sequence of means of partial sums $\{t_n\}$ are here

$$\frac{1}{1}, \frac{4}{2}, \frac{10}{3}, \frac{20}{4}, \ldots.$$

Thus, this sequence diverges to infinity as well as *G*, and *G* is now not Cesàro summable. In fact, any series which diverges to (positive or negative) infinity the Cesàro method also leads to a sequence that diverges likewise, and hence such a series is not Cesàro summable.

(C, α) summation :

In 1890, Ernesto Cesàro stated a broader family of summation methods which have since been called (C, α) for non-negative integers α. The (C, 0) method is just ordinary summation, and (C, 1) is Cesàro summation as described above.

The higher-order methods can be described as follows: given a series Σa_n, define the quantities

$$A_n^{-1} = a_n; \quad A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}$$

(where the upper indices do not denote exponents) and define E_n^α to be A_n^α for the series 1 + 0 + 0 + 0 + · · ·. Then the (C, α) sum of Σa_n is denoted by (C, α) - Σa_n and has the value

$$(C,\alpha) - \sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \frac{A_n^\alpha}{E_n^\alpha}$$

if it exists (Shawyer & Watson 1994, pp.16-17). This description represents an α-times iterated application of the initial summation method and can be restated as

$$(C,\alpha) - \sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+\alpha}{j}} a_j$$

Even more generally, for $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$, let A_n^α be implicitly given by the coefficients of the series

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} a_n x^n}{(1-x)^{1+\alpha}},$$

and E_n^α as above. In particular, E_n^α are the binomial coefficients of power −1 − α. Then the (C, α) sum of Σ a_n is defined as above. The generalized (N, p, q) transform T n,p,q of the (E, 1) transform t. n of the sequence {Sn} of partial sums of the series £u, n , i.e., the (N, p, q) S (E, 1) transform of {Sn} is given by, following Borwein (1958) [7]. If Tn,M ->S, as n -> o, then the series £u n or the sequence {Sn} of its partial sums is said to be summable (N, p, q) (E, 1) to the sum S.

If Σa_n has a (C, α) sum, then it also has a (N,p, q) sum for every β>α, and the sums agree; furthermore we have $a_n = o(n^\alpha)$ if α > −1.

3. MAIN RESULTS

In this section, we have the following theorems on the relative strength between $|\mathrm{N}, p, q|_p$ and $|(\mathrm{N}, p, q)(C, \alpha)|_p$.

Theorem 3.1: Let $\alpha > \gamma \geq 0, p \geq 1, \beta > -1$. If $f(x)$ is summable$|\mathrm{N}, p, q|_p$, then it is summable$|\mathrm{N}, p, q|1$.

To prove this theorem, we require proof of following Lemma.

Lemma 3.1: Let $p \geq 1, \gamma > 1$. Suppose that $f(x) \in L(0,x)$ for finite $x > 0$ .Suppose that $f(x) \in |C,\gamma,\beta|_p$,according to the definition (2.5).

Define $$\bar{f}(x)=\begin{cases} f(x) & for\ x\geq T \\ 0 & for\ x<T \end{cases} \tag{3.1}$$

Let $\bar{\partial}_{\gamma,\beta}(y)$ denote the expression corresponding to $\partial_{\gamma,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$.

Then $$\int_0^\infty y^{p-1}\left|\frac{d}{dy}\partial_{\gamma,\beta}(y)\right|^p dy<\infty. \tag{3.2}$$

Thus $\bar{f}(x)$ is summable $|C,\gamma,\beta|_p$ under the definition (2.7).(By a result Mishra and Mishra [8]).

Proof of Lemma 3.1:

$$\frac{d}{dy}U_{k,\alpha,\beta}(y)=C\int_0^\infty \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)x^{\alpha+\beta}\partial_{\alpha,\beta}(x)dx. \tag{3.11}$$

Integrating (3.11) by parts k times,we deduce with the help of (3.12) that

$$\frac{d}{dy}U_{k,\alpha,\beta}(y)=(-1)^k C\int_0^\infty x^{\alpha+\beta+k}\partial_{\alpha+k,\beta}(x)\left\{\frac{d^k}{dx^k}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right]\right\}dx. \tag{3.12}$$

It is verified that the expression in curly brackets (3.16) is $o\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}\right)$. (3.13)

Let $R(x,y)=\int_0^x t^{\alpha+\beta+k}\frac{d^k}{dx^k}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt.$

In fact, for fixed $k>0$, we have uniformly in $x>0, y>0$,

$$R(x,y)=0\left(\frac{x^k}{(x+y)^{k+1}}\right) \quad . \qquad (3.14)$$

This may be proved by induction on k, if $k=0$,we have

$$R(x,y)=\int_0^x t^{\alpha+\beta}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt$$

$$=\frac{x^k}{(x+y)^{k+1}} \quad ,$$

hence the result is evident. Suppose that $k \geq 1,$ and assume the result true for $k-1.$ Integrating by parts ,we have

$$R(x,y) = x^{\alpha+\beta+k} \frac{d^{k-1}}{dx^{k-1}}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right] - (\alpha+\beta+k)\int_0^x t^{\alpha+\beta+k+1} \frac{\partial^{k-1}}{\partial t^{k-1}}\left\{\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right\}dt.$$

Using (3.12) and putting $x = t + 3y$, we see that the expression in curly brackets

$$\leq C\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}dx = \frac{C}{y}\int_0^x \frac{t^{k-1}}{(1+t)^{k+1}}dt = \frac{C}{y} \quad ,$$

Again using (3.18) , the inner integral

$$\leq C\, x^k \int_0^\infty \frac{1}{(x+y)^{k+1}}\, dy \ , \tag{3.15}$$

on putting $y = xt$, the expression on the right of (3.19) is equal to

$$C\int_0^\infty \frac{1}{(1+t)^{k+1}}\, dt = C$$

Since we are supposing that k is an integer, we have k =1 in the last clause of the theorem.

Now also
$$\partial_{\alpha+1,\beta}(x) = \partial_{\alpha+1,\beta}(1) + \int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right) dx\,. \tag{3.16}$$

By Holder's inequality with indices p and q , we have

$$\left|\int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right) dx\right| \leq \left(\int_1^x x^{p-1}\left|\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right|^p dx\right)^{\frac{1}{p}} \left(\int_1^x \frac{1}{x}dx\right)^{\frac{1}{q}}. \tag{3.17}$$

$$= O(\log x)^{\frac{1}{q}}\ . \tag{3.18}$$

From (3.15), (3.16) and (3.17), we see that

$$\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx \text{ is convergent .}$$

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