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Bayesian Estimation Under Different Loss Functions in Competitive Risks

Didier Alain NJAMEN NJOMEN[1], **Thiery DONFACK**[1]
Donald WANDJI TANGUEP[2]

[1]*University of Maroua, Faculty of Sciences, Department of Mathematics and Computer Sciences, P.O. Box 814, Maroua-Cameroon.*
[2]*University of Maroua, Department of Mathematics, Higher Teachers' Training College, P.O. Box 55 Maroua-Cameroon.*

Abstract

In this article, in taking up the work of Wu et al. (2016) on the Bayesian estimation of a parameter of the competitive risk model, we obtain new estimators which generalize the various existing loss functions : generalized quadratic, entropy and DeGroot in the case of the Gompertz distribution as well as their estimators of the survival function.
Keywords : Bayesian estimation, Gompertz distribution, Type-I progressively censoring, Maximum likelihood estimation.

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1. INTRODUCTION AND BACKGROUND

The Gompertz distribution model is used in survival analysis (Somda and al. 2012 ; Somda, 2015), in aging biology (Witten and Satzer, 1992), demography and engineering (Bergstrom and Bagnoli, 2004). This distribution incorporates the notion of competitive risks which highlights other competitive events that prevent observing the event of interest. Several different methods of estimating the unknown parameters of this model have been presented such as maximum likelihood estimation (Lenart, 2014), Bayesian estimation, hierarchical bayesian estimation (Lindley and Smith, 1972 ; Berger, 1985 ; Robert, 1992, 2001 ; Wang and Wand, 2004 ; Yousefzadeh and Hadi, 2016). However, the latter method encounters a complicated integration and the complexity of the integration is very difficult to put into practice.

However, we asked ourselves whether under the same conditions of the competitive risk model as defined by Wu and al (2016), and under other loss functions (entropy loss functions, generalized quadratic loss and DeGroot) the result remained unchanged ? It is in this spirit that this article is written.

In this article, we assume that the survival time X is a positive or zero variable, and absolutely continuous.Instead of observing independent and identically distributed realizations (i.i.d) of duration X, we observe the realization of the variable X subjected to various disturbances independent or not of the studied phenomenon. In the presence of right-hand random censorship, the lifetimes are not all observed ; for some of them, we only know that they are greater than a certain known value. There are several types of censorship : Type I, II, and III censorship. The reader interested in the notions of censorship can consult the works of Andersen and al., (1993) ; Fleming and Harringthon, (1991). Type I censorship describes the situation where a test ends at a certain period and we know that the remaining individuals have not yet been observed. In this case, the censorship time is fixed in advance and the number of individuals not having been observed is a random variable. Let C be a fixed value, instead of observing the variables $X_1, \cdots, X_n$, we observe X_i when $X_i \leq C$ if not, we know that $X_i > C_i$. We use the following notation $T_i = X_i \wedge C_i = \min(X_i, C_i)$, with $i = 1, \cdots, n$.

In the case of simple type I censorship, all individuals are censored after the same date of time while in the case of progressive type I censorship which is used in this article, all individuals are censored on the same date whatever the length of time they were followed. Progressively censored data type I were first proposed by Kundu and al. (2006). Indeed, progressively censored type I data is described as follows : suppose that n units are wagered in a progressive life-test censorship scheme : $(R_1, R_2, \cdots R_r), 1 \leq r \leq n$. The experiment is finished on the date τ where $\tau \in (0, \infty)$, R_i $(i = 1, 2....R)$ and r is fixed in advance. At the time of the first failure t_1, R_1 of the remaining units are randomly removed, at the time of the second failure t_2, R_2 of the remaining units are randomly removed and so on. If the Rth failure time t_R occurs before time τ, all the remaining units $R_R = n - R - (R_1 + ... + R_{R-1})$ are removed and the terminal time of the experiment is t_R. On the other hand, if the Rth failure time t_R does not occur before time τ and only J failures occur before time τ, where $0 \leq J \leq R$. Then at the time τ, all the remaining R^*_τ units are removed, where $R^*_\tau = n - J - (R_1 + ... + R_J)$ and the terminal time of the experiment is τ. We denote the two cases as :

— Case 1

$$t_1 < t_2 < ..... < t_R, \quad t_R < \tau;$$

— Case 2

$$t_1 < t_2 < ..... < t_J < \tau < t_{J+1} < ... < t_R, \quad t_R > \tau.$$

2. LOSS FUNCTIONS

The term loss function was first used by Wald (1939). It represents some measure of the difference between the observed values of the data and the values calculated using the adjustment function. This is the function that is minimized in the procedure of fitting a model (Boudjerda Khawla, 2017).

2.1. Quadratic loss function

The quadratic loss function proposed by Legendre (1805) and Gauss (1810) is defined by :

$$L(\theta, d) = (\theta - d)^2.$$

A variant of this loss function is a weighted loss function (**generalized quadratic loss function**) defined by :

$$L(\theta, d) = w(\theta)(\theta - d)^2.$$

Under the assumption of a quadratic cost, the Bayes estimator $\delta^\pi(x)$ of θ associated with the prior distribution π is the posterior mean of θ is defined by :

$$\delta^\pi(x) = E_\pi(.|x)(\theta) = \int_{\theta\in\Theta} L(\theta, \delta(x))\pi(\theta|x)d\theta.$$

Indeed, the Bayes estimator minimizes the a posteriori cost, that is to say :

$$\rho(\pi, \delta) = E^{\pi(.|x)}(L(\theta, \delta(x))).$$

Under the assumption of a quadratic cost, we have :

$$\begin{aligned} \rho(\pi, \delta) &= E^{\pi(.|x)}(\theta - \delta(x))^2; \\ &= E^{\pi(.|x)}(\theta)^2 - 2\delta(x)E^{\pi(.|x)}(\theta) + \delta(x)^2. \end{aligned}$$

It is therefore a second degree polynomial in $\delta(x)$. It will be minimum in $E^{\pi(.|x)}(\theta)$.

2.2. Linex loss function

A very handy asymmetric loss function is the Linex loss function. It was introduced by Varian (1975). This almost exponential loss function on one side of zero under the assumption that the minimum loss is obtained for $\widehat{\delta}(x) = \theta$. Let a be a real number other than zero, then linex's loss function for θ is defined by :

$$L(\Delta) \propto \exp(a\Delta) - a\Delta - 1, \quad a \neq 0,$$

oï¿½ $\Delta = (\delta(x) - \theta)$ and $\delta(x)$ is an estimator of θ.
The sign of a representing respectively the direction and the degree of symmetry ($a > 0$, the overestimation is more serious than the underestimation and vice versa). For a close to zero, the loss function of linex is approximately the quadratic loss function :

$$E_\theta(L(\delta(x) - \theta)) \propto \exp(a\delta(x))E_\theta(\exp(-a\theta)) - a(\delta(x) - E_\theta(\theta) - 1), \quad (1*)$$

where $E_\theta(.)$ represents the posterior expectation relative to the posterior density of θ. To find the Bayes estimator $\delta_{\ pi}(x)$ which minimizes $(1*)$, we driff the equation $(1*)$ with respect to $\delta(x)$ and we get

$$\frac{d}{d\delta(x)}(E_\theta(L(\delta(x) - \theta))) = a\exp(a\delta(x))E_\theta(\exp(-a\theta)) - a.$$

By equaling this expression to zero, we get :

$$\exp(-a\delta(x))E_\theta(\exp(-a\theta)) = a.$$

Then, the Bayes estimator $\widehat{\delta_L}(x)$ under the Linex loss function is :

$$\delta(x) = \frac{-1}{a}\log(E_\theta(\exp(-a\theta)),$$

given that $E_\theta(\exp(-a\theta)$ exists and is finite.

2.3. DeGroot loss function

DeGroot (1970) a introduit plusieurs types des fonctions de perte et il obtient les estimateurs de Bayes sous cette fonction de perte . Un exemple de fonction de perte symï¿½trique est la fonction de perte de DeGroot dï¿½finie par :

$$L(\theta, \delta(x)) = \left(\frac{\theta - \delta(x)}{\delta(x)}\right)^2.$$

Sous cette fonction de perte, l'estimateur de Bayes est dï¿½finie par :

$$\delta_\pi(x) = \frac{E_\pi(\theta^2|x)}{E_\pi(\theta|x)}.$$

2.4. Entropy loss function

Galabria and Pulcini (1994) proposed a loss function which derives from the Linex loss function called the entropy loss function. It is defined by :

$$L_E(\theta, d) \propto \left(\frac{d}{\theta}\right)^p - p \ln\left(\frac{d}{\theta}\right) - 1.$$

This entropy loss function is minimal when $d = \theta$.
The Bayes estimator of parameter θ under this loss function is defined by :

$$\delta(x) = \left(E_\theta(\theta)^{-p}\right)^{\frac{-1}{p}} .$$

— When $p = 1$, the Bayes estimator coincides with the Bayes estimator under the weighted quadratic loss function $\frac{(\theta-d)^2}{\theta}$.

— When $p = -1$, the Bayes estimator coincides with the Bayes estimator under the quadratic loss function.

3. BAYESIAN ESTIMATION UNDER DIFFERENT LOSS FUNCTIONS

In many random experience situations, it seems reasonable to imagine that the practitioner has some idea of the random phenomenon he is observing. However, the classic approach is based essentially on a principle of likelihood, that is to say that it considers that what has been observed gives an exhaustive account of the phenomenon. However, observation only gives a glimpse and this can be bad. Nevertheless, this problem is solved by a certain number of theorems allowing to evaluate the good quality of the estimators if the number of observations is sufficient.
The Bayesian analysis of statistical problems introduces into the inference process the information which the practitioner has a priori. In the context of parametric statistics, this results in the choice of a law on the parameter of interest. The Bayesian approach differs from the classical approach in the sense that the unknown is no longer considered to be completely unknown, it has become a random variable whose behavior is assumed to be known. we do to intervene in the statistical analysis a distribution associated with a parameter. In our case, the Gamma distribution will be associated with the unknown parameter β_k.
In this article, we will estimate the unknown parameter β_k and the survival function $s_k(t)$ under different loss functions.

3.1. Competitive risks

3.1.1 Introduction

In survival analyzes, a competitive risk (or competing risk) is an event that prevents observation of the event of interest; most often called death. The concept of competing risks or competitive risks arose in the $18^{\textbf{rd}}$ century, when Daniel Bernoulli studied the impact of smallpox eradication on death rates in England. More and more, this subject is the subject of much debate nowadays for estimating the probability of a particular event in the presence of other events, or after the modification or elimination of another event. In some cases, a subject may be at risk of undergo experiencing several different events : the events are in competition with each other; we are then in a situation of competitive risks. In demography, the competitive risk situation is observed to study the probability of a couple to marry or to live in cohabitation. In medical research, which is the field that interests us the most, this situation is also common in various fields such as in gynecology, in infectiology, in cancerology ..., to name just a few. In gynecology, it is used to study the probability of giving birth by natural route or by cesarean section (birth by natural route is then in competition with cesarean section); in infectiology, to study the probability of dying or of contracting a nosocomial infection (death is here in competes with nosocomial infection, that is to say the infection caught during a hospital stay ;in cancerology, to study the probability of recurrence of cancer or of dying from it (here, the death being in competition with the recurrence). Several authors have used distributions in the presence of competitive risks : Gompertz distributions (Somda and al., 2014; Somda, 2015), exponential distribution (Mao and al., 2014), Lindley distribution (Mazucheli and al., 2011), stochastic process (Njamen and Ngatchou, 2014).

3.1.2 Functions used in competitive risks

An examination of the literature has shown that for a given subject, at most one event denoted by $\delta(k)$ among K events will be observed. If no event occurs, then the subject is censored at the end of its tracking ($\delta_k = 0$). Most often, the various competing events are causes of death or failure. For example, a subject may die due to cancer, a heart problem, an accident, ... The use of competitive risks is also necessary when an event prevents observation of the event of interest. For example, death may occur before one observes a relapse of a disease. In summary, competitive risk models are used when the occurrence of another event modifies the probability of observing the event of interest. In practice, we fix only one event of interest ($\delta_k = 1$) among the possible Ks.
To analyze the data in the context of competitive risks, two types of probabilities can be defined (see Tsiatis, 1975) :

— The raw probability of events of type k in the presence of other event risks, also known as cumulative incidence function (CIF), denoted $F_k(k = 1; 2; ...; K)$. This function is given by :

$$F_k(t) = P[T \leq t, \delta_k = 1], \tag{1}$$

where T is the time between the diagnosis and the occurrence of the first event and $\delta_k = 1$ the indicator of the type of event.

— The raw probability of events of type k, in the situation where only this risk would act on the population.

In the context of concurrent risks, the central object is the cause-specific risk function of events of type k, which is interpreted as the probability of occurrence of the event of type k in an infinitesimal interval , knowing that this event has not yet occurred at the start of the interval.

In this article, we use the same conditions as Wu et al. (2016), i.e, it is assumed that :

— there are K independent concurrent failure modes ;

— a system failure only occurs in one the K modes competing with durations $T_1.....T_K$;

— the system failure time is $T = \min\{T_1...T_K\}$ which is a latent time ;

— the lifetime of the concurrent failure mode $k(k = 1, ..., K)$ denoted T_k follows a Gompertz distribution (α_k, β_k).

3.2. Modeling of Gompertz distribution by competitive risks

Jeon and Fine (2006) used the Gompertz (1825) distribution to model the cumulative incidence function associated with an event.

The cumulative incidence function (CIF) associated with a type K event denoted $F_k(t, \Psi_k)$ is defined by :

$$F_k(t, \Psi_k) = 1 - \exp\left[-\frac{\beta_k}{\alpha_k}\{\exp(\alpha_k t) - 1\}\right] ; \tag{2}$$

wiht $\Psi_k = (\alpha_k; \beta_k) \in \mathbb{R}^* \times \mathbb{R}$.

It is used as survival data in aging biology where α_k denotes the shape parameter or the coefficient of the dependent mortality-age rate while β_k denotes the parameter of scale or coefficient of the independent mortality-age rate (Witten and Satzer, 1992). These models are also widely used in demography where they make it possible to estimate the lifespans of populations (Bergstrom and Bagnoli, 2004). The density function is obtained by deriving the cumulative incidence function with respect to time and we obtain :

$$f_k(t;\Psi_k) = \frac{\partial F_k(t;\Psi_k)}{\partial t} = \beta_k \exp(\alpha_k t) \exp\left[-\frac{\beta_k}{\alpha_k}\{\exp(\alpha_k t) - 1\}\right]. \quad (3)$$

The instantaneous risk function associated with the event of type k is then obtained as follows :

$$\lambda_k(t;\Psi_k) = \frac{f_k(t;\Psi_k)}{1 - F_k(t;\Psi_k)} = \beta_k \exp(\alpha_k t). \quad (4)$$

Finally, the cumulative risk function is obtained immediately from the instantaneous risk function :

$$\Lambda_k(t;\Psi_k) = \int_0^t \lambda_k(u;\Psi_k)du = -\frac{\beta_k}{\alpha_k}\left(1 - \exp(\alpha_k t)\right). \quad (5)$$

The Gompertz distribution is improper when $\alpha_k < 0$ and $\beta_k > 0$.
The survival function for an event of type k is given by :

$$S_k(t;\Psi_k) = 1 - F_k(t;\Psi_k) = \exp\left[-\frac{\beta_k}{\alpha_k}\{\exp(\alpha_k t) - 1\}\right]. \quad (6)$$

3.3. Estimation by the maximum likelihood method

Under progressive type I censorship, we consider a population of K competitive risks. Let k be a fixed constant, $k \in \{1, 2, ..., K\}$. Let $r^* = \min\{t_R, r\}$ and $R^* = R$, $t_R \leq r$, $R^* = J$, $t_R > r$ where r^* is the final time of the experiment, R^* is the number of failures before time r^*.
$(t_1,\ \alpha_1),\ \cdots,\ (t_{R^*},\ \alpha_{R^*})$ are the observed chess data where $t_1, t_2, ..., t_{R^*}$ are the chess times in statistical order $\alpha_i \in \{1, 2, ..., K\}$, and where $\alpha_i = k$ $(k = 1, 2, ..., K)$ indicates the failure mode caused by the k^{rd} event.

Let

$$\delta_k(\alpha_i) = \begin{cases} 1 & \text{if } \alpha_i = k \\ 0 & \text{if } \alpha_i \neq k, \end{cases}$$

and $n_k = \sum_{i=1}^{R^*} \delta_k(\alpha_i) \geq 0$ is the number of failures caused by the $k^{\mathbf{rd}}$ event.
Under type I progressive censorship as defined in the introduction, the maximum likelihood function is given by :

$$L_k(t|\alpha_k, \beta_k) \propto \prod_{k=1}^{K}\left[\prod_{i=1}^{R^*} f_K(t_i)^{\delta_k(\alpha_i)}[1 - F_k(t_i)]^{1-\delta_k(\alpha_i)}[1 - F_k(t_i)]^{R_i}[1 - F_k(r^*)]^{n-R^*-\sum_{i=1}^{R^*} R_i}\right] \quad (7)$$

where R_i is the i^{rd} remainder in the random variables
The relation (7) breaks down as follows :

$$L_k(t|\alpha_k,\beta_k) = \prod_{k=1}^{K}\left[\prod_{i=1}^{R^*}\left(\beta_k e^{\alpha_k t_i}\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{\delta_k(\alpha_i)}\left(\exp\{-\frac{\beta_K}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{1-\delta_k(\alpha_i)}\right]$$

$$\times \quad \left[\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{R_i}\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k r^*}-1)\}\right)^{n-R^*-\sum_{i=1}^{R^*}R_i}\right]$$

$$= \prod_{k=1}^{K}\left[\prod_{i=1}^{R^*}\beta_k^{\delta_k(\alpha_i)}e^{\delta_k(\alpha_i)\alpha_k t_i}\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{R_i}\right]$$

$$\times \quad \left[\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k r^*}-1)\}\right)^{n-R^*-\sum_{i=1}^{R^*}R_i}\right]$$

$$= \prod_{k=1}^{K}\left[\prod_{i=1}^{R^*}\beta_k^{\delta_k(\alpha_i)}e^{\delta_k(\alpha_i)\alpha_k t_i}\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{R_i+1}\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k r^*}-1)\}\right)^{n-R^*-\sum_{i=1}^{R^*}R_i}\right]$$

$$= \prod_{k=1}^{K}\left[\beta_k^{\sum_{i=1}^{R^*}\delta_k(\alpha_i)}e^{\sum_{i=1}^{R^*}\delta_k(\alpha_i)\alpha_k t_i}\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k t_i}-1)\}\right)^{\sum_{i=1}^{R^*}(R_i+1)}\right]$$

$$\times \quad \left[\left(\exp\{-\frac{\beta_k}{\alpha_k}(e^{\alpha_k r^*}-1)\}\right)^{n-R^*-\sum_{i=1}^{r^*}R_i}\right]$$

$$= \prod_{k=1}^{K}\left[\beta_k^{n_k}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i-(\frac{\beta_k}{\alpha_k})\left[\sum_{i=1}^{R^*}(R_i+1)(e^{\alpha_k t_i}-1)+(n-R^*-\sum_{i=1}^{R^*}r_i)(e^{\alpha_k r^*}-1)\right]\right\}\right]$$

$$= \prod_{k=1}^{K}\left[\beta_k^{n_k}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i-(\frac{\beta_k}{\alpha_k})\times A_k\right\}\right],$$

with

$$A_k = \sum_{i=1}^{R^*}(R_i+1)\left(e^{\alpha_k t_i}-1\right)+\left(n-R^*-\sum_{i=1}^{R^*}r_i\right)\left(e^{\alpha_k r^*}-1\right) \qquad \text{and} \qquad \alpha_k>0. \qquad (*)$$

3.4. Bayesian estimation under different loss functions

3.4.1 A priori and a posteriori distribution

3.4.1.1 Distribution a priori In this subsection, for the failure event k, we assume α_k known, and we use the conjugate Gamma distribution as a priori distribution for the parameter β_k with the hyper-parameters a_k and b_k as defined in Wu et al. (2016). Then the a priori distribution is defined as follows :

$$\pi(\beta_k) = (b_k^{a_k}/\Gamma(a_k))\beta_k^{a_k-1}\exp(-b_k\beta_k), \qquad (8)$$

with $\beta_k > 0$, $a_k > 0$, $b_k > 0$ and $\Gamma(a_k) = \int_0^\infty t^{a_k-1}e^{-t}dt$.

3.4.1.2 A posteriori distribution A posteriori density is calculated using the following Bayes formula :

$$\pi(\beta_k|t) = \frac{\pi(\beta_k) \times L_k(t|\alpha_k, \beta_k)}{\int_0^\infty \pi(\beta_k) \times L_k(t|\alpha_k, \beta_k) d\beta_k}. \quad (9)$$

By calculating the numerator quantity $\pi(\beta_k) \times L_k(t|\alpha_k, \beta_k)$, we have :

$$\begin{aligned}
\pi(\beta_k) \times L_k(t|\alpha_k, \beta_k) &= (b_k^{a_k}/\Gamma(a_k))\beta_k^{a_k-1}\exp(-b_k\beta_k) \\
&\times \prod_{k=1}^{K}\left[\beta_k^{n_k}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i - (\frac{\beta_k}{\alpha_k}) \times A_k\right\}\right]; \\
&= \prod_{k=1}^{K}\beta_k^{n_k+a_k-1}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i - \frac{\beta_k}{\alpha_k} \times A_k - b_k\beta_k\right\}; \\
&= \prod_{k=1}^{K}\beta_k^{n_k+a_k-1}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\} \times \exp\left\{-(\frac{\beta_k}{\alpha_k} \times A_k + b_k\beta_k)\right\}. \quad (2*)
\end{aligned}$$

The same, by calculating the denominator of (9), we have :

$$\begin{aligned}
\int_0^\infty \pi(\beta_k) \times L_k(t|\alpha_k, \beta_k) d\beta_k &= \int_0^\infty \prod_{k=1}^{K}\beta_k^{n_k+a_k-1}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\} \\
&\times \exp\left\{-(\frac{\beta_k}{\alpha_k} \times A_k + b_k\beta_k)\right\} d\beta_k \\
&= \prod_{k=1}^{K}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}\int_0^\infty \beta_k^{n_K+a_k-1} \\
&\times \exp\left\{-(\frac{\beta_k}{\alpha_k} \times A_k + b_k\beta_k)\right\} d\beta_k; \\
&= \prod_{k=1}^{k}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}\int_0^\infty \beta_k^{n_k+a_k-1} \\
&\times \exp\left\{-\beta_k(\frac{A_k}{\alpha_k} + b_k)\right\} d\beta_k; \\
&= A\int_0^\infty \beta_k^{n_k+a_k-1}\exp\left\{-\beta_k(\frac{A_k}{\alpha_k} + b_k)\right\} d\beta_k \\
&= A\int_0^\infty \left(\frac{b_k + \frac{A_k}{\alpha_k}}{b_k + \frac{A_k}{\alpha_k}}\right)^{n_k+a_k-1}\beta_k^{n_k+a_k-1}\exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k
\end{aligned}$$

$$\begin{aligned}
&= \frac{A}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-1}}\int_0^{\infty}\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-1}\beta_k^{n_k+a_k-1}\\
&\times \exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}d\beta_k\\
&= \frac{A}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-1}}\int_0^{\infty}\left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k-1}\\
&\times \exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}d\beta_k\\
&= \frac{A\times\Gamma(n_k+a_k)}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}}, \qquad (3*)
\end{aligned}$$

where $\Gamma(n_k+a_k)\times\frac{1}{b_k+\frac{A_k}{\alpha_k}}=\int_0^{\infty}\left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}d\beta_k$;
and

$$A=\prod_{k=1}^{K}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}.$$

By dividing (2*) by (3*), we get :

$$\begin{aligned}
\pi(\beta_k|t) &= \frac{\prod_{k=1}^{K}\beta_k^{n_k+a_k-1}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}\times\exp\left\{-\frac{\beta_k}{\alpha_k}\times A_k-b_k\beta_k\right\}}{\frac{A\times\Gamma(n_k+a_k)}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}}}\\
&= \frac{\prod_{k=1}^{K}\beta_k^{n_k+a_k-1}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}\times\exp\left\{-\frac{\beta_k}{\alpha_k}\times A_k-b_k\beta_k\right\}}{\prod_{k=1}^{K}\frac{b_k^{a_k}}{\Gamma(a_k)}\exp\left\{\alpha_k\sum_{i=1}^{R^*}\delta_k(\alpha_i)t_i\right\}\times\frac{\Gamma(n_k+a_k)}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}}}\\
&= \frac{\beta_k^{n_k+a_k-1}}{\Gamma(n_k+a_k)}\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}.
\end{aligned}$$

Finally, the a posteriori density is given by :

$$\pi(\beta_k|t)=\frac{\beta_k^{n_k+a_k-1}}{\Gamma(n_k+a_k)}\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}, \qquad (10)$$

with α_k and A_k define in (*).

3.4.2 Loss functions

Wu and al., (2016) considered the quadratic loss function and the Linex loss function (Varian, 1975). In this paper, we use the functions of generalized squared loss, entropy loss (Galabria and Pulcini, 1994), and DeGroot loss (DeGroot, 1970).

The following table presents the three loss functions and the expression of each Bayesian estimator with the corresponding a postriori risk (Boudjerda, 2017)

Loss functions	Expression	Bayes estimator	A posteriori risk
Generalized quadratic	$L(\beta,\delta)=\Gamma(\beta)(\beta-\delta)^2$	$\widehat{\delta}_{QG}=\frac{\mathbb{E}_\pi(\Gamma(\beta)\beta)}{\mathbb{E}_\pi(\Gamma(\beta))}$	$\mathbb{E}_\pi(\Gamma(\beta)(\beta-\widehat{\delta}_{QG}))$
Entropy	$L(\beta,\delta)=(\frac{\delta}{\beta})^p-p\ln(\frac{\delta}{\beta})-1$	$\widehat{\delta}_E=[\mathbb{E}_\pi(\theta)^{-p}]^{-1/p}$	$P[\mathbb{E}_\pi(\ln\beta)-\ln(\widehat{\delta}_E)]$
DeGroot	$L(\beta,\delta)=\left(\frac{\beta-\delta}{\delta}\right)^2$	$\widehat{\delta}_D=\frac{\mathbb{E}_\pi(\beta^2\vert x)}{\mathbb{E}_\pi(\theta\vert x)}$	

TABLE 1 – The loss functions and the corresponding Bayesian estimators (with a posteriori risks)

4. MAINS RESULTS

4.1. Bayesian estimate of β_k under the generalized squared loss function

The following result gives the expression of the estimator of β_k under the generalized squared loss function.

Theorem 4.1. *Considering the a priori distribution* (8) *and the a posteriori distribution* (10)*, the estimator of* $beta_k$ *under the generalized squared loss function is defined in the case of the Gompertz distribution by :*

$$\widehat{\beta}_{k(BQG)}(a_k,b_k) \quad = \quad \frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}}, \qquad avec \quad \alpha_k>0.$$

Proof: Under the generalized quadratic loss function, and using the hypothesis of Boudjerda (2017) : $\Gamma(\beta_k)=\beta_k^{\alpha-1}, \quad \alpha\in\mathbb{R}$, we have the Bayesian estimator of the parameter $beta_k$ which is given by the following formula :

$$\begin{aligned}\widehat{\beta}_{k(BQG)}(a_k,b_k) &= \frac{\mathbb{E}_\pi(\Gamma(\beta_k)\beta_k)}{\mathbb{E}_\pi(\Gamma(\beta_k))}\\ &= \frac{\mathbb{E}_\pi(\beta_k^\alpha)}{\mathbb{E}_\pi(\beta_k^{\alpha-1})}\\ &= \frac{\int_0^\infty \beta_k^\alpha\pi(\beta_k|t)d\beta_k}{\int_0^\infty \beta_k^{\alpha-1}\pi(\beta_k|t)d\beta_k}, \quad (**)\end{aligned}$$

Let's calculate the numerator and the denominator separately.
For the numerator of (**), we have :

$$\int_0^\infty \beta_k^\alpha\pi(\beta_k|t)d\beta_k=\int_0^\infty \beta_k^\alpha\frac{\beta_k^{n_k+a_k}}{\Gamma(n_k+a_k)}\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}d\beta_k$$

$$
\begin{aligned}
&= \frac{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k}}{\Gamma(n_k+a_k)} \int_0^{\infty} \beta_k^{n_k+a_k+\alpha-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= L \int_0^{\infty} \beta_k^{n_k+a_k+\alpha-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \quad \text{with} \quad L = \frac{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k}}{\Gamma(n_k+a_k)} \\
&= L \int_0^{\infty} \beta_k^{n_k+a_k+\alpha-1} \left(\frac{b_k + \frac{A_k}{\alpha_k}}{b_k + \frac{A_k}{\alpha_k}}\right)^{n_k+a_k+\alpha-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-1}} \int_0^{\infty} \left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+\alpha-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha}} \times \Gamma(n_k+a_k+\alpha),
\end{aligned}
$$

so

$$
\int_0^{\infty} \beta_k^{\alpha} \pi(\beta_k|t) d\beta_k = \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha}} \times \Gamma(n_k+a_k+\alpha),
$$

with

$$
\Gamma(n_k+a_k+\alpha) \times \frac{1}{b_k + \frac{A_k}{\alpha_k}} = \int_0^{\infty} \left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+\alpha-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k.
$$

The quantity in the denominator of (**) breaks down as follows :

$$
\int_0^{\infty} \beta_k^{\alpha-1} \pi(\beta_k|t) d\beta_k
$$

$$
\begin{aligned}
&= \int_0^{\infty} \beta_k^{\alpha-1} \pi(\beta_k|t) d\beta_k \\
&= \int_0^{\infty} \beta_k^{n_k+a_k+\alpha-2} \times L \times \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= L \int_0^{\infty} \beta_k^{n_k+a_k+\alpha-2} \times \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= L \int_0^{\infty} \frac{\left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+\alpha-2}}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-2}} \times \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-2}} \int_0^{\infty} \left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+\alpha-2} \times \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-1}} \times \Gamma(n_k+a_k+\alpha-1),
\end{aligned}
$$

with

$$\Gamma(n_k+a_k+\alpha-1)\times\frac{1}{b_k+\frac{A_k}{\alpha_k}}=\int_0^{\infty}\left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+\alpha-2}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}d\beta_k.$$

By making the ratio of the numerator and the denominator, we obtain :

$$\begin{aligned}
\widehat{\beta}_{k(BQG)}(a_k,b_k) &= \frac{\frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha}}\times\Gamma(n_k+a_k+\alpha)}{\frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-1}}\times\Gamma(n_k+a_k+\alpha-1)}\\
&= \frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha}}\times\Gamma(n_k+a_k+\alpha)\times\frac{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+\alpha-1}}{L\times\Gamma(n_k+a_k+\alpha-1)}\\
&= \left(b_k+\frac{A_k}{\alpha_k}\right)^{-1}\times\frac{\Gamma(n_k+a_k+\alpha)}{\Gamma(n_k+a_k+\alpha-1)}\\
&= \frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}}.
\end{aligned}$$

So, the estimator of β_k under the generalized squared loss function is :

$$\widehat{\beta}_{k(BQG)}(a_k,b_k) = \frac{n_k+a_k+\alpha-1}{b_k+\frac{A_k}{\alpha_k}} \quad \text{with} \quad \alpha_k>0. \tag{11}$$

■

4.2. Estimate of β_k under DeGroot's loss function

The following result gives the expression of the estimator of β_k under the DeGroot loss function.

Theorem 4.2. *Considering the a priori distribution* (8) *and the a posterior distribution* (10)*, the estimator of* β_k *under the DeGroot loss function is defined in the case of the distribution of Gompertz by :*

$$\widehat{\beta}_{k(BD)}(a_k,b_k) = \frac{n_k+a_k+1}{b_k+\frac{A_k}{\alpha_k}} \quad with \quad \alpha_k>0.$$

Proof: The Bayesian estimator of the parameter β_k under the DeGroot loss function is given by the following formula :

$$\widehat{\beta}_{k(BD)}(a_k,b_k)=\frac{\mathbb{E}_\pi(\beta_k^2|t)}{\mathbb{E}_\pi(\beta_k|t)} \tag{12}$$

where

$$\mathbb{E}_\pi(\beta_k^2|t) = \int_0^\infty \beta_k^2 \pi(\beta_k^2|t) d\beta_k,$$

and

$$\mathbb{E}_\pi(\beta_k|t) = \int_0^\infty \beta_k \pi(\beta_k^2|t) d\beta_k.$$

The quantity $\mathbb{E}_\pi(\beta_k^2|t)$ is defined by :

$$\begin{aligned}
\mathbb{E}_\pi(\beta_k^2|t) &= \int_0^\infty \beta_k^2 \beta_k^{n_k+a_k-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} \times L d\beta_k \\
&= L \int_0^\infty \beta_k^{n_k+a_k+1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+1}} \int_0^\infty \left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k+1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k.
\end{aligned}$$

So,

$$\mathbb{E}_\pi(\beta_k^2|t) = \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+2}} \times \Gamma(n_k + a_k + 2).$$

The same, the quantity $E_\pi(\beta_k|t)$ is defined by :

$$\begin{aligned}
\mathbb{E}_\pi(\beta_k|t) &= \int_0^\infty \beta_k \beta_k^{n_k+a_k-1} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} \times L d\beta_k \\
&= L \int_0^\infty \beta_k^{n_k+a_k} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k}} \int_0^\infty \left[\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k} \exp\left\{-\beta_k(b_k + \frac{A_k}{\alpha_k})\right\} d\beta_k.
\end{aligned}$$

So,

$$E_\pi(\beta_k|t) = \frac{L}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k+a_k+1}} \times \Gamma(n_k + a_k + 1).$$

Thus, the estimator of β_k is given by :

$$\begin{aligned}
\widehat{\beta}_{k(BD)}(a_k,b_k) &= \frac{E_\pi(\beta_k^2|t)}{E_\pi(\beta_k|t)}\\
&= \frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+2}}\times\Gamma(n_k+a_k+2)\times\frac{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k+1}}{L\times\Gamma(n_k+a_k+1)}\\
&= \frac{1}{\left(b_k+\frac{A_k}{\alpha_k}\right)}\times\frac{\Gamma(n_k+a_k+2)}{\Gamma(n_k+a_k+1)}\\
&= \frac{n_k+a_k+1}{b_k+\frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k>0
\end{aligned}$$

So, the estimator of β_k under the DeGroot loss function is :

$$\widehat{\beta}_{k(BD)}(a_k,b_k) = \frac{n_k+a_k+1}{b_k+\frac{A_k}{\alpha_k}} \qquad \textbf{with} \quad \alpha_k>0. \tag{13}$$

■

4.3. Estimate of β_k under the Entropy loss function

The following result gives the expression of the estimator of β_k under the loss function Entropy

Theorem 4.3. *Considering the a priori distribution* (8) *and the a posterior distribution* (10)*, the estimator of* β_k *under the Entropy loss function is defined in the case of the Gompertz distribution by :*

$$\widehat{\beta}_{k(BE)}(a_k,b_k) = \left[\frac{1}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{-p}}\times\frac{\Gamma(n_k+a_k-p)}{\Gamma(n_k+a_k)}\right]^{-1/p} \qquad \text{with} \quad \alpha_k>0.$$

Proof: The Bayesian estimator of β_k under the entropy loss function is given by the following formula :

$$\widehat{\beta}_{k(BE)}(a_k,b_k) = \left[\mathbb{E}_\pi(\beta_k)^{-p}\right]^{-1/p}. \tag{14}$$

The quantity $\mathbb{E}_\pi(\beta_k)^{-p}$ is defined by :

$$
\begin{aligned}
\mathbb{E}_{\pi}(\beta_k)^{-p} &= \int_0^{\infty} \beta_k^{-p}\pi(\beta_k|t)d\beta_k \\
&= \int_0^{\infty} \beta_k^{-p}\beta_k^{n_k+a_k-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\}\times L d\beta_k \\
&= L\int_0^{\infty} \beta_k^{n_k+a_k-p-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= L\int_0^{\infty} \frac{\left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k-p-1}}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-p-1}}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-p-1}} \\
&\times \int_0^{\infty} \left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k-p-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{L}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-p}}\times\Gamma(n_k+a_k-p) \\
\text{with} \quad & \Gamma(n_k+a_k-p)\times\frac{1}{b_k+\frac{A_k}{\alpha_k}} \\
&= \left[\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right]^{n_k+a_k-p-1}\exp\left\{-\beta_k(b_k+\frac{A_k}{\alpha_k})\right\} d\beta_k \\
&= \frac{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k-p}}\times\frac{\Gamma(n_k+a_k-p)}{\Gamma(n_k+a_k)} \\
&= \frac{1}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{-p}}\times\frac{\Gamma(n_k+a_k-p)}{\Gamma(n_k+a_k)}.
\end{aligned}
$$

So,
$[\mathbb{E}_{\pi}(\beta_k)^{-p}]^{-1/p} = \left[\frac{1}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{-p}}\times\frac{\Gamma(n_k+a_k-p)}{\Gamma(n_k+a_k)}\right]^{-1/p}.$
Thus, the estimator of β_k is therefore :

$$\widehat{\beta}_{k(BE)}(a_k,b_k) = \left[\frac{1}{\left(b_k+\frac{A_k}{\alpha_k}\right)^{-p}}\times\frac{\Gamma(n_k+a_k-p)}{\Gamma(n_k+a_k)}\right]^{-1/p} \qquad \text{with} \quad \alpha_k>0. \quad (15)$$

■

4.4. Estimation of the survival function under the generalized squared loss function

The following result gives the expression of the estimator of the survival function under the generalized squared loss function.

Theorem 4.4. *By considering the a priori distribution* (8) *and the posterior distribution* (10)*, the estimator of the survival function* $\widehat{S}_{BQG(t)}$ *under the generalized squared loss function is defined in the case of the Gompertz distribution by :*

$$\widehat{S}_{BQG(t)} = \prod_{k=1}^{K} \left(\frac{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q-1}{\alpha_k}\{1 - \exp(\alpha_k t)\}}{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\}} \right)^{n_k + a_k} \qquad \text{with} \quad \alpha_k > 0.$$

Proof: This estimate is given by the following formula as defined in Wu et al. (2016) :

$$\widehat{S}_{BQG(t)} = \prod_{k=1}^{K} \left(\frac{\mathbb{E}_{\pi}(r(s(t))s(t))}{\mathbb{E}_{\pi}(r(s(t))} \right). \tag{16}$$

By posing : $r(s(t)) = (s(t))^{Q-1}$, $\quad r(s(t))s(t) = (s(t))^{Q}$ and $L = \frac{\left(b_k + \frac{A_k}{\alpha_k}\right)^{n_k + a_k}}{\Gamma(n_k + a_k)}$;
We have :

$$\begin{aligned}
\mathbb{E}_{\pi}(r(s(t))s(t)) &= \int_0^{\infty} (s(t))^{Q} \pi(\beta_k/t) d\beta_k. \\
&= L \int_0^{\infty} \exp\left[\frac{\beta_k}{\alpha_k}\{1 - \exp(\alpha_k t)\}\right]^{Q} \times \beta_k^{n_k + a_k - 1} \times \exp\left\{-\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right\} d\beta_k. \\
&= L \int_0^{\infty} \beta_k^{n_k + a_k - 1} \times \exp\left[\frac{Q\beta_k}{\alpha_k}\{1 - \exp(\alpha_k t)\}\right] \times \exp\left\{-\left(b_k + \frac{A_k}{\alpha_k}\right)\beta_k\right\} d\beta_k. \\
&= L \int_0^{\infty} \beta_k^{n_k + a_k - 1} \times \exp\left[\left(\frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\} - \left(b_k + \frac{A_k}{\alpha_k}\right)\right)\beta_k\right] d\beta_k. \\
&= L \int_0^{\infty} \beta_k^{n_k + a_k - 1} \times \exp\left[-\left(\frac{-Q}{\alpha_k}\{1 - \exp(\alpha_k t)\} + \left(b_k + \frac{A_k}{\alpha_k}\right)\right)\beta_k\right] d\beta_k. \\
&= L \int_0^{\infty} \beta_k^{n_k + a_k - 1} \times \exp\left[\left(b_k + \frac{A_k}{\alpha_k}\right) - \left(\frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\}\right)\beta_k\right] d\beta_k.(***)
\end{aligned}$$

By posing :

$$N = \left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\} \qquad \text{with} \ \ \alpha_k > 0,$$

then the previous equation (***) becomes :

$$\begin{aligned}
\mathbb{E}_\pi(r(s(t))s(t)) &= L\int_0^\infty \beta_k^{n_k+a_k-1} \times \exp\left[-N\beta_k\right] d\beta_k;\\
&= L\int_0^\infty \frac{(N\beta_k)^{n_k+a_k-1}}{(N)^{n_k+a_k-1}} \times \exp\left[-N\beta_k\right] d\beta_k;\\
&= \frac{L}{(N)^{n_k+a_k-1}}\int_0^\infty (N\beta_k)^{n_k+a_k-1} \times \exp\left[-N\beta_k\right] d\beta_k. \quad (****)
\end{aligned}$$

The same, by posing

$$C = N\beta_k,$$

we have

$$dC = N d\beta_k.$$

Thus, the equation (****) therefore becomes :

$$\begin{aligned}
\mathbb{E}_\pi(r(s(t))s(t)) &= \frac{L}{(N)^{n_k+a_k}}\int_0^\infty (C)^{n_k+a_k-1} \times \exp\left[-C\right] dC;\\
&= \frac{L}{(N)^{n_k+a_k}} \times \Gamma(n_k+a_k),
\end{aligned}$$

with

$$\Gamma(n_k+a_k) = \int_0^\infty (C)^{n_k+a_k-1} \times \exp\left[-C\right] dC.$$

So

$$(\mathbb{E}_\pi(r(s(t))s(t)) = \frac{\left(b_k+\frac{A_k}{\alpha_k}\right)^{n_k+a_k}}{\left(\left(b_k+\frac{A_k}{\alpha_k}\right) - \frac{Q}{\alpha_k}\{1-\exp(\alpha_k t)\}\right)^{n_k+a_k}}.$$

The same, let's calculate the quantity $\mathbb{E}_\pi(r(s(t))$.

$$\begin{aligned}
\mathbb{E}_\pi(r(s(t))) &= \mathbb{E}_\pi s(t)^{Q-1}\\
&= \int_0^\infty (s(t))^{Q-1}\pi(\beta_k/t) d\beta_k\\
&= L\int_0^\infty \exp\left[\frac{\beta_k}{\alpha_k}\{1-\exp(\alpha_k t)\}\right]^{Q-1} \times \beta_k^{n_k+a_k-1} \times \exp\left\{-\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right\} d\beta_k\\
&= L\int_0^\infty \beta_k^{n_k+a_k-1} \times \exp\left[\frac{(Q-1)\beta_k}{\alpha_k}\{1-\exp(\alpha_k t)\}\right] \times \exp\left\{-\left(b_k+\frac{A_k}{\alpha_k}\right)\beta_k\right\} d\beta_k\\
&= L\int_0^\infty \beta_k^{n_k+a_k-1} \times \exp\left[\left(\frac{Q-1}{\alpha_k}\{1-\exp(\alpha_k t)\} - \left(b_k+\frac{A_k}{\alpha_k}\right)\right)\beta_k\right] d\beta_k\\
&= L\int_0^\infty \beta_k^{n_k+a_k-1} \times \exp\left[\left(b_k+\frac{A_k}{\alpha_k}\right) - \left(\frac{Q-1}{\alpha_k}\{1-\exp(\alpha_k t)\}\right)\beta_k\right] d\beta_k(i)
\end{aligned}$$

By posing :

$$N^{'} = \left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q-1}{\alpha_k}\{1 - \exp(\alpha_k t)\} \quad \text{ with } \alpha_k > 0.$$

Equation (i) becomes :

$$\begin{aligned}
\mathbb{E}_\pi(r(s(t))) &= L\int_0^\infty \beta_k^{n_k+a_k-1} \times \exp\left[-N^{'}\beta_k\right] d\beta_k; \\
&= L\int_0^\infty \frac{(N^{'}\beta_k)^{n_k+a_k-1}}{(N)^{n_k+a_k-1}} \times \exp\left[-N^{'}\beta_k\right] d\beta_k; \\
&= \frac{L}{(N^{'})^{n_k+a_k-1}}\int_0^\infty (N^{'}\beta_k)^{n_k+a_k-1} \times \exp\left[-N^{'}\beta_k\right] d\beta_k. \quad (ii)
\end{aligned}$$

By posing :

$$C^{'} = N^{'}\beta_k,$$

we have

$$dC^{'} = N^{'}d\beta_k,$$

and equation (ii) therefore becomes :

$$\begin{aligned}
\mathbb{E}_\pi(r(s(t))) &= \frac{L}{(N^{'})^{n_k+a_k}}\int_0^\infty (C^{'})^{n_k+a_k-1} \times \exp\left[-C^{'}\right] dC^{'}; \\
&= \frac{L}{(N^{'})^{n_k+a_k}} \times \Gamma(n_k + a_k);
\end{aligned}$$

with

$$\Gamma(n_k + a_k) = \int_0^\infty (C^{'})^{n_k+a_k-1} \times \exp\left[-C^{'}\right] dC^{'}.$$

So

$$\mathbb{E}_\pi(r(s(t))) = \frac{L}{(N^{'})^{n_k+a_k}} \times \Gamma(n_k + a_k).$$

Thus, we have :

$$\begin{aligned}
\frac{E_\pi(r(s(t))s(t))}{E_\pi(r(s(t))} &= \frac{\frac{L}{(N)^{n_k+a_k}} \times \Gamma(n_k + a_k)}{\frac{L}{(N^{'})^{n_k+a_k}} \times \Gamma(n_k + a_k)} \\
&= \frac{(N^{'})^{n_k+a_k}}{(N)^{n_k+a_k}} \\
&= \left(\frac{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q-1}{\alpha_k}\{1 - \exp(\alpha_k t)\}}{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\}}\right)^{n_k+a_k}.
\end{aligned}$$

Finally, the estimator of the survival function under the generalized squared loss function is :

$$\widehat{S}_{BQG(t)} = \prod_{k=1}^{K} \left(\frac{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q-1}{\alpha_k}\{1 - \exp(\alpha_k t)\}}{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{Q}{\alpha_k}\{1 - \exp(\alpha_k t)\}} \right)^{n_k + a_k}, \tag{17}$$

with $\alpha_k > 0$.

■

4.5. Estimation of the survival function under the loss function of DeGroot (1970)

The following result gives the expression of the survival function under the DeGroot loss function.

Theorem 4.5. *By considering the a priori distribution* (8) *and the a posterior distribution* (10)*, the estimator of the survival function* $\widehat{S}_{BD(t)}$ *under DeGroot's loss function is defined in the case of the Gompertz distribution by :*

$$\widehat{S}_{BD} = \prod_{k=1}^{K} \left(\frac{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{1}{\alpha_k}\{1 - \exp(\alpha_k t)\}}{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{2}{\alpha_k}\{1 - \exp(\alpha_k t)\}} \right)^{n_k + a_k} \quad \text{with} \quad \alpha_k > 0.$$

Proof: The estimator of the survival function under the DeGroot loss function is given by the following formula as defined in Wu et al. (2016) :

$$\widehat{S}_{BD(t)} = \prod_{k=1}^{K} \frac{E_\pi(s(t)^2)}{E_\pi s(t)}.$$

By replacing $Q = 2$ in (17) above, we get the estimate of the survival function under DeGroot's loss function :

$$\widehat{S}_{BD} = \prod_{k=1}^{K} \left(\frac{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{1}{\alpha_k}\{1 - \exp(\alpha_k t)\}}{\left(b_k + \frac{A_k}{\alpha_k}\right) - \frac{2}{\alpha_k}\{1 - \exp(\alpha_k t)\}} \right)^{n_k + a_k} \quad \text{with} \quad \alpha_k > 0. \tag{18}$$

■

4.6. Estimation of the survival function under the entropy loss function

The following result gives the expression of the estimator of the survival function under the entropy loss function.

Theorem 4.6. *By considering the a priori distribution* (8) *and the posterior distribution* (10)*, the estimator of the survival function* $\widehat{S}_{BE(t)}$ *under the entropy loss function is* *defined in the case of the Gompertz distribution by :*

$$\mathbb{E}_\pi(s(t)^{-p})^{\frac{-1}{p}} = \prod_{k=1}^{K}\left[\left(\frac{b_k+\frac{A_k}{\alpha_k}}{(b_k+\frac{A_k}{\alpha_k})+\frac{p}{\alpha_k}\{1-\exp(\alpha_k t)\}}\right)^{n_k+a_k}\right]^{\frac{-1}{p}} \quad \text{with} \quad \alpha_k>0.$$

Proof: Under the entropy loss function, the Bayesian estimator of the survival function is given by :

$$\widehat{S}_{BE}(t) = \left(\prod_{k=1}^{K} E_\pi(s(t)^{-p})\right)^{\frac{-1}{p}}. \tag{19}$$

By posing : $Q=-p$, then the quantity $E_\pi(s(t)^{-p})$ is defined by :

$$\begin{aligned} E_\pi(s(t)^{-p}) &= \left[\int_0^\infty s(t)^{-p}\pi(\beta_k/t)d\beta_k\right]^{\frac{-1}{p}} \\ &= E_\pi(s(t)^Q) \\ &= E_\pi(r(s(t))s(t)) \\ &= \frac{L}{N_2^{n_k+a_k}}\times\Gamma(n_k+a_k) \end{aligned}$$

with

$$N_2 = (b_k+\frac{A_k}{\alpha_k})+\frac{p}{\alpha_k}\{1-\exp(\alpha_k t)\}.$$

So

$$\begin{aligned} E_\pi(s(t)^{-p}) &= \frac{(b_k+\frac{A_k}{\alpha_k})}{N_2^{n_k+a_k}} \\ &= \left(\frac{b_k+\frac{A_k}{\alpha_k}}{(b_k+\frac{A_k}{\alpha_k})+\frac{p}{\alpha_k}\{1-\exp(\alpha_k t)\}}\right)^{n_k+a_k}. \end{aligned}$$

Hense

$$\mathbb{E}_\pi(s(t)^{-p})^{\frac{-1}{p}} = \left[\prod_{k=1}^{K}\left(\frac{b_k+\frac{A_k}{\alpha_k}}{(b_k+\frac{A_k}{\alpha_k})+\frac{p}{\alpha_k}\{1-\exp(\alpha_k t)\}}\right)^{n_k+a_k}\right]^{\frac{-1}{p}} \tag{20}$$

with $\alpha_k>0$.

■

5. RELATIONSHIP BETWEEN THE DIFFERENT ESTIMATORS

In this section, we present the relationship between the different estimates under the three loss functions.

5.1. Under the generalized quadratic loss function

Under the generalized squared loss function, the estimate of β_k varies with the values of α.

$$\widehat{\beta}_{k(BQG)}(a_k, b_k) = \frac{n_k + a_k + \alpha - 1}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0.$$

For $\alpha = 0$, we have :

$$\widehat{\beta}_{k(BQG)}(a_k, b_k) = \frac{n_k + a_k - 1}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0. \tag{21}$$

For $\alpha = 1$, we have :

$$\begin{aligned} \widehat{\beta}_{k(BQG)}(a_k, b_k) &= \left(b_k + \frac{A_k}{\alpha_k}\right)^{-1} \times \frac{\Gamma(n_k + a_k + 1)}{\Gamma(n_k + a_k)} \\ &= \left(b_k + \frac{A_k}{\alpha_k}\right)^{-1} \times (n_k + a_k) \\ &\text{because} \quad \Gamma(k+1) = k\Gamma(k). \end{aligned}$$

So

$$\widehat{\beta}_{k(BQG)}(a_k, b_k) = \frac{n_k + a_k}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0. \tag{22}$$

For $\alpha = 2$, we have :

$$\widehat{\beta}_{k(BQG)}(a_k, b_k) = \frac{n_k + a_k + 1}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0. \tag{23}$$

For $\alpha = d$, we have :

$$\widehat{\beta}_{k(BQG)}(a_k, b_k) = \frac{n_k + a_k + d - 1}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0. \tag{24}$$

We find that, the larger d, the more the value of the estimator of β_k under the generalized squared loss function increases.

5.2. Under DeGroot's loss function

Under DeGroot's loss function, the estimate of β_k is :

$$\widehat{\beta}_{k(BD)}(a_k, b_k) = \frac{n_k + a_k + 1}{b_k + \frac{A_k}{\alpha_k}} \qquad \text{with} \quad \alpha_k > 0.$$

5.3. Under the entropy loss function

Under the loss function, the estimate of β_k is :

$$\widehat{\beta}_{k(BE)}(a_k, b_k) = \left[\frac{1}{\left(b_k + \frac{A_k}{\alpha_k}\right)^{-p}} \times \frac{\Gamma(n_k + a_k - p)}{\Gamma(n_k + a_k)} \right]^{-1/p} \quad \text{with} \quad \alpha_k > 0.$$

For $p = -1$, we have :

$$\begin{aligned} \widehat{\beta}_{k(BE)}(a_k, b_k) &= \frac{\Gamma(n_k + a_k) + 1}{\Gamma(n_k + a_k)} \times \frac{1}{b_k + \frac{A_k}{\alpha_k}} \\ &= \frac{(n_k + a_k)\Gamma(n_k + a_k)}{\Gamma(n_k + a_k)} \times \frac{1}{b_k + \frac{A_k}{\alpha_k}} \\ &= \frac{n_k + a_k}{b_k + \frac{A_k}{\alpha_k}} \\ \widehat{\beta}_{k(BE)}(a_k, b_k) &= \frac{n_k + a_k}{b_k + \frac{A_k}{\alpha_k}} \quad \text{with} \quad \alpha_k > 0. \end{aligned}$$

For $p = 1$, we have :

$$\begin{aligned} \widehat{\beta}_{k(BE)}(a_k, b_k) &= \left[\frac{\Gamma(n_k + a_k) - 1}{\Gamma(n_k + a_k)} \times \frac{1}{(b_k + \frac{A_k}{\alpha_k})^{-1}} \right]^{-1} \\ &= \left[\frac{(n_k + a_k - 1)\Gamma(n_k + a_k)}{\Gamma(n_k + a_k)} \times \frac{1}{(b_k + \frac{A_k}{\alpha_k})^{-1}} \right]^{-1} \\ &= \frac{n_k + a_k - 1}{b_k + \frac{A_k}{\alpha_k}} \\ \widehat{\beta}_{k(BE)}(a_k, b_k) &= \frac{n_k + a_k - 1}{b_k + \frac{A_k}{\alpha_k}} \quad \text{with} \quad \alpha_k > 0. \end{aligned}$$

Thus, we notice that :

- For $\alpha = 1$, the estimate of $beta_k$ under the generalized squared loss function is equal to the estimate of β_k under the quadratic loss function (Wu et al., 2016);
- For $\alpha = 2$, the estimate of β_k under the Degroot loss function is equal to the estimate β_k under the generalized quadratic loss function;
- For $\alpha = 0$ and $p = 1$, the estimate β_k under the entropy loss function is equal to the estimate β_k under the generalized quadratic loss function.

6. CONCLUSION AND PERSPECTIVES

In this article, we studied the Bayesian estimation of a parameter of the competitive risk model, in particular on the quadratic loss and Lines functions. At the end of the tedious decompositions, we obtain new estimators under different loss functions : generalized quadratic loss function, entropy loss function and DeGroot loss function in the case of the Gompertz distribution as well as their survival function estimators. We also obtain the estimator of β_k under other loss functions using only the estimator of β_k under the generalized quadratic loss function, this by varying the parameters constant α and p.

As a perspective, we intend in the near future to determine the estimator of $beta_k$ within the framework of the Bayesian expectation under different loss functions and make a comparison between all the estimators obtained with those in the literature. Finally, we are going to perform numerical simulations of all the estimators obtained in order to check their robustness.

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