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Perfect Domination Stable Graphs upon Vertex Removal

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Abstract

A dominating set S of a graph is said to be perfect dominating set if each vertex not in S is adjacent to exactly one vertex in S. When a vertex is removed from the graph, the perfect domination number of a graph may be altered or remain unchanged. We say that a graph is perfect domination stable if $\gamma_p(G-v) = \gamma_p(G)$, for any vertex $v \in V(G)$. In this article, we study the perfect domination stable graphs and characterize perfect domination stable trees.

Keywords: Domination critical, Perfect domination, Perfect domination critical, Perfect domination stable.

AMS Subject classification: 05C69

1. INTRODUCTION

For any graph G having n vertices, we say that any subset S of vertices dominates the vertex set of G if each vertex not in S is adjacent to at least one vertex in S. The idea of domination dates back to 1850 when the chessboard problem was considered. Since then, it has developed in many directions by introducing and studying different domination parameters. Let S be now a dominating set of G. We say that S is a perfect dominating set if each vertex not in S is adjacent to exactly one vertex in S. The minimum cardinality of a perfect dominating set is called the perfect domination number, denoted by $\gamma_p(G)$. In any graph, by a γ_p-set, we mean a perfect dominating set of minimum cardinality.

When a domination parameter is defined, the classical way is to determine value of parameter for different graphs and obtain various bounds, characterizations. Another direction of research is to determine the effect of graph operations on the value of the parameter. For this, we may consider operations like vertex or edge removal, edge lifting, edge splitting and so on. Generally, any graph operation can alter the value of the parameter or remains unchanged. In this article, we consider the problem of determining effect of the graph operation; vertex removal on the perfect domination number of the graph.

Consider a simple graph G of order n and choose an arbitrary vertex v in G. We say that vertex v is $P-$critical if removal of that vertex changes the value of the parameter P. If all the vertices of G are critical, then we say that graph itself $P-$critical. If removal of any vertex does not alter the value of the parameter P, then we say that, corresponding graph is $P-$stable. This problem was considered and studied for the domination parameter by Brigham et. al [1]. Throughout this article, by a graph we mean a simple graph with finite number of vertices. The graph $G \circ K_1$ is obtained by attaching a vertex to vertex of G.

Consider two graphs G and H of order a and b respectively. Then the graph denoted by $G \circ H$ is called corona of G and H obtained by attaching a copy of H to each vertex of G. The join of graphs G and H is obtained by attaching every vertex in H to each vertex in G. A graph $P_n \times P_m$ is called a grid graph and it is said to be ladder graph if $m = 2$.

2. EFFECT OF VERTEX REMOVAL ON PERFECT DOMINATION NUMBER

Let G be any graph and S be a perfect dominating set of G. An arbitrary vertex $v \in V(G)$ is said to be perfect domination critical, simply γ_p- critical if $\gamma_p(G - v) \neq \gamma_p(G)$. A graph G itself called perfect domination critical if every vertex in G is perfect domination critical. We say that G is perfect domination stable if $\gamma_p(G - v) = \gamma_p(G)$, for every vertex in G. In this article, we characterize trees that are perfect domination stable.

Example 2.1. *Consider a cycle on four vertices, then removal of any vertex decreases perfect domination number by one, so* $\gamma_p(C_4 - v) = \gamma(C_4)$*. Hence,* C_4 *is perfect domination critical. Again, if we consider a path on six vertices, then removal of any vertex except end vertices increases perfect domination number by one. Therefore, all vertices of degree two in* P_6 *are perfect domination critical but not* P_6*. Any complete graph on at least two vertices is perfect domination stable.*

3. MAIN RESULTS

Consider a star $K_{1,n}, n \geq 3$ of order $n+1$. Let $\mathcal{K}$ be the family of star graphs obtained by subdividing all its edges.

Theorem 3.1. *Let G be any graph in the family $\mathcal{K}$. Then G is perfect domination stable.*

Proof. Let G be any graph in the family $\mathcal{K}$. Then obviously G will be a star with all its edges subdivided. Further, the minimum perfect dominating set S of G is obtained by taking all leaves but one support vertex. Hence, $\gamma_p(G) = n.$ For an arbitrary vertex v of G, there are three possible cases:

Case 1: Suppose v is the vertex at the center. Then the graph $G - \{v\}$ results a graph with n components each which is isomorphic to K_2, therefore $\gamma_p(G - v) = n.$

Case 2: Suppose v is a support vertex. Then $G - v$ results in a disconnected graph with K_1 and a star H with $n-1$ edges subdivided. Therefore, $\gamma_p(G - v) = 1 + \gamma_p(H) = 1 + n - 1$, so $\gamma_p(G - v) = n.$

Case 3: Suppose v is a leaf of G and let $uv \in E(G)$. Removal of v results in a star with $n-1$ edges subdivided. In fact the γ_p-set of G itself will be a γ_p-set of $G - v$. hence, $\gamma_p(G - v) = \gamma_p(G).$ □

Proposition 3.1. *A ladder graph G of order $2n$ is perfect domination stable if n is odd.*

Proof. Let G be a ladder graph of order $2n$, then by definition $G \cong P_2 \times P_n$. We shall assume n an odd integer and G contains to two copies of path P_n with corresponding vertices adjacent. Let the vertex set of path be $V(P_n) = \{v_1, v_2, \ldots, v_{\frac{n-1}{2}} \ldots v_n\}$. The graph is symmetric with respect to copies of P_n and also the effect of removal vertex v_k for $k > \frac{n-1}{2}$ is exactly same as the removal of vertex v_k for $k < \frac{n-1}{2}$. The minimum perfect dominating set of G will be given by $\{v_1, v_3', v_5, v_7' \ldots v_n'\}$ and each vertex in the γ_p-set except end vertices (vertices of degree two) dominates exactly three vertices of G. Let us choose an arbitrary vertex v of G, since the graph symmetric, it enough if we take $v = v_k$, $k \leq \frac{n-1}{2}$. Consider $G' = G - v$, graph obtained by removing v, the minimum perfect dominating set S of G itself will be a perfect dominating set of G unless u is not in S. For if $u \in S$, adding v' corresponding vertex of v in G leads to γ_p-set of G'. Hence, in both the cases, $\gamma_p(G) = \gamma_p(G')$ implying that G is perfect domination stable. □

Proposition 3.2. *Let P_n be a path of order $n = 3k + 5$, then P_n is perfect domination stable.*

Proof. Consider a path P_n of order $n = 3k + 5$, where k is an integer. We prove this proposition by induction on k. If $k = 0$, then P_5 is perfect domination stable since $\gamma_p(P_n) = 2$ and removal of any vertex results union of paths of order fewer than 5. Hence, $\gamma_p(P_5 - v) = \gamma_p(P_5)$ and so P_5 is perfect domination stable. Assume the result for $k - 1$ i.e., $n = 3k + 2$ and consider a path of order $n = 3k + 5$. For any vertex v of P_n, its removal results a union of P_r and P_s so that $s, r \geq 1$ and $r + s = n$. Then, obviously $r, s < n$ and by induction hypothesis both P_r and P_s are perfect domination stable. In symbols, $\gamma_p(P_r - v) = \gamma_p(P_r)$ and $\gamma_p(P_s - v) = \gamma_p(P_s)$, for any vertex v of P_n. Further, $\gamma_p(P_n - v) = \gamma_p(P_r - v) + \gamma_p(P_s) = \gamma_p(P_r) + \gamma_p(P_s) = \gamma_p(P_n)$. Hence the result is true for k and hence for any positive integer n. □

3.1. The family $\mathcal{T}$

Let $\mathcal{T}$ be the family of trees constructed recursively by a tree T_1 which is a path P_5 on five vertices. For $n \geq 1$, a tree T_{n+1} is obtained recursively one of the two operations listed below:

We first label the vertices of $T \in \mathcal{T}$ as follows. Initially if $T = P_5$, then $sta(v) = C$ if v is a leaf of T and $sta(v) = B$ if v is a support vertex of T and $sta(v) = A$, otherwise. Once the status is assigned for a vertex, it remains unchanged as the tree is constructed. Let $T_1 = T$ be a path P_5 in T.

Operation $\mathcal{O}_1$: Assume $v \in T_n$ and $sta(v) = A$. The tree T_{n+1} is obtained from T_n by attaching a path u, w and the edge uv. Let $sta(u) = B$ and $sta(w) = C$.

Operation $\mathcal{O}_2$: Assume $v \in T_n$ and $sta(v) = C$. The tree T_{n+1} is obtained from T_n by attaching a path u, w, x and the edge uv. Let $sta(u) = A$, $sta(w) = B$ and $sta(x) = C$.

Observation 1. *The set of vertices having status A, B, C are respectively denoted by $A(T)$, $B(T)$ and $C(T)$. Following are the observations from the definition of operations.*

- *Every vertex of status B is adjacent to exactly one vertex status A and one of status C. That is if $sta(v) = B$, then $N(v) \subseteq A(T) \cup C(T)$.*
- *If v is a pendant vertex, then $sta(v) = C$.*
- *If $sta(v) = A$ or B, then $\deg v$ is at least two and if $\deg(v \geq 3)$ then $sta(v) = A$.*

- *$B(T) \cup \{v\}$ where $sta(v) = C$ such that $N(v) \cap B(T) = \phi$ AND $C(T) \cup \{v\}$ where $sta(v) = B$ such that $N(v) \cap C(T) = \phi$ constitutes two perfect dominating sets in T.*
- *For any γ_p-set S, there exists at least one vertex v of status C such that degree of v is one.*
- *Suppose $sta(v) = A$, then v does not belong to any γ_p-set.*
- *For any tree T, we have $|B(T)| = |C(T)|$ and $|A(T)| < |B(T)|$.*

Lemma 3.1. *Let $T \in \mathcal{T}$ be arbitrary. Then following are true.*

1. *$A(T) \cap S = \phi$, for any γ_p-set S.*
2. *Every vertex of status B or C belongs to some γ_p-set of T.*
3. *If T is obtained from $T' \in \mathcal{T}$, then $\gamma_p(T) = \gamma(T') + 1$.*

Proof. Let T be any tree in the family $\mathcal{T}$. From observation six, it follows that any vertex of status A does not lie in any minimum perfect dominating set. Hence, statement (i) is obvious. From observation four, it is clear that the set $B(T)$ along with a vertex v of status C, such that v is not adjacent to a vertex of status B forms a perfect dominating set. Similar argument proves that $C(T) \cup \{v\}$ forms a perfect dominating set of T. Since we have $|B(T)| = |C(T)|$, from observation 6, it follows that $\gamma_p(T) \leq |B(T)| + 1$.

Conversely, choose a minimum perfect dominating set S of T and assume that S neither equals to $B(T) \cup \{v\}$ nor $C(T) \cup \{v\}$, where vertices of status B and C are non-adjacent. This implies that there is at least one vertex in S having status A or there are at least two vertices of status B and C respectively. From (i), we must have for any vertex v in a minimum perfect dominating set S, $sta(v) \neq A$, hence first case is not possible. For the next case, assume $u, v, w, x \in S$ such that $sta(u) = sta(v) = B$ and $sta(w) = sta(x) = C$. Without loss of generality, we may assume that neither of u, v adjacent to w, x and at least one of w or x is a pendant vertex of T. The distance between any two vertices of S in G is must be at least 3 and never exceed 4. For if the distance between two vertices is four then there will be at least one vertex at distance two from the vertex of S not dominated by any vertex of S, this is impossible.

Without loss of generality, we may assume that w is a pendant vertex of G. The vertex of status B near w has distance 3, since each vertex of status B adjacent to exactly one vertex of status A and one of status C, from observation 1, $B(T) \cup w$ clearly dominates T, which is not possible since $x \in S$. This is a contradiction for our assumption and hence we have $\gamma_p(G) = |B(T)| + 1$, and so (iii) is obvious. Finally, let T may be any tree obtain from $T' \in F$. Irrespective of the operation through which T is generated from T', number of vertices of status B or C will be incremented by one, so $|B(T)| = |B(T')| + 1$. Therefore, it follows that $\gamma_p(T) = |B(T)| + 1 = |B(T')| + 2 = \gamma_p(T') + 1$.

□

Theorem 3.2. *For any tree T, T is perfect domination stable if and only if $T \in \mathcal{T}$.*

Proof. It is enough to prove that if T is perfect domination stable then $T \in mathcalT$. We shall prove this by induction on n, the order of T. If $n = 5$, then there are four different trees among which $T = P_5$ is perfect domination stable and by construction $P_5 \in \mathcal{T}$. Now assume the result for all graphs of order less than or equal to n. Let T be any tree of order n, choose a longest path $P : v_0, v_1, v_2 \ldots v_k$. Then $\deg v_0 = \deg v_k = 1$, $\deg v_1 = \deg v_{k-1} = 2$ and $\deg v_i \geq 2, 2 \leq i \leq k-2$. Now we shall consider following possible cases. **Case 1:** Suppose $\deg v_2 > 2$. Then consider T of order n, which is perfect domination stable. Let $T' = T - \{v_0, v_1\}$, then $\gamma_p(T) = \gamma_p(T') + 1$.

Claim: T' is perfect domination stable. We first note that $\gamma_p(T) = \gamma_p(T') + 1$. Choose an arbitrary vertex w from the vertex set of T' and let $T^* = T - \{w\}$. Adding the path $\{v_0, v_1\}$ that are removed earlier, we get the tree T^{**}, that is $T^*{*} = T \cup \{v_0, v_1\}$. In fact, the tree T^{**} can be viewed as a tree obtained from T removing the vertex w. That is, $T^{**} = T - \{w\}$. Further, we have $\gamma_p(T^{**}) = \gamma_p(T^*) + 1$. To prove this, we consider following sub-cases depending on the γ_p-set S of T^*.

Sub Case 1: Suppose v_1 is adjacent to a vertex in S. In such case since v_1 is already dominated, the set $S \cup \{v_1\}$ will be minimum perfect dominating set of T^{**}. **Sub case 2:** Suppose v_1 is adjacent to $x \notin S$, then obviously there will be a vertex in S dominating x. Therefore, v_1 cannot be chosen to dominate v_0, due to which x will be having two neighbors in the perfect dominating set of T^{**}. Therefore, $S \cup \{v_0\}$ will be minimum perfect dominating set of T^{**}. But, we have, $\gamma_p(T^{**}) = \gamma_p(T - \{w\}) = \gamma_p(T)$. Therefore, $\gamma_p(T^*) + 1 = \gamma_p(T^{**}) = \gamma_p(T)$. It is already observed that $\gamma_p(T) = \gamma_p(T') + 1$. Combining these two, we get $\gamma_p(T^*) + 1 = \gamma_p(T) = \gamma_p(T') + 1$. Implying that $\gamma_p(T^*) = \gamma_p(T')$ and so T' is perfect domination stable. Now, T' is in $\mathcal{T}$ and so T is obtained from T' by applying operation $\mathcal{O}_1$. Therefore, $T \in \mathcal{T}$. **Case 2:** Suppose $\deg v_2 = 2$. Then consider T of order n, which is perfect domination stable. Let $T' = T - \{v_0, v_1, v_2\}$, then $\gamma_p(T) = \gamma_p(T') + 1$. As in case 1, it is easy to check that T' is perfect domination stable. $\mathcal{T}$. Further T is obtained from T' by using operation $\mathcal{O}_2$. Also, we must have $\deg v_3 = 2$ and it is a leaf of T'. If not then degree of v_3 is at least three and so the status of v_3 is A. This is not true since T is obtained from T' attaching a path $P_3 : v_0, v_1, v_2$ to a vertex v_3, which must have status C, follows from the construct of tress. Therefore, $T \in \mathcal{T}$. This establishes the proof of the theorem. □

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