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Penalization of Brownian Motion Paths with Opposite of a Maximum Unilateral Function

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Abstract

In this paper, using Roynette et al. (2006b) developed by Belabbaci (2014), we establish through reflection's principle, and Markov strong property, a new penalization of Brownian motion paths using opposite of one-sided maximum from stochastic process defined by $\Gamma_t = \psi(I_t)$, with $I_t = -S_t$, where $S_t = \sup(X_s, t \geq s)$, $(X_t)_{t\geq 0}$ being a canonical process of $\mathcal{C}(\mathbb{R}^+;\mathbb{R})$ and ψ a Borelian's function and integral of $\mathbb{R}$ with values in $\mathbb{R}^+$.

Keywords : Stochastic process, Brownian motion, Maximum principle, Stopping times, Martingales with continuous parameter.

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1. INTRODUCTION AND BACKGROUND

Literature review (see Oliu-Barton, 2006) defines a stochastic process as a phenomenon changing randomly over time. Different sets of these occurrences appear in nature, life or science. A city population, the weather, prices of financial assets, figures of people gathered in a "queue" or for a bus as well as the position of a given particle of pollen within a fluid are stochastic processes. The latter was studied for the first time by Robert Brown in 1827 and received the appellation : Brownian Motion (MB). MB plays a fundamental role in the theory of random process, much like Gaussian distribution for the theory of probabilities.

Formally, a stochastic process is the data of a measurable space $(E;\varepsilon)$, a probability space $(\Omega;\ \mathcal{F};\ \mathbb{P})$ and a family of random variables

$$\{X_t : (\Omega;\ \mathcal{F};\ \mathbb{P}) \to (E;\varepsilon); t \in \mathbb{R}^+\}.$$

This paper, focuses in continuous stochastic process, with values in $E = \mathbb{R}$. According to their properties, several families of stochastic processes might be found : standard Brownian motion or Wiener's process, the martingale, Markov's process and more. In 1923, Norbert Wiener defined a unique law of probability in which canonical process of $\mathcal{C}(\mathbb{R}^+;\mathbb{R})$ appears to be a Brownian motion. He labeled it Wiener's measure, recorded $\mathbb{P}_x$ or $\mathbb{W}_x$; this being very important to proof results of paper. The Martingale ranges among the most important stochastic process in probability field, due to its wide use in finance, especially as regards to modeling asset prices in markets. There are ways to construct such process. One method of construction consists in taking a $(\Gamma_t, t \geq 0)$ stochastic process and apply penalization principle to it. Indeed, one would want a stochastic process like Brownian motion, but with radically different properties. In other words, Wiener's process law would no longer be stand as a martingale in it, but rather as a "weight" to acting to change its properties. This ordinary weight $(\Gamma_t)_{t\geq 0}$ is not necessary $\mathcal{F}_t$-adapted, with values in $\mathbb{R}^+$, and such that $\mathbb{E}(\Gamma_t) < \infty$, for all t. One may for instance, find a Brownian pseudo-motion whose global maximum is distributed at random.

In other words, let $(\Gamma_t, t \geq 0)$ be a stochastic process. We define for $t \geq 0$ and for $x \in \mathbb{R}$, a probability measure $W^{\Gamma}_{x,t}$ on $(\Omega, \mathcal{F}_\infty)$ by

$$W^{\Gamma}_{x,t} = \frac{\Gamma_t}{\mathbb{E}_x(\Gamma_t)}\mathbb{P}_x, \;\; t \geq 0, x \in \mathbb{R}. \tag{1}$$

The process Γ not being a martingale, the result $(W^{\Gamma}_{x,t})_{t\geq 0}$ is not consistent. Therefore, it is almost impossible to define a law of probability. However, this law can emerge within the limit when $t \to \infty$. When this limit exists, it will be labeled "penalization associated" with P_x and $(\Gamma_t)_{t\geq 0}$.

We look for weights $(\Gamma_t)_{t\geq 0}$ and martingales M^{Γ} such that, when $t \to \infty$,

$$\mathbb{E}_{W^{\Gamma}_{x,t}}[\mathbb{1}_{\Lambda_s}] \to \mathbb{E}_P[M^{\Gamma}_s \mathbb{1}_{\Lambda_s}] := \widetilde{P}(\Lambda_s), \quad \text{for all } \Lambda_t \in \mathcal{F}_t.$$

Thus, the process $(\Gamma_t, t \geq 0)$ has been penalized.

Since the works of Azéma & Yor (1979a, 1979b), several examples of penalization were studied during the last few years, notably by Roynette et al. (2005) ; Roynette et al. (2006a, 2006b, 2006c, 2006d) ; Oliu-Barton (2006), Roynette et al. (2007), Roynette et al. (2008a, 2008b), Roynette & Yor (2008a, 2008b), Roynette et al. (2009a, 2009b), Belabbaci (2014) and Roynette & Vallois (2014).

In this paper, we choose a noticeable example of a penalty for which we we find a major interest and which was initiated by Roynette et al. (2006b) and reiterated by (Belabbaci, 2014). It is the penalization by a function of the unilateral maximum $\varphi(S_t)$ where $S_t := \max_{s\leq t} X_s$, φ being a positive integral function.

In order to obtain our fundamental result, we set : $I_t = -S_t$, with $\psi : \mathbb{R} \to \mathbb{R}_+$, Borel and integral function and $\Gamma_t = \psi(I_t)$. We study the convergence defined in (1). This paper aims at showing that for$\Gamma_t = \psi(I_t)$, the convergence defined in (1) takes place.

The article is organized as follows : The following section introduces fundamentals of Brownian motion and stochastic process' basic properties ; in section 3, we recall the notions of penalization of a stochastic process, their functionalities as well as the principle of penalization. In section 4, we present the fundamental result of penalizing a Brownian motion by a function of the opposite of one-sided maximum. Finally, in section 5, we make a brief conclusion.

2. PRELIMINARY CONCEPTS

This section, presents basic concepts useful for the understanding of this paper. They include the Brownian movement, the principle of reflection and the Wiener measure. The main definitions and properties of this section are drawn in Lévêque (2005) and Mourrat (2014).

Definition 2.1. *Standard Brownian motion is defined as any stochastic process* $(B_t, t \geq 0)$ *such as :*

(i) $B_0 = 0$ $\mathbb{P}$. *p.s ;*

(ii) $\forall s, t \geq 0$, $B_{t+s} - B_t \sim \mathcal{N}(0, s)$ *;*

(iii) $(B_t, t \geq 0)$ *has independent increments ;*

(iv) $(B_t, t \geq 0)$ *is continuous* $\mathbb{P} - p.s.$

One of the fundamental properties of Brownian motion is based on its covariance function, hence the following result :

Theorem 2.1. $(\mathbf{Mourrat},\ \mathbf{2014})$
Let $B = (B_t, t \geq 0)$ be a standard Brownian motion. The following processes are also standard Brownian motion :

- *Symmetry : $(B = -B_t, t \geq 0)$,*
- *Simple Markov Property : $B_t^s = (B_{t+s} - B_s)_{t \geq 0}$,*
- *Scaling : $B_t^\lambda = (\frac{1}{\lambda} B_{\lambda^2 t},\ \ t \geq 0),\ \ with\ \ \lambda > 0.$*

The following definition is no other than Wiener measure :

Definition 2.2. $(\mathbf{Wiener\ \ measure})$ *Let $(B_t, t \geq 0)$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Wiener measure is the measure of probability $\mathbb{P}_0$ on $C(\mathbb{R}^+, \mathbb{R})$ defined as measurement image of $\mathbb{P}$ by the application :*

$$\begin{cases} \chi : \Omega \to \mathcal{C}(\mathbb{R}^+, \mathbb{R}) \\ \quad \omega \mapsto (B_t(\omega))_{t \in \mathbb{R}^+} . \end{cases} .$$

The application χ is measurable due to the extension of χ with each of the coordinated applications $\omega \mapsto \omega_t$. It is measurable. This extension gives random variables B_t. The measure of probability $\mathbb{P}_0$ is solely determined, independent of the choice of a Brownian motion $(B_t, t \geq 0)$, that is, all Brownian movements from 0 have the same law that of Wiener measure. If $x \in \mathbb{R}$, we also note $\mathbb{P}_x$ the image measurement of $\mathbb{P}_0$ by the translation $\omega \mapsto x + \omega$ (this is the law of Brownian motion from x).

Definition 2.3. $(\mathbf{Stopping\ \ Time})$ *A T random variable with values in $[0, +\infty]$ is a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$ if $\forall t \geq 0$,*

$$\{T \leq t\} \in \mathcal{F}_t.$$

The following result states the Markov strong property which is an extension of the simple Markov's property :

Theorem 2.2. $(\mathbf{Mourrat}, (\mathbf{2014}))$ *Let T be a stopping time, such as $\mathbb{P}(T < \infty) > 0$. Then conditionally at $T < \infty$, the process $B^{(T)}$ defined by*

$$B_t^{(T)} = (B_{T+t} - B_T, t \geq 0),$$

is a Brownian movement independent of $\mathcal{F}_T$. In equivalent manner, the property is stated as follows : conditionally at $B_T = x$, the process $(B_{T+s})_{s \geq 0}$, is independent of $\mathcal{F}_T$ and has as law $\mathbb{P}_x$.

Currently, we introduce the principle of reflection which is a direct consequence of the strong Markov property. It reads as follows :

Proposition 2.1. (**Reflection Principle**) *Let* $t \in \mathbb{R}^+$, $(a,b) \in \mathbb{R}^2$, *with* $b > 0$ *and* $a \leq b$. *So we have :*

$$\forall t > 0, \ \ \mathbb{P}[S_t \geq b, X_t \leq a] = \mathbb{P}[X_t \geq 2b - a]. \tag{2}$$

In particular, S_t *has the same law as* $|B_t|$.

To demonstrate this proposition, we use the definition of pause time and Markov strong property stated above.

Proof: We apply Markov strong property to time given at definition 2 above :

$$T_b = \inf\{t \geq 0, B_t = b\}. \tag{3}$$

We know that, $T_b < \infty$ p.s. In continuation,

$$\begin{aligned} \mathbb{P}\left[S_t \geq b, B_t \leq a\right] &= \mathbb{P}\left[T_b \leq t, B_t \leq a\right] \\ &= \mathbb{P}\left[T_b \leq t, B_t - b \leq a - b\right] \\ &= \mathbb{P}[T_b \leq t, B_t - B_{T_b} \leq a - b] \\ &= \mathbb{P}\left[T_b \leq t, B^{(T_b)}_{t-T_b} \leq a - b\right]; \end{aligned} \tag{4}$$

where

$$B^{(T_b)}_{t-T_b} = B_t - B_{T_b} = B_t - b.$$

Let $B^{(T_b)}_{t-T_b}$ by B' so that, according to Theorem 2, the process B' is a motion Brownian independent $\mathcal{F}_{T_b}$ so in particular T_b. As B' has the same law as $-B'$, the couple $(T_b; -B')$ has the same law as $(T_b; B')$.

Let

$$H = \left\{(s,w) \in \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}); s \leq t, w(t-s) \leq a - b\right\}.$$

So we have :

$$\begin{aligned} \mathbb{P}[T_b \leq t, B^{(T_b)}_{t-T_b}] &= \mathbb{P}[(T_b, B') \in H] \\ &= \mathbb{P}[(T_b, -B') \in H] \\ &= \mathbb{P}[T_b \leq t, -B^{(T_b)}_{t-T_b} \leq a - b] \\ &= \mathbb{P}[T_b \leq t, B_t \geq 2b - a] \\ &= \mathbb{P}[B_t \geq 2b - a], \end{aligned}$$

since the event $\{B_t \geq 2b - a\}$ is contained in $\{T_b \leq t\}$.

Indeed, are $a > 0$ et $b \leq a$, that is $\omega \in \{B_t \geq 2b - a\}$, then :

$$B_t(\omega) \geq 2b - a = b + (b - a) \geq b \Rightarrow S_t(\omega) \geq b \Rightarrow T_b(\omega) \leq t.$$

For the second assertion, we observe that :

$$\begin{aligned} \mathbb{P}[S_t \geq a] &= \mathbb{P}[S_t \geq a, B_t \geq a] \\ &= \mathbb{P}[S_t \geq a, B_t \leq a] \\ &= 2\mathbb{P}[B_t \geq a] \\ &= \mathbb{P}[|B_t| \geq a]. \end{aligned}$$

So

$$S_t \stackrel{d}{=} |B_t|, \quad \forall t > 0.$$

So, the density of S_t is equal to

$$f_{S_t}(x) = \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \mathbb{1}_{[0,+\infty[}(x), \quad \forall t \geq 0.$$

This ends the proof of the theorem. ■

Following results provide a good understanding of this paper. Interested reader might consult Revuz & Yor (1993, 1999) or Le Gall (2011).

The following corollary 1 establishes the equality in random laws variables S_t, $|B_t|$, and $|B_1|$, for all $t \in \mathbb{R}^+$.

Corollary 2.1. *Let* $(B_t, t \geq 0)$ *be a Brownian motion,* $S_t = \sup(X_s, t \geq s)$. *We have :*

$$\forall t \geq 0, S_t \stackrel{d}{=} |B_t| \stackrel{d}{=} \sqrt{t}|B_1|. \tag{5}$$

The density of the couple (S_t, B_t) is given by the following corollary 2 :

Corollary 2.2. *For* $t > 0$, $a \leq b$ *and* $b \geq 0$, *the pair* (S_t, B_t) *has for density the function* $f(S_t, B_t)$ *defined by*

$$f(S_t, B_t)(b, a) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2b - a)^2}{2t}\right\}. \tag{6}$$

As regards to the density of T_a with respect to Wiener measure, it is given by the following proposition 2 :

Proposition 2.2. *For every $a \in \mathbb{R}$, and according to the Wiener's measure $\mathbb{P}_x$, the density of T_a is*

$$f_{T_a}(t) = \frac{|x-a|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(x-a)^2}{2t}\right\} 1\!\!1_{\{t>0\}}. \tag{7}$$

The behavior of $\mathbb{P}_x(T_0 > t)$ is given by the following result :

Proposition 2.3. *For all $x > 0$, we have :*

$$\mathbb{P}_x(T_0 > t) \sim_{t+\infty} \sqrt{\frac{2}{\pi t}} x. \tag{8}$$

The following result will be useful to proof our fundamental result. It was established by (Mourrat (2014)).

Proposition 2.4. $(\mathbf{Mourrat},\ (\mathbf{2014}))$ *Let $\mathcal{B}$ be a sub-set of $\mathcal{F}$ and $Y\ (resp. X)$ a variable random $\mathcal{B}$-mesurable (resp. independent of $\mathcal{B}$). So for any measurable function $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\mathbb{E}[h(Y,X)/\mathcal{B}] = \phi(Y),\ \mathbb{P}.p.s, \tag{9}$$

where $\phi(t) = \mathbb{E}(h(t,X))$.

Proof: Let h be a bounded measurable function, $Z \in \mathcal{B}$ and $Z \geq 0$. We have :

$$\phi(y) = \int h(y,x) f_X(x) dx,$$

and

$$\begin{aligned}
\mathbb{E}\left[Zh(Y,X))\right] &= \int\int\int zh(y,x) f_{Y,Z}(y,z) f_X(x) dx dz dy \\
&= \int z \left[\int (\int h(y,x) f_X(x) dx) f_Y, Z(y,z) dy\right] dz \\
&= \int z\phi(y) f_{Y,Z}, (y,z) dy dz \\
&= \mathbb{E}[Z\phi(Y)],
\end{aligned}$$

where $\phi(t) = \mathbb{E}[h(t,X)]$. ■

3. PENALIZATION OF A STOCHASTIC PROCESS

3.1. Notations

In this section, we use the following notations that will help us in this paper :

- $\Omega = C(\mathbb{R}^+, \mathbb{R})$, the space of continuous functions on $\mathbb{R}^+$;
- $(\Omega, (\mathcal{F})_{t\geq 0})$ the canonical space with $(X_t)_{t\geq 0})$ the application defined by

$$X_t(w) = w(t) \text{ and } \mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t, t \geq 0\};$$

- $(\mathbb{P}_x)_{x\in\mathbb{R}}$, the family of measurement of Wiener on the canonical space $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$;
- $(X_t)_{t\geq 0})$ is a Brownian motion from x ;

3.2. Penalization's functional

It is worth remembering that for several years, until about 2009, Marc Yor and his co-authors were interested in other penalization's functionalities (see Roynette & Vallois, 2014, p.3). These are :

- functions of unilateral maximum or maximum of the absolute value, or the local time in 0 or number of descents, linear Brownian motion (see Roynette et al. (2006b)) ;
- functions of the unilateral maximum of the Brownian bridge (see Roynette et al. (2005)) ;
- local time functions in 0 for recurrent Bessel processes (see Roynette et al. (2008a)) ;
- functions depending on the number of turns or module for Brownian motion multidimensional (see Roynette et al. (2009a)) ;
- functions related to the length of excursions, or the maximum unilateral after a first time pass, or functions of additive functionalities Brownian, and so on (see Roynette et al. (2008b), Roynette & Yor (2008a)).

In Roynette & Vallois (2014), a penalization functional is a family of positive random variables (not necessary adapted) $\{\Gamma_t, t \geq 0\}$ such as

$$0 \;<\; \mathbb{E}_x(\Gamma_t) < \infty, \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

This defines the new probability $W^{\Gamma}_{x,\,t}$ on $(\Omega, \mathcal{F}_\infty)$, for all $t \geq 0, x \in \mathbb{R}$,

$$W^{\Gamma}_{x,t} \;=\; \frac{\Gamma_t}{\mathbb{E}_x(\Gamma_t)}\mathbb{P}_x \,.$$

By construction, $W^{\Gamma}_{x,t}$ is absolutely continuous with respect to $\mathbb{P}_x$, but the family of probabilities $\big(W^{\Gamma}_{x,\,t},\ t \geq 0\big)$ is not, in general, projective.
If for everything $s \geq 0$ and $F_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} W^{\Gamma}_{x,\,t}(F_s) \quad = \quad W^{\Gamma}_{x,\,\infty}(F_s), \tag{10}$$

then, we say that $(\Gamma_t,\ t \geq 0)$ is a functional of penalization.

3.3. Principle of penalization of a stochastic process

The principle of penalization is described globally in Roynette & Vallois (2014) in considering that $\{\Gamma_t, t \geq 0\}$ as a family of suitable positive random variables such as $0 < \mathbb{E}_x(\Gamma_t) < \infty$. If for all $s \geq 0$, and $F_s \in \mathcal{F}_s$, the family $(W^{\Gamma}_{x,t}, t \geq 0)$ admits a limit $W^{\Gamma}_{x,\infty}$ when $t \to \infty$, so $W^{\Gamma}_{x,\infty}$ induces a probability on $(\Omega, \mathcal{F}_\infty)$ which is called probability obtained by the penalization $\{\Gamma_t, t \geq 0\}$. For all $s \geq 0$, $F_s \in \mathcal{F}$, the restriction of $W^{\Gamma}_{x,\infty}$ to $\mathcal{F}_s$ admits a density M^x_s and $(M^x_s, s \geq 0)$ is a positive $(\mathcal{F}_s)_{s\geq 0}$ W_x-martingale and

$$W^{\Gamma}_{x,\infty} = \mathbb{E}_x(\mathbb{1}_{\mathcal{F}_s} M^x_s). \tag{11}$$

A penalization study consists in proving that the relation (11) above takes place. To do this, we exhibit an equivalent of $\mathbb{E}_x(\Gamma_t)$ when $t \to \infty$. The quantity $\mathbb{E}_x(\Gamma_t)$ is then called the normalization factor.

4. PENALISATION BY A FUNCTION OF THE OPPOSITE OF THE MAXIMUM UNILATERAL

In this section, it is a question of constructing a penalization model based from the opposited of the one-sided maximum. For that, we start by defining the continuous time stochastic process :

$$\Gamma_t \quad = \quad \psi(I_t)_{t\in\mathbb{R}^+} \ \text{ with } \ I_t = -S_t,\ \ \forall t \geq 0, \tag{12}$$

where $S_t = \sup_{0\leq u\leq t} X_u$, and $\psi : \mathbb{R} \to]0, +\infty]$ is a Borelian's function and integrable on $\mathbb{R}$.

Since $(X_t)_{t\in\mathbb{R}^+}$ is a Brownian motion then $(-X_t)_{t\in\mathbb{R}^+}$ is also a motion Brownian. Moreover, the trajectories of $(X_t)_{t\in\mathbb{R}^+}$ are continuous. So with regard to these two hypotheses, we conclude that (I_t) is well defined.

4.1. Previous results

Roynette et al. (2006b) state the result on the penalization by a function of the unilateral maximum :

Theorem 4.1. (**Roynette et al. (2006b)**) *Let φ be as above.*

1. *Let $u \geq 0$ and $x \in \mathbb{R}$. For any Γ_u in $\mathcal{F}_u$, we have :*

$$\lim_{t\to\infty} \frac{\mathbb{E}_x\left[\mathbb{1}_{\Gamma_u}\varphi(S_t)\right]}{\mathbb{E}_x[\varphi(S_t)]} = \frac{1}{1-\Phi(x)}\mathbb{E}_x[\mathbb{1}_{\Gamma_u}M_u^{\varphi}], \tag{13}$$

where $(M_u^{\varphi})_{u\geq 0}$ is the martingale defined by :

$$M_t^{\varphi} : = (S_t - X_t)\varphi(S_t) + 1 - \Phi(S_t). \tag{14}$$

2. *Let $(Q_x^{\varphi})_{x\in\mathbb{R}}$ be the family of probabilities on $(\Omega, \mathcal{F}_\infty)$:*

$$Q_x^{\varphi}(\Gamma_u) = \frac{1}{1-\Phi(x)}\mathbb{E}_x[\mathbb{1}_{\Gamma_u}M_u^{\varphi}], for \;\; any \;\; u \geq 0, \;\; and \;\; \Gamma_u \in \mathcal{F}_u. \tag{15}$$

Proof: See Theorem 3.6, p. 11 of **Roynette et al.(2006b)**, ■

4.2. Main results

4.2.1. Penalization by the process $\psi(I)$ where $\psi : \mathbb{R} \to]0, +\infty]$ is a Borelian function and integrable

The following preliminary result is the key ingredient to state the result relative to the penalization by a function of the opposite of the one-sided maximum. This next technical lemma is the first basic result of this paper :

Lemme 4.1. *Let $\psi : \mathbb{R} \to]0, +\infty]$ a Borelian and integrable function. Let $x, a \in \mathbb{R}$ such that $a \leq x$.. So*

(i)

$$\mathbb{E}_0[\psi(a \vee (x + I_u))] \sim_{u\longmapsto\infty} \sqrt{\frac{2}{\pi u}}\{(x-a)\}\psi(a) + \int_a^{+\infty} \psi(y)dy; \tag{16}$$

(ii)

$$\mathbb{E}_0[\psi(a \vee (x + I_u))] \leq \sqrt{\frac{2}{\pi u}}\{(x-a)\psi(a) + \int_a^{+\infty} \psi(y)dy\}; \tag{17}$$

(iii)

$$\mathbb{E}_0[\psi(a \vee (x + I_u))] \geq \sqrt{\frac{2}{\pi u}} \int_a^{+\infty} \psi(y) e^{(-\frac{(y-x)^2}{2})} dy\}, \forall\, u \geq 1. \tag{18}$$

Proof: We have :

$$\begin{aligned}
\mathbb{E}_0[\psi(a \vee (x + I_u))] &= \int_\Omega \psi(a \vee (x + y)) f_{I_u}(y) dy \\
&= \int_{x+I_u \leq a} \psi(a \vee (x + y)) f_{I_u}(y) dy \\
&+ \int_{x+I_u \geq a} \psi(a \vee (x + y)) f_{I_u}(y) dy \\
&= \psi(a) \int \mathbb{1}_{\{x+I_u \leq a\}}(y) f_{I_u}(y) dy \\
&+ \int \mathbb{1}_{\{x+I_u \geq a\}}(y) \psi(x + y) f_{I_u}(y) dy.
\end{aligned}$$

Thus, according to Corollary 1, we have :

$$\begin{aligned}
\mathbb{E}_0[\psi(a \vee (x + I_u))] &= \psi(a)\mathbb{E}_0[\mathbb{1}_{\{x+I_u \leq a\}}] + \mathbb{E}_0[\psi(x + I_u)\mathbb{1}_{\{x+I_u \geq a\}})] \\
&= \psi(a)\mathbb{P}_0[x + I_u \leq a] + \mathbb{E}_0[\psi(x + I_u)\mathbb{1}_{\{x+I_u \geq a\}})] \\
&= \psi(a)\mathbb{P}_0[S_u \geq -a + x] + \mathbb{E}_0[\psi(x - S_u)\mathbb{1}_{\{x-S_u \geq a\}})] \\
&= \psi(a)\mathbb{P}_0(\sqrt{u}|X_1| \geq -a + x) \\
&+ \mathbb{E}_0[\psi(x - S_u)\mathbb{1}_{\{S_u \leq -a+x\}})] \\
&= \psi(a)\mathbb{P}_0(|X_1| \geq \frac{a - x}{\sqrt{u}}) \\
&+ \mathbb{E}_0[\psi(x - S_u)\mathbb{1}_{\{S_u \leq -a+x\}})] \\
&= 2\psi(a)\mathbb{P}_0(X_1 \geq \frac{a - x}{\sqrt{u}}) + \mathbb{E}_0[\psi(x - S_u)\mathbb{1}_{\{S_u \leq -a+x\}})] \\
&= 2\psi(a)\frac{1}{\sqrt{2\pi}} \int_{\frac{a-x}{\sqrt{u}}}^0 e^{(-\frac{(\zeta)^2}{2})} d\zeta \\
&+ \sqrt{\frac{2}{\pi u}} \int_{-\infty}^{-a+x} \psi(x - y) e^{(-\frac{(y)^2}{2u})} dy.
\end{aligned}$$

By changing the variables $z = x - y$ and $v = \sqrt{u}\zeta$, we have :

$$\mathbb{E}_0[\psi(a \vee (x + I_u))] = 2\psi(a)\frac{1}{\sqrt{2\pi}} \int_{a-x}^0 e^{(-\frac{(\zeta)^2}{2})} d\zeta + \sqrt{\frac{2}{\pi u}} \int_a^{+\infty} \psi(z) e^{(-\frac{(z-x)^2}{2u})} dz.$$

So, we have :

$$\begin{aligned}
\mathbb{E}_0[\psi(a \vee (x+I_u))] &= \psi(a)\sqrt{\frac{2}{\pi u}}\int_{a-x}^{0} e^{(-\frac{(v)^2}{2u})}dv \\
&+ \sqrt{\frac{2}{\pi u}}\int_{a}^{+\infty} \psi(z)e^{(-\frac{(z-x)^2}{2u})}dz \\
&\sim_{u\to+\infty} \sqrt{\frac{2}{\pi}}\left[\psi(a)(x-a)+\int_{a}^{+\infty}\psi(z)dz\right]. \qquad (19)
\end{aligned}$$

Let's move to the proof of inequality (17), since $e^{(-\frac{(\theta)^2}{2u})} \leq 1, \forall\theta \in \mathbb{R}$, then the equality (19) given :

$$\begin{aligned}
\mathbb{E}_0[\psi(a \vee (x+I_u))] &\leq \sqrt{\frac{2}{\pi u}}\left[\psi(a)\int_{x-a}^{0} dv + \int_{a}^{+\infty}\psi(y)dy\right] \\
&\leq \sqrt{\frac{2}{\pi u}}\left[\psi(a)(x-a)+\int_{a}^{+\infty}\psi(y)dy\right],
\end{aligned}$$

hence inequality (17).

Finally, we have according to (16) :

$$\begin{aligned}
\mathbb{E}_0[\psi(a \vee (x+I_u))] &= \psi(a)\sqrt{\frac{2}{\pi u}}\int_{a-x}^{0} e^{(-\frac{(v)^2}{2u})}dv \\
&+ \sqrt{\frac{2}{\pi u}}\int_{a}^{+\infty} \psi(y)e^{(-\frac{(y-x)^2}{2u})}dy \\
&\geq \sqrt{\frac{2}{\pi u}}[\int_{a}^{+\infty} \psi(y)e^{(-\frac{(y-x)^2}{2u})}dy,
\end{aligned}$$

and if $u \geq 1$, then

$$\sqrt{\frac{2}{\pi u}}\left[\int_{a}^{+\infty} \psi(y)e^{(-\frac{(y-x)^2}{2u})}dy\right] \geq \sqrt{\frac{2}{\pi u}}\left[\int_{a}^{+\infty} \psi(y)e^{(-\frac{(y-x)^2}{2})}dy\right]$$

This ends the proof of the theorem. ■

4.2.2. Penalisation by a function of the opposited of the one-sided maximum

Since the process $\Gamma_t = \psi(I_t)$ is defined, the following theorem (4) gives the penalization by the process $\psi(I)$ and is the second basic result of this paper.

Theorem 4.2. *Let $\psi : \mathbb{R} \to \mathbb{R}_+$ a Borelian function and integrable. Let $I_t = -S_t$, where S_t is the one-sided maximum. Are $u \geq 0$ and $x \in \mathbb{R}$. For all Γ_u in $\mathcal{F}_u$, we have :*

$$\lim_{t\to\infty} \frac{\mathbb{E}_x\left[\mathbb{1}_{\Gamma_u}\psi(I_t)\right]}{\mathbb{E}_x[\psi(I_t)]} = \frac{1}{\int_x^{+\infty}\psi(y)dy}\mathbb{E}_x[\mathbb{1}_{S_u}M_u^{\psi}], \qquad (20)$$

where $M^{\psi} = (M_t^{\psi})_{t\geq 0}$ *is the martingale defined by :*

$$M_t^{\psi} \;=\; (X_t - I_t)\psi(I_t) + \int_{I_t}^{+\infty} \psi(y)dy.$$

Proof: The proof of this theorem is based on Lemma 1 and the strong Markov property. Indeed : are $s \geq 0$, $x \in \mathbb{R}$ and $\Gamma_s \in \mathcal{F}_s$. Suppose that $t > s$, according to Proposition 4, where

$$\widetilde{\psi}(a, x; r) \;=\; \mathbb{E}_0[\psi(a \vee (x + I_r)].$$

So, applying the first result of the previous lemma 2, we have :

$$\begin{aligned}
\mathbb{E}_x[\psi(I_t)/\mathcal{F}_s] &= \mathbb{E}_0[\psi(I_s \vee (X_s + I_{t-s}]\\
&\sim_{t\to+\infty} \sqrt{\frac{2}{\pi(t-s)}}\{(-I_s + X_s)\psi(I_s) + \int_{I_s}^{+\infty} \psi(y)dy\}\\
&\sim_{t\to+\infty} \sqrt{\frac{2}{\pi(t-s)}} M_s^{\psi}
\end{aligned}$$

Again, applying the first result of Lemma 2, with "$x = a$", we get :

$$\begin{aligned}
\mathbb{E}_x[\psi(I_t)] &= \mathbb{E}_0[\psi(x + I_t)]\\
&= \mathbb{E}_0[\psi(x \vee (x + I_t))]\\
&\sim_{t\to+\infty} \sqrt{\frac{2}{\pi t}} \int_x^{+\infty} \psi(y)dy.
\end{aligned}$$

Let's evaluate the following quantity :

$$\frac{\mathbb{E}_x[1_{\Gamma_s}\psi(I_t)]}{\mathbb{E}_x[\psi(I_t)]}.$$

By inequality (17), we have :

$$\mathbb{E}_x\left[\psi(I_t)/\mathcal{F}_s\right] \leq \sqrt{\tfrac{2}{\pi(t-s)}} M_s^{\psi},$$

and, according to the inequality (18), by putting $x = a$, we have :

$$\mathbb{E}_x[\psi(I_t)] \;=\; \mathbb{E}_0[\psi(x + I_t)] \geq \sqrt{\frac{2}{\pi t}} \int_x^{+\infty} \psi(y) e^{-\frac{(y-x)^2}{2}} dy.$$

So

$$\begin{aligned}\mathbb{E}_x[\psi(I_t/\mathcal{F}_s)] &\leq \frac{\sqrt{\frac{2}{\pi(t-s)}}M_s^\psi}{\sqrt{\frac{2}{\pi t}}\int_x^{+\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy} \\ &\leq \frac{\sqrt{\frac{2}{\pi(t-s)}}}{\sqrt{\frac{2}{\pi t}}}\left[\frac{M_s^\psi}{\int_x^{+\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy}\right] \\ &\leq \sqrt{\frac{t}{t-s}}\left[\frac{M_s^\psi}{\int_x^{+\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy}\right].\end{aligned}$$

Since $(M_s^\psi)_{t\geq 0}$ is a martingale, then $\mathbb{E}_x[M_s^\psi] < \infty$.

Moreover, $\forall\, x \in \mathbb{R}$,

$$\left|\int_x^{+\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy\right| \leq \int_x^{+\infty}\left|\psi(y)e^{-\frac{(y-x)^2}{2}}\right|dy \leq \int_x^{+\infty}\psi(y)dy < \infty.$$

So

$$\mathbb{E}_x\left[\frac{M_s^\psi}{\int_x^{\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy}\right] = \frac{1}{\int_x^{\infty}\psi(y)e^{-\frac{(y-x)^2}{2}}dy}\mathbb{E}_x[M_s^\psi] < \infty.$$

Now, using the dominated convergence theorem, we get :

$$\begin{aligned}\lim_{t\to+\infty}\frac{\mathbb{E}_x[\mathbb{1}_{\Gamma_s}\psi(I_t)]}{\mathbb{E}_x[\psi(I_t)]} &= \lim_{t\to\infty}\left[\frac{\mathbb{E}_x[\mathbb{E}_x[\mathbb{1}_{\Gamma_s}\psi(I_t)/\mathcal{F}_t]}{\mathbb{E}_x[\psi(I_t)]}\right] \\ &= \lim_{t\to\infty}\mathbb{E}_x\left[\mathbb{1}_{\Gamma_s}\frac{\mathbb{E}_x\mathbb{E}_x[1_{\Gamma_s}\psi(I_t)/\mathcal{F}_t]}{\mathbb{E}_x[\psi(I_t)]}\right] \\ &= \mathbb{E}_x\left[\mathbb{1}_{\Gamma_s}\lim_{t\to\infty}\frac{\sqrt{\frac{2}{\pi(t-s)}}M_s^\psi}{\sqrt{\frac{2}{\pi t}}(\int_x^{+\infty}\psi(y)dy)}\right] \\ &= \mathbb{E}_x\left[\mathbb{1}_{\Gamma_s}\frac{M_s^\psi}{\int_x^{+\infty}\psi(y)dy}\right] \\ &= \frac{1}{\int_x^{+\infty}\psi(y)dy}\mathbb{E}_x\left[\mathbb{1}_{\Gamma_s}M_s^\psi\right].\end{aligned}$$

This ends the proof of the theorem. ■

5. CONCLUSION

In this paper, we used the works Roynette et al. (2006b) developed by Belabbaci (2014), to establish through reflection's principle, and Markov strong property, a new penalization of Brownian motion paths using opposite of one-sided maximum from stochastic process.

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