Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 17, Number 2 (2021), pp. 183-196

http://www.ripublication.com/gjpam.htm

Two Dimensional Volterra Integral Equation with Singular Kernels in Contact Problems and Its Numerical Computations

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Abstract

we study the Two-dimensional Volterra integral equation with weakly kernels in contact problems (**TDVIE**) of the second kind, where the goal here is to establish the solution of **TDVIE** of the second kind which investigated from the three-dimensional contact problem in the theory of elasticity with singular kernel, where the coefficients bed of the compressible materials are neglected. Then, the existence of a unique solution is discussed and proved. The **TDVIE** with singular kernel is solved numerically, using Toeplitz matrix (**TMM**) and Product Nyström methods (**PNM**). These numerical methods are adapted to determine the pressure between the two surfaces under certain conditions. Moreover, numerical results when the kernel takes logarithmic form and Carleman function are obtained. In addition, Error analysis are presented.

Keywords: Volterra integral equation, contact problems, logarithmic form, Carleman function.

1. INTRODUCTION

Integral equations play an important role in many other sciences, such as contact problems in the theory of elasticity, vicodynamics fluid, and others [1-4]. In addition, the numerical methods takes an important place in solving **IEs**. Abdalkhani [5], obtained a numerical solution of the nonlinear **VIE** of the second kind when the kernel taken the form of Abel's function form. Guoqiang et. al. [6], obtained numerically the solution of two dimensional of the nonlinear **VIE** by collocation method and iterated collocation method. In [7], Al-Bugami and Al-Wagdani discussed Runge-Kutta and Block-by-Block methods to solve linear **TDVIE** with continuous kernel. Al-Bugami, and Al-Wagdani in [8], solved Linear **TDVIE** with continuous kernel by using

Trapeziodal and Simsons rules. In [9], the author solved **L-VIE's** using the trapezoidal rule and Simpson's rule. In [10], the author has solved **NL-VIE's** using the trapezoidal rule and Simpson's rule.

Consider the three- dimensional semi-symmetric problem (Hertz contact problem) of two rigid surfaces having two different elastic materials occupying the domain

$$\Omega = \{\Omega : (s,t) \in \Omega : \sqrt{s^2 + t^2} \le a,\ z = 0\}.$$

The problem is investigated from the semi-symmetric contact problem of two different elastic materials in three-dimensional when the modules of elasticity is changing, in the lower layer, according to the power law $\sigma_i = K_0 \varepsilon_i^{\upsilon}$ $(0 \le \upsilon < 1)$ where σ_i and ε_i, i=1, 2, 3, are the stress and strain rate intensities respectively, while K_0, υ are physical constant. Assume the upper surface is impressed into the lower surface by a constants force $P < \infty$, whose eccentricity of application *e*, and the frictional forces in the contact domain Ω between the two surfaces are so small in which it can be neglected. In the absence of the external forces, the components of the stresses and the strain satisfy both Hook's and Lame's laws.

Consider the singular IE of the second kind:

$$(\lambda_1 + \lambda_2) w(s,t) - (\theta_1 + \theta_2) \iint_{\Omega\,\Omega} k(s-u, t-v) w(u,v)\,du\,dv = f(s,t) = [\delta - f_1(s,t) - f_2(s,t)] \qquad (1)$$

where

$$\iint_{\Omega\,\Omega} w(s,t)\,dx\,dy = P < \infty, \quad (P \text{ is a constant}) \qquad (2)$$

where $f_i(s,t)$, $i = 1,2$ are the known functions describing the two surface, Ω is the contact domain between the two surfaces, *w*(*s*,*t*) are the unknown normal stresses between $f_1(s,t)$ and $f_2(s,t)$, $\lambda_i, (i = 1,2)$ are the coefficients bed of the compressible materials that depend on its geometry. δ is the rigid displacement under the action of a force *P*, $\theta_i = 1 - \mu_i^2 / \pi E_i$, $i = 1,2$.

We write (1) in the form

$$w(s,t) - \lambda \int_0^s \int_0^t k(s-u, t-v) w(u,v)\,du\,dv = f(s,t) \qquad (3)$$

where

$$\lambda = \frac{\theta_1 + \theta_2}{\lambda_1 + \lambda_2}, f(s,t) = \frac{\delta - f_1(s,t) - f_2(s,t)}{\lambda_1 + \lambda_2}.$$

Equation (3) represents a linear **TDVIE** with the singular kernel. λ is constant, $k(s-u, t-v)$ is called the kernel of **IE**.

If $(\lambda_1+\lambda_2)=0, \theta_1+\theta_2=\frac{1}{2\pi\theta}$, $(\theta=G(1-\upsilon)^{-1})$, $f_2(s,t)=0$, we have the following **IE**

$$\int_0^s\int_0^t k(s-u,t-v)w(u,v)dudv = f^*(s,t), \quad [f^*(s,t)=2\pi\theta(\delta-f_1(s,t)] \tag{4}$$

Where G is the displacement magnitude, υ is Poison's coefficient, $f(s,t)$ is a known function, which describes the shape of stamp base. Eq.(4) represents a **VIE** of the first kind with singular kernel in two dimensional, which deduced from the three-dimensional contact problem of frictionless impression of a rigid stamp in the surface of an elastic (G,υ) half-space.

We can write **IE** (3) in the form

$$\mu w(s,t)=\lambda\int_0^s\int_0^t k(s-u,t-v)w(u,v)dudv = f(s,t), \tag{5}$$

2. EXISTENCS AND UNIQUENESS SOLUTIO

We assume the following conditions:

(i) $k(s-u,t-v)\in C([0,S]\times[0,T])$, and satisfies:

$$[\int_0^s\int_0^t |k(s-u,t-v)|^2 dudv]^{\frac{1}{2}} = A<\infty \qquad (A \text{ is a constant})$$

(ii) $f(s,t)$ is continuous with its derivatives and belongs to $J=C([0,S]\times[0,T])$ and its norm is defined as

$$\|f(s,t)\| = \max_{s,t\in J}\int_0^s[\int_0^t f^2(s,t)ds]^{\frac{1}{2}}dt = M,$$

(iii)

$$\|w(s,t)\| = [\int_0^s\int_0^t |w(s,t)|^2]^{\frac{1}{2}} \le C\|\phi\|_2$$

We write Eq. (5) in the integral operator form:

$$\bar{W}w(s,t)=\frac{1}{\mu}f(s,t)+Ww(s,t) \tag{6}$$

$$Ww(s,t)=\frac{\lambda}{\mu}\int_0^s\int_0^t k(s-u,t-v)w(u,v)dudv \tag{7}$$

Theorem 1. If the condition (i) – (iii) are verified, then eq. (5) has a unique solution

in $C([0,S]\times[0,T])$

Theorem 1 can be proven by the following two lemmas:

Lemma 1. Under the condition (i) – (iii), the operator $\bar{W}$ defined by (5), maps the space $C([0,S]\times[0,T])$ into itself.

Proof:

From formula (6) and (7), we get

$$\left\|\bar{W}w(s,t)\right\| \le \frac{1}{|\mu|}\left\|f(s,t)\right\| + \left|\frac{\lambda}{\mu}\right|\left\|\int_0^s\int_0^t |k(s-u,t-v)|\,|w(u,v)|\,dudv\right\|$$

$$\left\|\bar{W}w(s,t)\right\| \le \frac{M}{|\mu|} + \left|\frac{\lambda}{\mu}\right|\{\int_0^s\int_0^t |k(s-u,t-v)|^2\,dudv\}^{\frac{1}{2}}\{\int_0^s\int_0^t |w(u,v)|^2\,dudv\}^{\frac{1}{2}}$$

The above inequality take the form

$$\left\|\bar{W}w(s,t)\right\| \le \frac{M}{|\mu|} + \theta\left\|w(s,t)\right\|, \quad (\theta = \left|\frac{\lambda}{\mu}\right| AC) \tag{8}$$

The last inequality (8) shows that, the operator $\bar{W}$ maps the space $C([0,S]\times[0,T])$ into itself.

Moreover, the inequality (8) involves the roundedness of the operator W of Eq. (6) , where

$$\left\|Ww(s,t)\right\| \le \theta\left\|w(s,t)\right\|, \tag{9}$$

Lemma 2.

If the conditions (i) and (iii) are satisfied, then the operator $\bar{W}$ is contractive in $C([0,S]\times[0,T])$.

Proof:

For the two functions $w_1(s,t)$ and $w_2(s,t)$ in the space $C([0,S]\times[0,T])$, the formulas (6), (7) lead to

$$\left\|\bar{W}w_1 - \bar{W}w_2\right\| \le \left|\frac{\lambda}{\mu}\right|\left\|\int_0^s\int_0^t |k(s-u,t-v)|\,|w_1(u,v) - w_2(u,v)|\,dudv\right\|$$

Then, we have

$$\left\|\bar{W}w_1 - \bar{W}w_2\right\| \le \left|\frac{\lambda}{\mu}\right|(\int_0^s\int_0^t |k(s-u,t-v)|^2\,dudv)^{\frac{1}{2}}(\int_0^s\int_0^t |w_1(u,v) - w_2(u,v)|^2\,dudv)^{\frac{1}{2}}$$

Finally, we obtain

$$\left\|(\bar{W}w_1 - \bar{W}w_2)(s,t)\right\| \le \theta \left\|w_1(s,t) - w_2(s,t)\right\| \tag{10}$$

3. THE NUMERICAL METHODS FOR SOLVING T-DFIE

3.1 The (TMM).

Consider:

$$\mu w(s,t) - \lambda \int_0^a \int_0^b k(s-u,t-v)w(u,v)dudv = f(s,t) \tag{11}$$

the integral term of Eq. (11) can be written as :

$$\int_0^a \int_0^b k(s-u,t-v)w(u,v)dudv = \sum_{n=-N}^{N-1} \sum_{m=-M}^{M-1} k(s-u,t-v)w(u,v)dudv \tag{12}$$

Where $h = \frac{a}{N}$, we approximate the integral in the right hand side of Eq. (3.31), if $m=n$, by

$$\int_{nh}^{nh+h} \int_{mh}^{mh+h} k(s-u,t-v)w(u,v)dudv = A_{n,m}(s,t)w(nh,mh) + B_{n,m}(s,t)w(nh+h,mh+h) + R \tag{13}$$

If $w(u,v) = 1.1,\ uv$ in Eq. (13), then we obtain

$$A_{n,m}(x,y) = \frac{1}{h}\left[\frac{(nh+h)(mh+h)I}{(nh+mh+h)} - \frac{J}{(nh+mh+h)}\right] \tag{14}$$

$$B_{n,m}(x,y) = \frac{1}{h}\left[\frac{J}{(nh+mh+h)} - \frac{(nh)(mh)I}{(nh+mh+h)}\right] \tag{15}$$

where

$$I(s,t) = \int_{nh}^{nh+h} \int_{mh}^{mh+h} k(s-u,t-v)dudv$$

$$J(s,t) = \int_{nh}^{nh+h} \int_{mh}^{mh+h} uv.k(s-u,t-v)dudv$$

Hence, eq. (12) becomes

$$\int_0^a \int_0^b k(s-u,t-v)\phi(u,v)dudv = \sum_{n=-N}^{N-1}\sum_{m=-M}^{M-1}[A_{n,m}(s,t)w(nh,mh)+B_{n,m}(s,t)w(nh+h,mh+h)]$$

$$= \sum_{n=-N}^{N-1}\sum_{m=-M}^{M-1} A_{n,m}(s,t)w(nh,mh) + \sum_{n=-N}^{N}\sum_{m=-M}^{M} B_{(n-1)(m-1)}(s,t)w(nh,mh)$$

$$= \sum_{n=-N}^{N}\sum_{m=-M}^{M} D_{n,m}(s,t)w(nh,mh) \qquad (16)$$

where

$$D_{n,m}(s,t) = \begin{cases} A_{-N}(s,t) & n=m=-N \\ A_n(s,t)+B_{n-1}(s,t) & -N < n=m < N \\ B_{N-1}(s,t) & n=m=N \end{cases}$$

Thus, the **IE** (5) becomes:

$$\mu w(s,t) - \lambda \sum_{n=-N}^{N}\sum_{m=-M}^{M} D_{n,m}(s,t)w(nh,mh) = f(s,t)$$

If we put $s=kh,\ t=lh,$ then we get:

$$\mu w_{k,l} - \lambda \sum_{n=-N}^{N}\sum_{m=-M}^{M} D_{kln,m} w_{nm} = f_{kl} \qquad -N \le k \le N, -M \le l \le M \qquad (17)$$

where

$$D_{kln,m} = \begin{cases} A_{-N}(kh,lh) & n=m=-N \\ A_n(kh,lh)+B_{n-1}(kh,lh) & -N < n=m < N \\ B_{N-1}(kh,lh) & n=m=N \end{cases} \qquad (18)$$

The matrix $D_{kln,m}$ may be written as $D_{kln,m} = G_{kln,m} - E_{kln,m}$, where

$$G_{kln,m} = A_n(kh,lh)+B_{n-1}(kh,lh), \qquad -N \le k,l,n, \le N \qquad (19)$$

is the Toeplitz matrix of order $2N+1$, and the matrix

$$E_{kln,m} = \begin{cases} B_{-N-1}(kh,lh) & n=m=-N \\ 0 & -N < n=m < N \\ A_N(kh,lh) & n=m=N \end{cases} \qquad (20)$$

Then,

$$w_{k,l} = [\mu I - \lambda(G_{kln} - E_{kln})]^{-1} f_{kl} \qquad (21)$$

Where I is the identity matrix and $\left|\mu I - \lambda(G_{kln} - E_{kln})\right| \neq 0.$

3.2. The (PNM)

Consider

$$\mu w(s,t) - \lambda \int_0^s \int_0^t k(s-u,t-v) w(u,v) du dv = f(s,t) \tag{22}$$

When $k(s-u,t-v)$ is singular within the range of integration. We can often factor out the singularity in k by writing

$$k(s-u,t-v) = \bar{k}(s-u,t-v) p(s-u,t-v) \tag{23}$$

where $p(s-u,t-v), \bar{k}(s-u,y-v)$ are badly behaved and well-behaved functions of their arguments, respectively. Eq. (22) take the form

$$\mu w(s,t) - \lambda \int_0^s \int_0^t p(s-u,t-v) \bar{k}(s-u,t-v) w(u,v) du dv = f(s,t) \tag{24}$$

We approximate the integral term in (24) when $s = s_i, t = t_i$ by

$$\int_0^s \int_0^t p(s_i-u,t_i-v) \bar{k}(s_i-u,t_i-v) w(u,v) du dv \approx \sum_{j=0}^{N} \sum_{i=0}^{M} \kappa_{ij} \kappa_{il} \bar{k}(s_i-u_j,t_i-v_j) w(u_i,v_j) \tag{25}$$

Where κ_{ij}, κ_{il} are the weights. Also,

$$\int_0^s \int_0^t p(s_i-u,t_i-v) \bar{k}(s_i-u,t_i-v) w(u,v) du dv \approx \sum_{j=0}^{N} \sum_{i=0}^{M} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2j}}^{v_{2j+2}} p(s_i-u,t_i-v) \bar{k}(s_i-u,t_i-v) du dv,$$

where $s_i = u_i = t_i = v_i = a + ih,\ i = 0,1,...,N$ with $h = \dfrac{b-a}{N}$ and N even. Now if we approximate the nonsingular part of the integrand over each interval $[u_{2j}, u_{2j+2}]$, $[v_{2l}, v_{2l+2}]$ by the second degree Lagrange interpolation polynomial that interpolates it at the points $u_{2j}, u_{2j+1}, u_{2j+2}, v_{2j}, v_{2j+1}, v_{2j+2}$ we find

$$\begin{aligned}
\int_0^x \int_0^y p(u_i-u,v_i-v) \tilde{k}(u_i-u,v_i-v) w(u,v) du dv = & \sum_{j=0}^{\frac{N-2}{2}} \sum_{l=0}^{\frac{M-2}{2}} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(u_i-u,v_i-v) \\
& \times \{ \frac{(u_{2j+1}-u)(v_{2l+1}-v)(u_{2j+2}-u)(v_{2l+2}-v)}{(2h^2)(2h^2)} \tilde{k}(u_i-u_{2j},v_i-v_{2l}) w(u_{2j},v_{2l}) \\
& + \frac{(u-u_{2j})(v-v_{2l})(u_{2j+2}-u)(v_{2l+2}-v)}{(h^2)(h^2)} \tilde{k}(u_i-u_{2j+1},v_i-v_{2l+1}) w(u_{2j+1},v_{2l+1}) \\
& + \frac{(u-u_{2j})(v-v_{2l})(u-u_{2j+1})(v-v_{2l+1})}{(2h^2)(2h^2)} \tilde{k}(u_i-u_{2j+2},v_i-v_{2l+2}) w(u_{2j+2},v_{2l+2}) \} du dv \\
= & \sum_{j=0}^{N} \sum_{l=0}^{M} \kappa_{ij} \kappa_{il} \tilde{k}(u_i-u_j,v_i-v_l) \phi(u_i,v_l)
\end{aligned}$$

Where $u_j = jh,\ u_{j+1} = (j+1)h,\ u_j - u_{j+1} = v_l - v_{l+1} = -h,$ and the weight functions

κ_{ij}, κ_{il} are given by

$$\kappa_{i,0}\kappa_{i,0} = \frac{1}{4h^2}\int_{u_0}^{u_2}\int_{v_0}^{v_2} p(u_i - u, v_i - v)(u_1 - u)(v_1 - v)(u_2 - u)(v_2 - v)dudv$$

$$\kappa_{i,2j+1}\kappa_{i,2l+1} = \frac{1}{h^4}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}} p(u_i - u, v_i - v)(u - u_{2j})(v - v_{2l})(u_{2j+2} - u)(v_{2l+2} - v)dudv$$

$$\kappa_{i,2j}\kappa_{i,2l} = \frac{1}{4h^4}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2l-2}}^{v_{2l}} p(u_i - u, v_i - v)(u - u_{2j-2})(v - v_{2j-2})(u - u_{2j-1})(v - v_{2j-1})dudv$$

$$+\frac{1}{4h^4}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}} p(u_i - u, v_i - v)(u_{2j+1} - u)(v_{2j+1} - v)(u_{2j+2} - u)(v_{2j+2} - v)dudv$$

$$\kappa_{i,N}\kappa_{i,M} = \frac{1}{4h^4}\int_{u_{N-2}}^{u_N}\int_{v_{M-2}}^{v_M} p(u_i - u, v_i - v)(u - u_{N-2})(v - v_{M-2})(u - u_{N-1})(v - v_{M-1})dudv \quad (26)$$

If we define

$$\alpha_{j,i}(u_i, v_i) = \frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}} p(u_i - u, v_i - v)(u - u_{2j-2})(v - v_{2j-2})(u - u_{2j-1})(v - v_{2j-1})dudv$$

$$\beta_{j,i}(u_i, v_i) = \frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}} p(u_i - u, v_i - v)(u_{2j-1} - u)(v_{2j-1} - v)(u_{2j} - u)(v_{2j} - v)dudv$$

$$\gamma_{j,i}(u_i, v_i) = \frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}} p(u_i - u, v_i - v)(u - u_{2j-2})(v - v_{2j-2})(u_{2j} - u)(v_{2j} - v)dudv \quad (27)$$

It follows that

$$\kappa_{i,0}\kappa_{i,0} = \beta_{1,1}(u_i, v_i), \quad \kappa_{i,2j+1}\kappa_{i,2j+1} = 4\gamma_{j+1,i+1}(u_i, v_i),$$
$$\kappa_{i,2j}\kappa_{i,2i} = \alpha_{j,i}(u_i, v_i) + \beta_{j+1,i+1}(u_i, v_i), \quad \kappa_{i,N}\kappa_{i,M} = \alpha_{\frac{N}{2},\frac{M}{2}}(u_i, v_i) \quad (28)$$

In general, assume $u = u_{2j-2} + \xi h$, $\mathrm{v} = v_{2l-2} + \delta h$, $0 \le \xi, \delta \le 2$, thus (27) become

$$\alpha_{j,l}(u_i, v_i) = \frac{h}{4}\int_0^2\int_0^2 \xi\delta(\xi - 1)(\delta - 1)p(u_i - (u_{2j-2} + \xi h), v_i - (v_{2l-2} + \delta h))d\xi d\delta$$

$$\beta_{j,l}(u_i, v_i) = \frac{h}{4}\int_0^2\int_0^2 (\xi - 1)(\xi - 2)(\delta - 1)(\delta - 2)p(u_i - (u_{2j-2} + \xi h), v_i - (v_{2l-2} + \delta h))d\xi d\delta,$$

$$\gamma_{j,l}(u_i, v_i) = \frac{h}{4}\int_0^2\int_0^2 \xi\delta(2 - \xi)(2 - \delta)p(u_i - (u_{2j-2} + \xi h), v_i - (v_{2l-2} + \delta h))d\xi d\delta \quad (29)$$

If we define $\psi_k = \int_0^2 \int_0^2 \xi^k \delta^k p(u_i - (u_{2j-2} + \xi h), v_i - (v_{2l-2} + \delta h)) d\xi d\delta, k = 0,1,2,$

and let $u_i - u_{2j-2} = (i - 2j + 2)h$, $v_i - v_{2l-2} = (i - 2l + 2)h$, we have

$$\psi_k = \int_0^2 \int_0^2 \xi^k \delta^k p((z - \xi)h, (g - \delta)h) d\xi d\delta, \quad k = 0,1,2, \; z = i - 2h + 2, \; g = i - 2i + 2 \tag{30}$$

Therefore, (22) transforms into the following system of **LAEs:**

$$\mu w(s_i, t_i) - \lambda \sum_{j=0}^{N} \sum_{l=0}^{M} \kappa_{ij} \kappa_{il} \tilde{k}(u_i - u_j, v_i - v_l) w(u_j, v_l) = f(s_i, t_i), \quad i = 0,1,...,N \tag{31}$$

4. APPLICATIONS AND NUMERICAL RESULTS

Here the **TMM** and **PNM** are used to get numerical solution for values of μ=1, and for different materials , Nickel $\upsilon_1, \upsilon_2 = 0.28, \lambda = 0.09003622491,$ and Fibber $\upsilon_1, \upsilon_2 = 0.22, \lambda = 0.1579974526$, where the Poisson ratio $0 \le \upsilon < 1,$ and we consider two rigid surfaces having two materials, from equation (1), we get $\lambda = \frac{\theta_1 + \theta_2}{\lambda_1 + \lambda_2}$, and $\theta_i = \frac{1 - \upsilon_i^2}{\pi E_i}$. $\lambda_i = \frac{2\mu\upsilon_i}{1 - 2\upsilon_i}$, $E_i = 2\mu(1 + \upsilon_i)$, i=1, 2, where *E* is called Young modulus and λ , μ are called Lame constants. We divided the position interval by *N*=10,20 units. Since $0 \le s,t \le S,T < \infty$, we choose the time *S*, *T* = 0.02, 0.5.

EXAMPLE 1.

$$w(s,t) - \lambda \int_0^s \int_0^t \ln|s - u| \ln|t - v| w(u,v) du dv = f(s,t)$$

The exact solution $w(s,t) = (st)^2$

T	*N*	*s*	*t*	*Aprro. T*	*Error T.*	*Aprro. N*	*Error N.*
0.02	**10**	**0.0**	**00.0**	**0.000509727**	**0.000509727**	**0.00005395**	**0.000053959**
		0.2	**0.2**	**0.002369712**	**0.000769725**	**0.00196486**	**0.000364866**
		0.4	**0.4**	**0.028112899**	**0.002512899**	**0.02601517**	**0.000415170**
		0.6	**0.6**	**0.136244777**	**0.006644777**	**0.12995993**	**0.000359938**
		0.8	**0.8**	**0.423467054**	**0.013867054**	**0.40961287**	**0.000012874**
		1.0	**1.0**	**1.009649481**	**0.009649481**	**0.99905451**	**0.000945487**
	20	**0.0**	**0.0**	**0.0002301491**	**0.000230149**	**0.000247747**	**0.000247747**
		0.2	**0.2**	**0.0019491625**	**0.000349162**	**0.002605172**	**0.001005172**

		0.4	**0.4**	**0.0267626252**	**0.001162625**	**0.027625099**	**0.002025099**
		0.6	**0.6**	**0.1327537734**	**0.031537734**	**0.130429967**	**0.000829967**
		0.8	**0.8**	**0.4162850315**	**0.006685031**	**0.409787864**	**0.000187864**
		1.0	**1.0**	**1.0045133080**	**0.004513308**	**0.999119041**	**0.000880959**
0.5	**10**	**0.0**	**0.0**	**0.00033123**	**0.00033123**	**0.00006925**	**0.000069253**
		0.2	**0.2**	**0.00177109**	**0.00017109**	**0.00200655**	**0.000406556**
		0.4	**0.4**	**0.02670636**	**0.00110636**	**0.02621856**	**0.000618561**
		0.6	**0.6**	**0.13583694**	**0.00623694**	**0.13002213**	**0.000422131**
		0.8	**0.8**	**0.42333867**	**0.01373867**	**0.40965948**	**0.000059748**
		1.0	**1.0**	**1.00961160**	**0.00961160**	**0.99908116**	**0.000918837**
	20	**0.0**	**0.0**	**0.00005875**	**0.000058758**	**0.00020323**	**0.00020323**
		0.2	**0.2**	**0.00136059**	**0.000239406**	**0.00159605**	**0.395×10^{-5}**
		0.4	**0.4**	**0.02537062**	**0.000229374**	**0.02488282**	**0.00071718**
		0.6	**0.6**	**0.13235491**	**0.002754914**	**0.12654010**	**0.00305990**
		0.8	**0.8**	**0.41616039**	**0.006560394**	**0.40248120**	**0.00711880**
		1.0	**1.0**	**1.00447636**	**0.004476364**	**0.99394592**	**0.00605408**

Table 1. The values of approximate and absolute error values by **TMM** and **PNM** at $\upsilon_1, \upsilon_2 = 0.28, \lambda = 0.09003622491$

T	***N***	***s***	***t***	***Aprro. T***	***Error T.***	***Aprro. N***	***Error N***
0.02	**10**	**0.0**	**00.0**	**0.000977610**	**0.000967761**	**0.00011391**	**0.000113912**
		0.2	**0.2**	**0.003025366**	**0.001425366**	**0.00225565**	**0.000655650**
		0.4	**0.4**	**0.030179334**	**0.004579334**	**0.02335572**	**0.000735572**
		0.6	**0.6**	**0.141599155**	**0.011999155**	**0.13023623**	**0.000636239**
		0.8	**0.8**	**0.434465400**	**0.024865400**	**0.40962440**	**0.000024402**
		1.0	**1.0**	**1.017164409**	**0.017164409**	**0.99834050**	**0.001659498**
	20	**0.0**	**0.0**	**0.000419395**	**0.000419395**	**0.00449280**	**0.00449280**
		0.2	**0.2**	**0.002229345**	**0.000629345**	**0.00006031**	**0.00153969**
		0.4	**0.4**	**0.027676901**	**0.002079011**	**0.02500721**	**0.00059278**
		0.6	**0.6**	**0.135214513**	**0.005614513**	**0.13036140**	**0.00076140**
		0.8	**0.8**	**0.421461658**	**0.011861658**	**0.41334917**	**0.00374917**
		1.0	**1.0**	**1.007975339**	**0.007975339**	**1.00505422**	**0.00505422**
0.5	**10**	**0.0**	**0.0**	**0.000634780**	**0.000634780**	**0.00016173**	**0.00016173**
		0.2	**0.2**	**0.001946280**	**0.000346280**	**0.00244489**	**0.00084489**
		0.4	**0.4**	**0.027667673**	**0.002067673**	**0.02668524**	**0.00108524**
		0.6	**0.6**	**0.140858556**	**0.011258556**	**0.13035047**	**0.00075047**

		0.8	**0.8**	**0.434229594**	**0.024629594**	**0.40970753**	**0.00010753**
		1.0	**1.0**	**1.017095573**	**0.017095573**	**0.99838352**	**0.00161647**
	20	**0.0**	**0.0**	**0.000109308**	**0.000109308**	**0.00015267**	**0.000152672**
		0.2	**0.2**	**0.001182348**	**0.000414651**	**0.00141780**	**0.000182192**
		0.4	**0.4**	**0.025213552**	**0.000386447**	**0.02472575**	**0.000874248**
		0.6	**0.6**	**0.134502622**	**0.004902622**	**0.12868781**	**0.000912188**
		0.8	**0.8**	**0.421237944**	**0.011637944**	**0.40755875**	**0.002041246**
		1.0	**1.0**	**1.007909502**	**0.007909502**	**0.99737906**	**0.002620938**

Table 2. The values of approximate and absolute error values by **TMM** and **PNM** at $\upsilon_1, \upsilon_2 = 0.22, \lambda = 0.1579974526$

EXAMPLE 2.

$$w(s,t) - \lambda \int_0^1 \int_0^1 |s-u|^{-\upsilon_1} |t-v|^{-\upsilon_2} w(u,v) du dv = f(s,t),$$

The exact solution $w(s,t) = (st)^2$

T	***N***	***s***	***t***	***Aprro. T***	***Error T.***	***Aprro. N***	***Error N***
0.02	**10**	**0.0**	**00.0**	**0.007665386**	**0.007665386**	**0.002920508**	**0.00292050**
		0.2	**0.2**	**0.006654016**	**0.005054016**	**0.004463092**	**0.00286309**
		0.4	**0.4**	**0.031250505**	**0.005650505**	**0.028493291**	**0.00289329**
		0.6	**0.6**	**0.136378298**	**0.006778298**	**0.132575184**	**0.00297518**
		0.8	**0.8**	**0.418318821**	**0.008718821**	**0.412777380**	**0.00317738**
		1.0	**1.0**	**1.005062513**	**0.005062513**	**1.004129766**	**0.00412976**
	20	**0.0**	**0.0**	**0.003377912**	**0.003377912**	**0.00153428**	**0.001534288**
		0.2	**0.2**	**0.003887172**	**0.002287172**	**0.00171813**	**0.000118137**
		0.4	**0.4**	**0.028121430**	**0.002521430**	**0.02545174**	**0.000148252**
		0.6	**0.6**	**0.132660935**	**0.003060935**	**0.12780782**	**0.001792178**
		0.8	**0.8**	**0.413613355**	**0.004013355**	**0.40550087**	**0.004099130**
		1.0	**1.0**	**1.002330767**	**0.002330767**	**0.99940964**	**0.000590351**
0.5	**10**	**0.0**	**0.0**	**0.007638260**	**0.007638260**	**0.00272606**	**0.00272606**
		0.2	**0.2**	**0.006374895**	**0.004774895**	**0.00420586**	**0.00260586**
		0.4	**0.4**	**0.030812962**	**0.005212962**	**0.02814328**	**0.00254328**
		0.6	**0.6**	**0.137200813**	**0.007600813**	**0.13234770**	**0.00274770**
		0.8	**0.8**	**0.420703105**	**0.011103105**	**0.41259062**	**0.00299062**
		1.0	**1.0**	**1.006883268**	**0.006883268**	**1.00396215**	**0.00396215**
	20	**0.0**	**0.0**	**0.003193799**	**0.003193799**	**0.002931819**	**0.002931819**

		0.2	**0.2**	**0.003481350**	**0.001881350**	**0.003716810**	**0.002116810**
		0.4	**0.4**	**0.027416568**	**0.001816568**	**0.026928768**	**0.001328768**
		0.6	**0.6**	**0.132855591**	**0.003255591**	**0.127040781**	**0.002559219**
		0.8	**0.8**	**0.414674171**	**0.005074171**	**0.400994981**	**0.008605019**
		1.0	**1.0**	**1.003146071**	**0.003146071**	**0.992615631**	**0.007384369**

Table 3. The values of approximate and absolute error values by **TMM** and **PNM** at $\upsilon_1, \upsilon_2 = 0.28, \lambda = 0.09003622491$

T	***N***	***s***	***t***	***Aprro. T***	***Error T.***	***Aprro. N***	***Error N***
0.02	**10**	**0.0**	**00.0**	**0.01387400**	**0.01387400**	**0.00546460**	**0.00546460**
		0.2	**0.2**	**0.01073462**	**0.00913462**	**0.00700136**	**0.00540136**
		0.4	**0.4**	**0.03579169**	**0.01019169**	**0.03104799**	**0.00544799**
		0.6	**0.6**	**0.14177688**	**0.01217688**	**0.13516955**	**0.00556955**
		0.8	**0.8**	**0.42518339**	**0.01558339**	**0.41544743**	**0.00584730**
		1.0	**1.0**	**1.00900741**	**0.00900741**	**1.00697705**	**0.00697705**
	20	**0.0**	**0.0**	**0.006019467**	**0.006019467**	**0.00110726**	**0.00110726**
		0.2	**0.2**	**0.005673522**	**0.004073522**	**0.00350448**	**0.00190448**
		0.4	**0.4**	**0.030085802**	**0.004485802**	**0.02741612**	**0.00181612**
		0.6	**0.6**	**0.135034099**	**0.005434099**	**0.13018098**	**0.00058098**
		0.8	**0.8**	**0.416706424**	**0.007106424**	**0.40859393**	**0.00100606**
		1.0	**1.0**	**1.004117684**	**0.004117684**	**1.00119656**	**0.00119656**
0.5	**10**	**0.0**	**0.0**	**0.01339855**	**0.01339855**	**0.00513428**	**0.00513428**
		0.2	**0.2**	**0.01003711**	**0.00843711**	**0.00659326**	**0.00499326**
		0.4	**0.4**	**0.03467794**	**0.00907794**	**0.03053102**	**0.00493102**
		0.6	**0.6**	**0.14119284**	**0.01159284**	**0.13479609**	**0.00519609**
		0.8	**0.8**	**0.42475280**	**0.01515280**	**0.41512660**	**0.00552660**
		1.0	**1.0**	**1.00865687**	**0.00865687**	**1.00668172**	**0.00668172**
	20	**0.0**	**0.0**	**0.005565712**	**0.00556571**	**0.00530373**	**0.00530373**
		0.2	**0.2**	**0.004991293**	**0.00339129**	**0.00522675**	**0.00362675**
		0.4	**0.4**	**0.028989068**	**0.00338906**	**0.02850126**	**0.00290126**
		0.6	**0.6**	**0.134465333**	**0.00486533**	**0.12865052**	**0.00094947**
		0.8	**0.8**	**0.416289078**	**0.00668907**	**0.40260988**	**0.00699011**
		1.0	**1.0**	**1.003773001**	**0.00377300**	**0.99324256**	**0.00675743**

Table 4. The values of approximate and absolute error values by **TMM** and **PNM** at $\upsilon_1, \upsilon_2 = 0.22, \lambda = 0.1579974526$

5. THE CONCLUSIONS

This paper proposed an effective numerical method to obtain the solution of T-DVIE of the second kind with weakly kernels. For this purpose, TMM and PNM has been presented. Error analysis and some numerical examples are presented to illustrate the effectiveness and accuracy of the method. In general, TMM and PNM are solved different values of which corresponding to the different materials. From the previous results, we note that:

1- When the values of λ and υ are fixed, the error values ***Error T***. and ***Error N.*** are decreases as well as N increases for the two different materials $(\upsilon_1, \upsilon_2 = 0.28, \lambda = 0.09003622491)$, $(\upsilon_1, \upsilon_2 = 0.22, \lambda = 0.1579974526)$.

2-The error in the evaluation of the approximate solution, by using the **PNM**, is less than the error in the evaluation of the approximate solution, using the **TMM** in all cases.

3-The error values ***Error T***. and ***Error N.*** of the **T-DVIE** with logarithmic form is less than the error values ***Error T***. and ***Error N.*** of **T-DVIE** with Carleman form.

4- By using the relation $\lambda = \dfrac{\theta_1 + \theta_2}{\lambda_1 + \lambda_2}$, we note that when the values of υ_1, and υ_2 are decrease the corresponding values of λ is increase. '

LIST OF ABBREVIATIONS

TDVIE	the Two-dimensional Volterra integral equation
TMM	Toeplitz matrix method
PNM	Product Nyström method
IEs	Integral equations
VIE	Volterra integral equation
L-VIE's	Linear Volterra integral equations
NL-VIE's	Nonlinear Volterra integral equations
LAEs	Linear algebraic equations

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