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The Energy Spectral Density of the Mertens Function

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Abstract

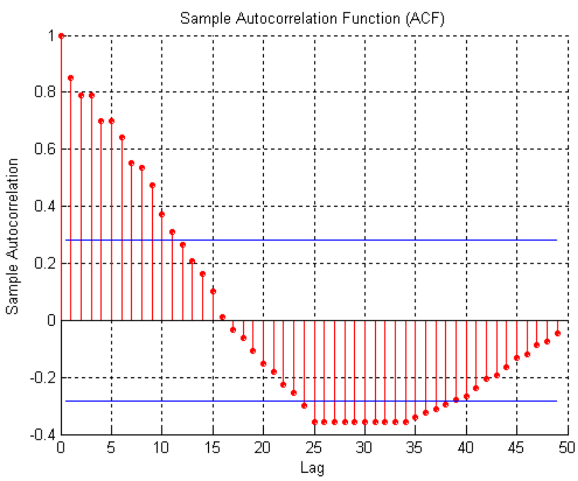
The Mertens function is the summatory Möbius function but the Mertens function can be generated recursively without using this definition. This recursive definition is the basis of autocorrelations that can be done on sequences of Mertens function values. Fourier transforms of the autocorrelations result in the energy spectral density. A likely upper bound of the absolute value of the Mertens function is determined.

1. INTRODUCTION

Lehman [1] proved that $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) = 1$ where $M(x)$ denotes the Mertens function. $M(x)$ is then equal to $1 - \sum_{i=1}^{x-1} M(\lfloor x/i \rfloor)$. It is only necessary to compute the first third ($\lfloor x/3 \rfloor$) of the sums - the sum of the remaining terms is $\lfloor (x+1)/2 \rfloor$.

2. AUTOCORRELATION

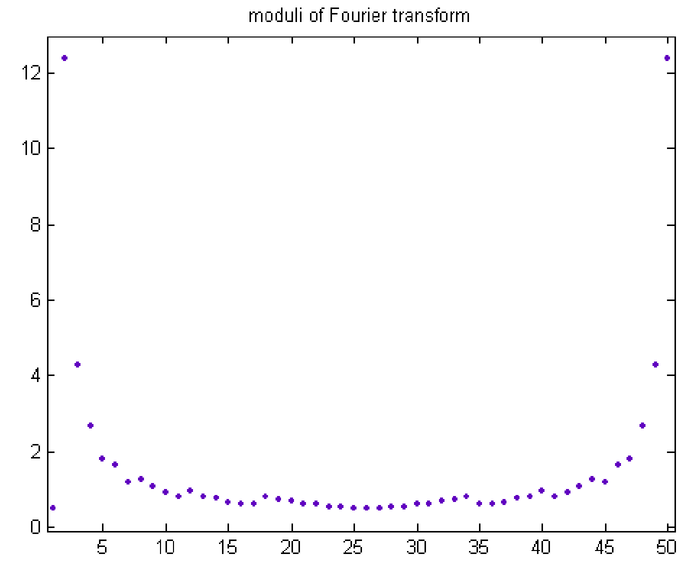
Let $a(x)$ denote $\{M(\lfloor x/1 \rfloor), M(\lfloor x/2 \rfloor), M(\lfloor x/3 \rfloor), \ldots, M(\lfloor x/x \rfloor)\}$, $x = 1, 2, 3, \ldots$. A plot of the autocorrelation of $a(x)$ for $x = 50$ is



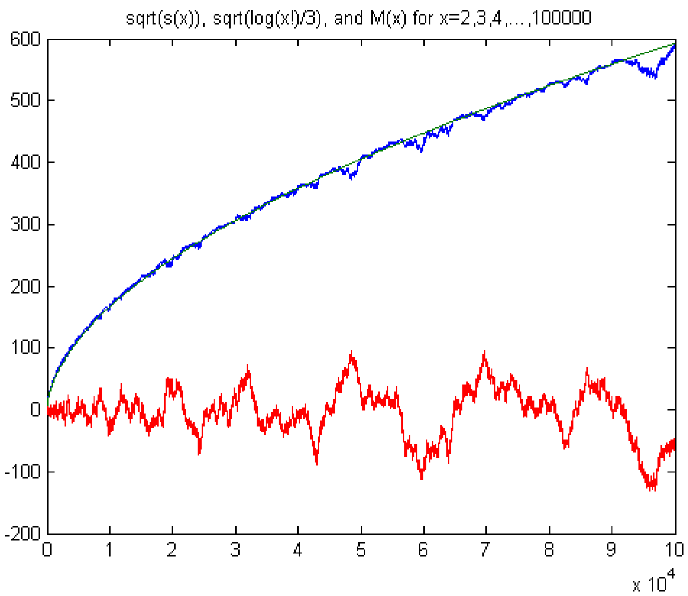
The autocorrelations for other x values are similar.

3. FOURIER TRANSFORMS

The energy spectral density is defined as $\int_{-\infty}^{\infty} |x(t)|^2$. The energy spectral density of $x(t)$ and the autocorrelation of $x(t)$ form a Fourier transform pair, a result known as the Wiener-Khinchin [2] [3] theorem. A plot of the moduli of the Fourier transform of the above autocorrelation is



Let $s(x)$ denote the sum of the moduli (the total energy). A plot of $\sqrt{s(x)}$, $\sqrt{\log(x!)/3.0}$, and $M(x)$ for $x = 2, 3, 4, \ldots, 100000$ is



A relationship between $\sqrt{s(x)}$ and $M(x)$ is to be expected since the energy spectral density is computed using the squares of the Mertens function values. The Mertens [4] conjecture states that $\sqrt{x}$ is greater than $|M(x)|$. Odlysko and te Riele [5] proved that $\lim_{x\to\infty} \sup |M(x)|x^{-1/2} > 1.06$ and $\lim_{x\to\infty} \inf M(x)x^{-1/2} < -1.009$. They say that their disproof of the Mertens conjecture can undoubtedly be used to produce larger values for $\lim_{x\to\infty} \sup |M(x)|x^{-1/2}$ than 1.06 with the use of more computer time and that their disproof provides some additional evidence that no such inequality of the form $|M(x)| \le cx^{1/2}$ holds for any fixed c. They also conjecture that $\lim_{x\to\infty} \sup |M(x)|x^{-1/2} = \infty$. An upper bound of $\sqrt{\log(x!)/3}$ for $|M(x)|$ is not inconsistent with this.

Cox [6] conjectured that $\log(x!) > \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2$ when $x > 7$. By Stirling's formula, $\log(x!) = x\log(x) - x + O(\log(x))$. (Stirling's approximation of $\log(x!)$ is $(x + \frac{1}{2})\log(x) - x + \frac{1}{2}\log(2\pi)$.) Since $\log(x)$ increases more slowly than any positive power of x, this is a better upper bound of $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2$ than $x^{1+\epsilon}$ for any $\epsilon > 0$.

REFERENCES

[1] Lehman, R. S., On Liouville's Function, *Math. Comput.* **14**:311-320 (1960)

[2] Wiener, Norbert (1964), *Time Series*. M.I.T. Press, Cambridge, Massachusetts, p.42

[3] Khintchine, Alexander (1934), "korrelationstheorie der stationären stochastischer Prozesse", *Mathematische Annalen*, **109** (1): 604-615

[4] Mertens, F. (1897) *Uber eine zahlentheoretische Funktion*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Abteilung 2a. **106**: 761-830

[5] Oklyzko, A.M. and te Riele, H.J.J. (1985), *Disproof of the Mertens Conjecture*, Journal für die reine und angewandte Mathematik, **357**: 138-160

[6] Cox, Darrell and Ghosh, Sourangshu and Sultanow, Eldar. (2021). Bounds of the Mertens Functions. Advances in Dynamical Systems and Applications. 16. 35-44