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Sign of the Mertens function

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Abstract

Lehman proved that the sum of certain Mertens function values is 1. Functions involving the sum of the signs of these Mertens function values are considered here. Specifically, upper bounds of these functions involving the number of Mertens function values equal to zero are determined.

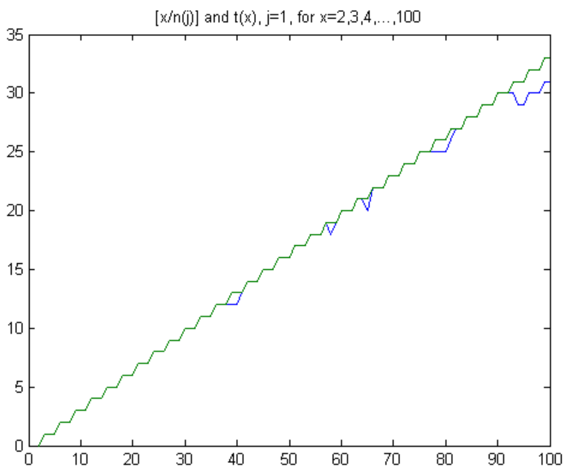
1. INTRODUCTION

Let $M(x)$ denote the Mertens function. Lehman [1] proved that

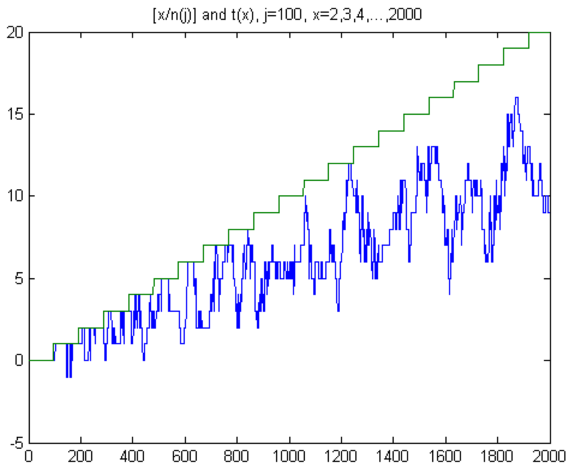
Theorem (1) $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) = 1$

Let $n(j)$ denote $2+\sum_{i=1}^{j} |\operatorname{sgn}(M(i))|$ where j is a natural number. Note that there is no contribution to the sum when $M(i)=0$. Let $t(x)$ denote $-\sum_{i=1}^{\lfloor x/(j+1) \rfloor} \operatorname{sgn}(M(\lfloor x/i \rfloor))$. Based on empirical evidence,
(2) $\lfloor x/n(j) \rfloor >= t(x)$, j=1,2,3,....

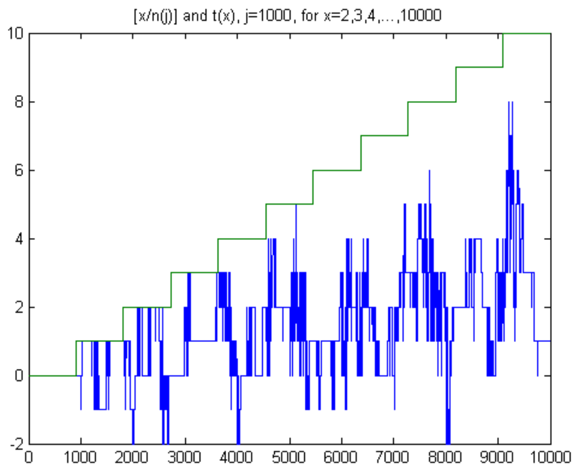
A plot of $t(x)$ and $\lfloor x/n(j) \rfloor$ for $j=1$ and $x \leq 100$ is



$M(x)$ equals 0 at $x = 2, 40, 41, 58, 65$, and 93. A plot of $t(x)$ and $\lfloor x/n(j) \rfloor$ for $j = 100$ and $x \leq 2000$ is



There are 92 $M(x)$ zeros less than 1000. A plot of $t(x)$ and $\lfloor x/n(j) \rfloor$ for $j = 1000$ and $x \leq 10000$ is



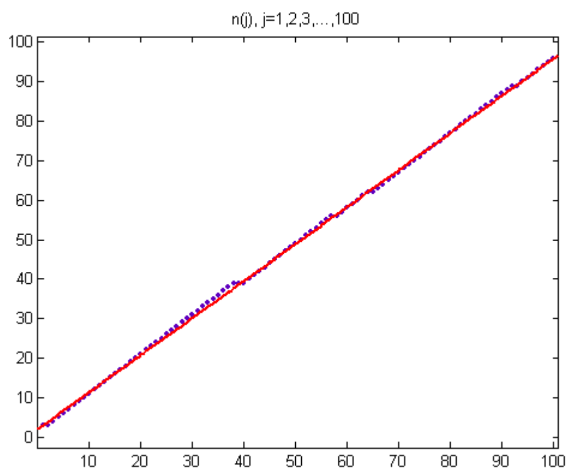
2. THE NUMBER OF MERTENS FUNCTION VALUES EQUAL TO ZERO

Let $V(x)$ denote the number of Mertens function zeros less than x. Landau proved that $V(x) = \Omega(\log(x))$. Hurst [2] computed $M(x)$ for x up to 10^{16} and concluded that this is a weak lower bound of $V(x)$. He determined that $3.5\sqrt{x}$ is a good estimate of $V(x)$ and compiled the following table of $V(x)$ values for x equal to powers of 10.

x =nth power of 10	V (x)
$n = 1:$	1
$n = 2:$	6
$n = 3:$	92
$n = 4:$	406
$n = 5:$	1549
$n = 6:$	5361
$n = 7:$	12546
$n = 8:$	41908
$n = 9:$	141121
$n = 10:$	431822
$n = 11:$	1628048
$n = 12:$	4657633
$n = 13:$	12917328
$n = 14:$	40604969
$n = 15:$	109205859
$n = 16:$	366567325

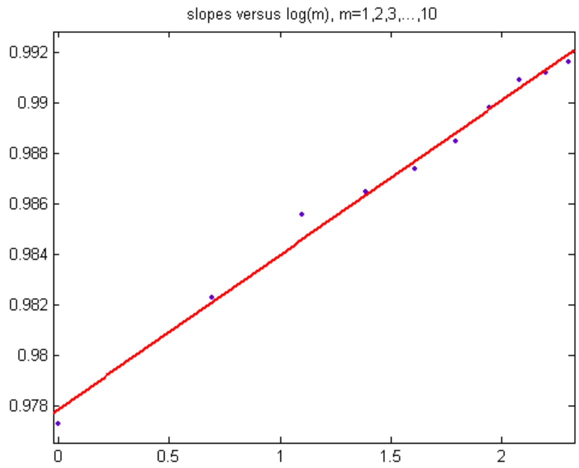
3. $N(J)$ VALUES

A plot of $n(j)$ for j = 1,2,3,...,100 is

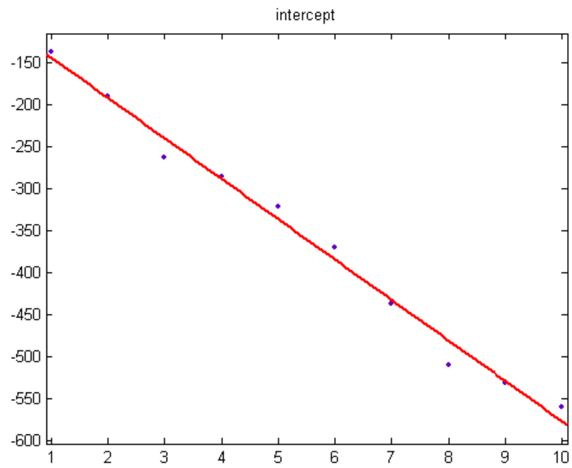


For a linear least-squares fit of the curve, $p_1 = 0.938$ with a 95% confidence interval of (0.9343, 0.9417), $p_2 = 2.039$ with a 95% confidence interval of (1.824, 2.254), SSE=28.34, R-squared=0.9996, and RMSE=0.5378. The slope is not 1 due to $M(i)$

values of 0. A plot of the slopes for $n(j)$ up to j equal 25000, 50000, 75000,..., 250000 versus $\log(m)$ for $m = 1, 2, 3, ..., 10$ is



The slopes increase at a logarithmic rate. The relative number of $M(i)$ values equal to 0 then decreases at a logarithmic rate. A plot of the corresponding intercepts is



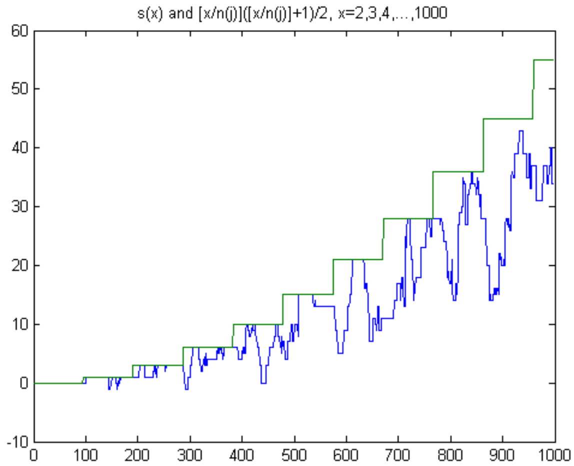
The intercepts decrease linearly.

4. A VARIANT OF $T(X)$

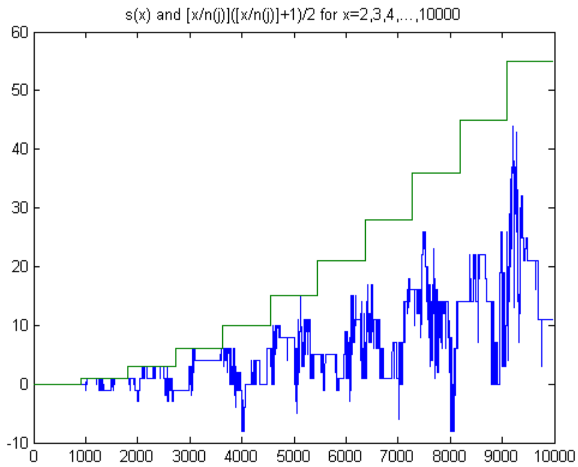
Let $s(x)$ denote $-\sum_{i=1}^{\lfloor x/(j+1)\rfloor} \mathrm{sgn}(M(\lfloor x/i\rfloor))i$. Based on empirical evidence,

(3) $(\lfloor x/n(j)\rfloor(\lfloor x/n(j)\rfloor + 1))/2 \geq= s(x)$, j=1,2,3,....

A plot of $s(x)$ and $(\lfloor x/n(j)\rfloor(\lfloor x/n(j)\rfloor+1))/2$ for $j=100$ and $x\leq 1000$ is



A plot of $s(x)$ and $(\lfloor x/n(j)\rfloor(\lfloor x/n(j)\rfloor+1))/2$ for $j=1000$ and $x\leq 10000$ is

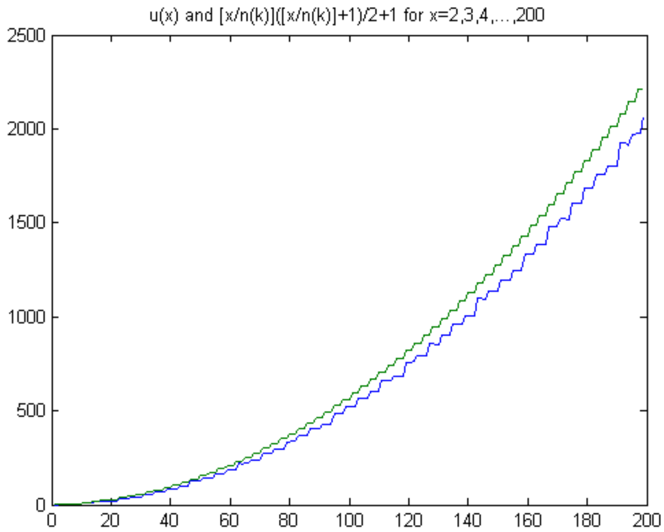


5. A VARIANT OF $S(X)$

Let $k=\sum_{i=1}^{j}|\operatorname{sgn}(M(i))|$ and let $r=(k-1)-\lfloor(k-1)/4\rfloor 4$. Let d=2, 3, 3, or 4 when r=0, 1, 2, or 3 respectively. Let $n(k)=3\lfloor(k-1)/4\rfloor+d$ and $\sigma_1(i)=\sum_{d|i}d$. Let $u(x)=-\sum_{i=1}^{\lfloor x/(j+1)\rfloor} sgn(M(\lfloor x/i\rfloor))\sigma_1(i)$. Based on empirical evidence,

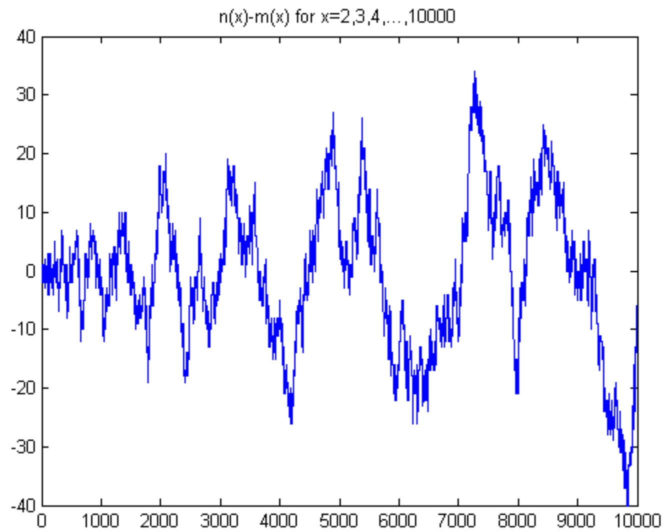
(4) $(\lfloor x/n(k)\rfloor(\lfloor x/n(k)\rfloor+1))/2+1\geq u(x)$, j=1,2,3....

A plot of $u(x)$ and $(\lfloor x/n(k)\rfloor(\lfloor x/n(k)\rfloor+1))/2+1$ for $j=3$ and $x\leq 200$ is

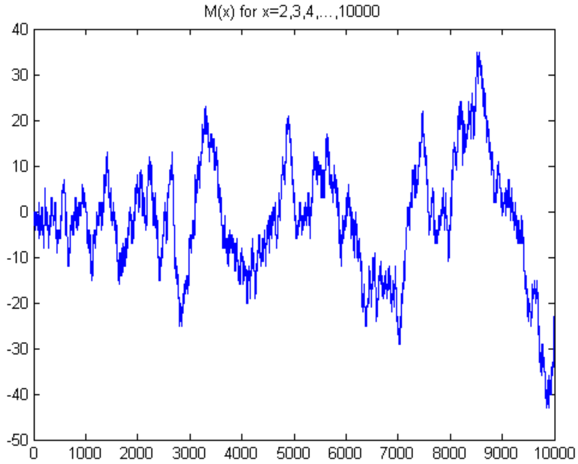


6. FAREY SEQUENCES AND THE MERTENS FUNCTION

The Farey sequence F_n of order n is the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. Let m_x denote the number of fractions before 1/4 and n_x the number of fractions between 1/4 and 1/2 in a Farey sequence of order x. A plot of $n_x - m_x$ for $x = 2, 3, 4, \ldots, 10000$ is



A plot of $M(x)$ for $x = 2, 3, 4, \ldots, 10000$ is



$n_x - m_x$ is an analogue of $M(x)$. Let $h(x)$ denote $\sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})$. (Although $1/4 < 1/3 < 1/2$, n_3 is set to 0 since 1/4 is not in a Farey sequence of order 3.) Based on empirical evidence,

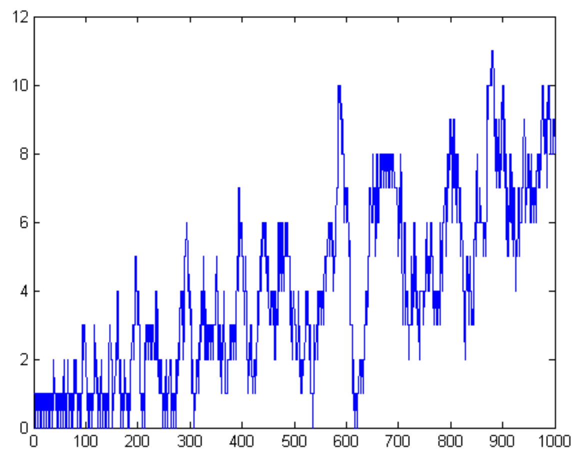
(5) $h(2), h(3), h(4), \ldots, h(13)$ equal 0, 0, 1, 1, 0, 1, 2, 1, 1, 2, 2, 2 respectively and $h(x + 12) = h(x) + 2$.

(6) $\sum_{i=1}^{x}(m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor} + \frac{1}{6})$ equals 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$, or $-\frac{2}{3}$.

Based on empirical evidence,

(7) $\sum_{i=1}^{x} \operatorname{sgn}(M(\lfloor x/i \rfloor)) \geq h(x)$

A plot of $\sum_{i=1}^{x} \operatorname{sgn}(M(\lfloor x/i \rfloor)) - h(x)$ for $x = 2, 3, 4, \ldots 1000$ is

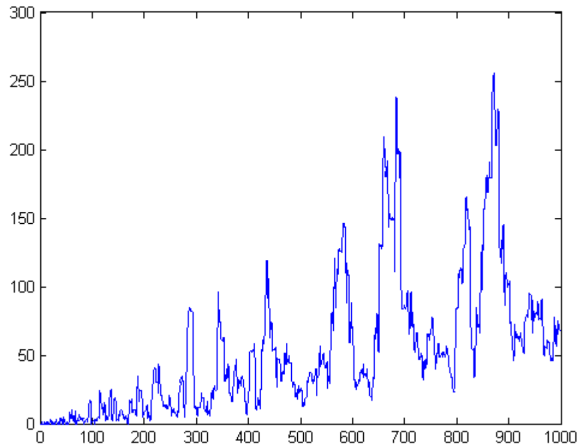


$\sum_{i=1}^{x} \operatorname{sgn}(M(\lfloor x/i \rfloor))$ equals $h(x)$ in 47 instances.

Based on empirical evidence,

(8) $1+\sum_{i=1}^{x}(n_{\lfloor x/i\rfloor}-m_{\lfloor x/i\rfloor})^2 \geq \sum_{i=1}^{x}\mathrm{sgn}(M(\lfloor x/i\rfloor))$

A plot of $1+\sum_{i=1}^{x}(n_{\lfloor x/i\rfloor}-m_{\lfloor x/i\rfloor})^2-\sum_{i=1}^{x}\mathrm{sgn}(M(\lfloor x/i\rfloor))$ for $x=2,3,4,\ldots,1000$ is



$1+\sum_{i=1}^{x}(n_{\lfloor x/i\rfloor}-m_{\lfloor x/i\rfloor})^2$ equals $\sum_{i=1}^{x}\mathrm{sgn}(M(\lfloor x/i\rfloor))$ in 39 instances.

REFERENCES

[1] R. S. Lehman, On Liouville's Function, *Math. Comput.* **14**:311-320 (1960)

[2] Greg Hurst, Computations of the Mertens Function and Improved Bounds on the Mertens Conjecture, arXiv:1610.08551v2 [math.NT] 1 Sep 2017