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Graph-Theoretic properties of Conjugate Graphs in Special Linear Groups

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Abstract

This paper investigates the graph-theoretic structure of conjugate graphs derived from special linear groups $SL_2(q)$, where q is an odd prime. The conjugate graph is defined with vertices corresponding to the non-central elements of the group, and edges representing conjugacy relations. We present detailed analyses of key properties, including the chromatic number, clique number, independence number, dominating number, and planarity. The spectrum, energy, Laplacian matrix, and common neighborhood matrix of the conjugate graph are also derived. These findings enrich the understanding of the interplay between algebraic and graph-theoretic properties in group theory.

Keywords: Conjugate graph, Special linear group.

Introduction

The interplay between algebraic structures and graph theory has been a subject of significant interest in recent mathematical research. Graphs associated with groups often reveal intricate connections between the two domains, offering a visual and structural perspective on abstract algebraic concepts. Among such graphs, the conjugate graph provides a fascinating way to analyze the conjugacy relations within a group. In this paper, we focus on the conjugate graph of special linear groups $SL_2(q)$, where q is an odd prime.

The conjugate graph, denoted by Γ_G^c was first introduced by Erfanian and Tolue in 2012[2]. It is a simple graph constructed from a group G by assigning vertices to the non-central elements of G and connecting two vertices if the corresponding elements are conjugate. This graph captures important group-theoretic properties and serves as a powerful tool for studying the underlying algebraic structure through the lens of graph theory. The study of conjugate graphs has garnered significant attention in recent years, with researchers investigating these graphs in the context of various groups, including p-groups, metacyclic groups, and others as discussed in [3],[8],[9] and [11]. These studies have provided valuable insights into the relationship between the algebraic properties of groups and the combinatorial characteristics of their associated graphs. Building upon the recent studies in this area, we aim to contribute to this area by examining the conjugate graph of the special linear group $SL_2(q)$, a group with a rich and well-understood conjugacy class structure. We begin by establishing its fundamental graph-theoretic parameters, including the chromatic number, clique number, independence number, and dominating number. Further, we investigate its planarity and demonstrate its disconnection into complete subgraphs. Spectral properties, such as the eigenvalues, graph energy, and Laplacian matrix, are also derived, highlighting the deep connections between the algebraic structure of $SL_2(q)$ and the graph-theoretic properties of $\Gamma^c_{SL_2(q)}$. Our results contribute to a growing body of literature exploring group-based graphs and their applications. They also provide new perspectives on the structural and spectral characteristics of conjugate graphs, which may inspire further research at the intersection of group theory and graph theory.

Preliminaries

In this section, some preliminary definitions and results about group theory and graph theory are mentioned which will be used throughout this paper.

(1.) **Conjugate of an element:** [6]Let G be a finite group. Two elements a and b of G are called conjugate if there exists g in G with $g^{-1}ag = b$.

(2.) **Conjugacy class:** [6]Conjugacy is an equivalence relation and the equivalence class that contains the element a in G denoted by $cl(a)$ is called the conjugacy class of a.

(3.) **Special linear group:** [5]The special linear group, denoted as $SL_n(F)$ is a mathematical group consisting of all $n \times n$ invertible matrices with determinant 1, where the entries of the matrices are elements of a field F. In this paper we shall consider the special linear group $SL_2(q)$ where q is an odd prime.

(4.) **Chromatic number:** [2]A chromatic number of a graph G, denoted as $\chi(G)$, is the least number of colours required to colour the vertices of G in such a way that no two adjacent vertices have the same colour.

(5.) **Clique number:** [2]A subset C of vertices of Γ is called a clique if the induced subgraph on C is a complete graph. The maximum size of a clique is called clique number of the graph G and is denoted by $\omega(G)$.

(6.) **Independent set:** [2]A subset X of the vertices of the graph G is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in the graph G is called the independence number of the graph denoted by $\alpha(G)$.

(7.) **Dominating number:** [2]For a graph Γ and a subset S of vertices, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is called a dominating set for Γ. The dominating number $\gamma(\Gamma)$ of Γ is the minimum size of a dominating set of the vertices of Γ.

(8.) **Planar graph:** [4]A graph is called planar if it can be drawn without crossing edges.

(9.) **Line graph:** [4] The line graph of a graph X is the graph $L(X)$ with the edges of X as its vertices, and where two edges of X are adjacent if and only if they are incident in X.

(10.) **Complement graph:** [4]The complement $\bar{X}$ of a graph X has the same vertex set as X, where vertices x and y are adjacent in $\bar{X}$ if and only if they are not adjacent in X.

(11.) **Spectrum:** [4]The spectrum of a matrix is the list of its eigenvalues together with their multiplicities.

(12.) **Laplacian matrix:** [4]The Laplacian matrix of a graph $G = (V, E)$ where V is the vertex set and E is the edge set is an $n \times n$ symmetric matrix with one row and column for each node defined by, $L = D - A$, where D is the degree matrix, which is the diagonal matrix formed from the vertex degrees and A is the adjacency matrix. The diagonal elements l_{ij} of L are therefore equal to the degree of vertex v_i and off-diagonal elements l_{ij} are -1 if vertex v_i is adjacent to v_j and 0 otherwise.

(13.) **Common neighbourhood graph matrix:** [1]Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For $i \neq j$, the common neighbourhood of the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$ is the set of vertices different from v_i and v_j which are adjacent to both v_i and v_j. The common neighbourhood matrix is then $CN = CN(G) = \|\gamma_{ij}\|$, where

$$\gamma_{ij} = \begin{cases} |\Gamma\{v_i, v_j\}|, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}.$$

(14.) **Conjugate Graph:** [2]A conjugate graph is a graph whose vertices are the non-centralelements of a group G and two distinct vertices are adjacent if they are conjugate. It is denoted by Γ_G^c.

(15.) **Result:** [5]The special linear group, denoted as $SL_n(F)$, is a mathematical group consisting of all $n \times n$ matrices with entries from a field F that have determinant equal to 1, under the operation of matrix multiplication. In this paper, we shall focus on $SL_2(q)$, which is a group consisting of 2×2 matrices with field of order q, where q is an odd prime. There are $q + 4$ conjugacy classes of this group which are listed as follows:
Number of conjugacy classes of size $1 = 2$.
Number of conjugacy classes of size $\frac{q^2-1}{2} = 4$.

Number of conjugacy classes of size $q(q-1) = \frac{q-1}{2}$.

Number of conjugacy classes of size $q(q+1) = \frac{q-3}{2}$.

(16.) **Result:** [10]Kuratowski's theorem: A graph is nonplanar if and only if it contains a subdivision of K_5 or $K_{3,3}$.

3. Results And Discussion

Theorem 3.1: The conjugate graph of $SL_2(q)$ denoted by $\Gamma^c_{SL_2(q)}$ is a union of $q+2$ complete graphs with q^3-q-2 vertices and $\frac{1}{2}[q^5-q^4-2q^3-4q^2+q+3]$ edges. Specifically,

$$\Gamma^c_{SL(2,q)} = \underbrace{K_{\frac{q^2-1}{2}} \cup \ldots \cup K_{\frac{q^2-1}{2}}}_{4} \cup \underbrace{K_{q(q-1)} \cup \ldots \cup K_{q(q-1)}}_{\frac{q-1}{2}} \cup \underbrace{K_{q(q+1)} \cup \ldots \cup K_{q(q+1)}}_{\frac{q-3}{2}}.$$

Proof: We know that for $SL_2(q)$ group, there are $q+4$ conjugacy classes out of which 2 are central, 4 conjugacy classes have $(q^2-1)/2$ elements each, $(q-1)/2$ conjugacy classes have $q(q-1)$ elements each and the other $(q-3)/2$ conjugacy classes have $q(q+1)$ elements each. Since all the elements within a given conjugacy class are mutually conjugate but not conjugate to any element in a different conjugacy class, hence by definition, $\Gamma^c_{SL_2(q)}$ is a union of $q+2$ complete graphs. Specifically, this includes four complete graphs of order $\frac{q^2-1}{2}$, $\frac{q-1}{2}$ complete graphs of order $q(q-1)$ and $\frac{q-3}{2}$ complete graphs of order $q(q+1)$. Also, we know,

$$|SL_n(q)| = \frac{|GL_n(q)|}{(q-1)} = \frac{(q^n-1)(q^n-q)\ldots(q^n-q^{n-1})}{q-1}.$$

For $n=2$, $|SL_2(q)| = q^3-q$. Since there are 2 elements in the centre of $SL_2(q)$, number of vertices of $\Gamma^c_{SL_2(q)} = q^3-q-2$.

Number of edges of $\Gamma^c_{SL_2(q)} = 4\binom{\frac{q^2-1}{2}}{2} + \left(\frac{q-1}{2}\right)\binom{q(q-1)}{2} + \left(\frac{q-3}{2}\right)\binom{q(q+1)}{2} = \frac{1}{2}[q^5 - q^4-2q^3-4q^2+q+3]$.

Example 3.2: Taking $q=3$, the conjugate graph $\Gamma^c_{SL_2(3)}$ is as given in figure 3.1,

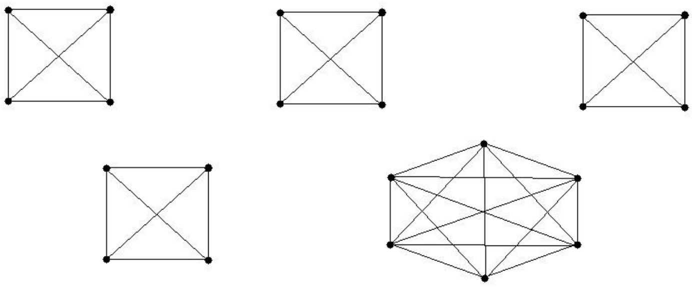

Figure 3.1

Corollary 3.3: $\Gamma^{c}_{SL_2(q)}$ is non planar.

Proof: For the smallest value of q i.e. $q = 3$, $\Gamma^{c}_{SL_2(q)}$ contains a subdivision of K_5 as shown in figure 3.1. Thus $\Gamma^{c}_{SL_2(q)}$ always contains a subdivision of K_5 for all values of q and hence is non planar.

Theorem 3.4: The clique number and the chromatic number of $\Gamma^{c}_{SL_2(q)}$ is $q(q+1)$.

i.e. $\chi\left(\Gamma^{c}_{SL_2(q)}\right) = \omega\left(\Gamma^{c}_{SL_2(q)}\right) = q(q+1)$.

Proof: We know that for a conjugate graph, clique number and chromatic number are equal. Now, since $\Gamma^{c}_{SL_2(q)}$ is a union of complete graphs, the clique number will be the greatest one between $\frac{q^2-1}{2}$, $q(q-1)$ and $q(q+1)$. Clearly, $q(q-1) < q(q+1)$ for all q. Now, to check between $\frac{q^2-1}{2}$ and $q(q+1)$.

Multiplying 2 on both these expressions we get, $2\left(\frac{q^2-1}{2}\right) = q^2 - 1$ and $2(q(q+1)) = 2q^2 + 2q$.

Clearly, $2q^2 + 2q > q^2 - 1$ and thus $q(q+1) > \frac{q^2-1}{2}$ for all q.

Hence $\chi\left(\Gamma^{c}_{SL_2(q)}\right) = \omega\left(\Gamma^{c}_{SL_2(q)}\right) = q(q+1)$.

Theorem 3.5: The independence number and dominating number of $\Gamma^{c}_{SL_2(q)}$ is $q+2$.

i.e. $\alpha\left(\Gamma^{c}_{SL_2(q)}\right) = \gamma\left(\Gamma^{c}_{SL_2(q)}\right) = q+2$.

Proof: We know that the independence number and dominating number of a conjugate graph are equal. Since $\Gamma^{c}_{SL_2(q)}$ is a union of $q+2$ components, each of which is a complete graph, we can form a subgraph by taking one vertex from each component so that the induced subgraph is an empty graph. Thus, $\alpha\left(\Gamma^{c}_{SL_2(q)}\right) = q + 2$ and hence the result.

Theorem 3.6: The line graph of $\Gamma^{c}_{SL_2(q)}$ is a union of $(q + 2)$ regular graphs. Four of these are $(q^2 - 5)$ regular graphs of order $\frac{1}{8}[q^4 - 4q^2 + 3]$, $(q - 1)/2$ are $2(q^2 - q - 2)$ regular graphs of order $\frac{1}{2}[q^4 - 2q^3 + q]$ and $(q - 3)/2$ are $2(q^2 + q - 2)$ regular graphs of order $\frac{1}{2}[q^4 + 2q^3 - q]$.

Proof: The line graph formed from a complete graph of order x is a $2(x - 2)$ regular graph of order $\binom{x}{2}$. This is easily verified by definition of a complete graph and regular graph. Thus, the line graph of complete graphs of size $\frac{q^2-1}{2}$ will form a $(q^2 - 5)$-regular graph of order $\frac{1}{8}(q^4 - 4q^2 + 3)$, complete graphs of size $q(q - 1)$ will form a $2(q^2 - q - 2)$-regular graph of order $\frac{1}{2}(q^4 - 2q^3 - q)$ and complete graph of size $q(q + 1)$ will form a $2(q^2 + q - 2)$-regular graph of order $\frac{1}{2}(q^4 + 2q^3 - q)$. Hence the result.

Theorem 3.7: $\overline{\Gamma^{c}_{SL_2(q)}}$ is a complete multipartite graph,

$$K_{\underbrace{\frac{q^2-1}{2},\ldots,\frac{q^2-1}{2}}_{4\ times},\underbrace{(q(q-1)),\ldots,(q(q-1))}_{\frac{q-1}{2}\ times},\underbrace{(q(q+1)),\ldots,(q(q+1))}_{\frac{q-3}{2}\ times}}.$$

Proof: We know that $\Gamma^{c}_{SL_2(q)}$ is a disconnected graph with $q + 2$ components. Thus, in $\overline{\Gamma^{c}_{SL_2(q)}}$, every vertex in each component is connected to all vertices in every other component except its own. In this way, we get a complete multipartite graph where each independent set corresponds to the vertices of a single component in $\Gamma^{c}_{SL_2(q)}$. Hence the theorem.

Theorem 3.8: The spectrum of $\Gamma^{c}_{SL_2(q)}$ is $\left\{(-1)^{q^3-2q-4}, \left(\frac{q^2-3}{2}\right)^4, (q^2 - q - 1)^{\frac{q-1}{2}}, (q^2 + q - 1)^{\frac{q-3}{2}}\right\}$.

Proof: We know that if $X = K_{m_1} \cup K_{m_2} \cup \ldots \cup K_{m_l}$, where K_{m_i}'s are complete graphs on m_i vertices for $1 \leq i \leq l$, then the spectrum of X is given by $\left\{(-1)^{\sum_{i=1}^{l} m_i - l}, (m_1 - 1)^1, \ldots, (m_l - 1)^1\right\}$. For $\Gamma^{c}_{SL_2(q)}$, $l = q + 2$. Thus, $\sum_{i=1}^{l} m_i = 4\left(\frac{q^2-1}{2}\right) + \left(\frac{q-1}{2}\right)\left(q(q - 1)\right) + \left(\frac{q-3}{2}\right)\left(q(q + 1)\right) = q^3 - q - 2$.

Hence, spectrum of $\Gamma^{c}_{SL_2(q)} = \left\{(-1)^{q^3-q-2-q-2}, \left(\frac{q^2-1}{2}-1\right)^4, (q(q-1)-1)^{\frac{q-1}{2}}, (q(q+1)-1)^{\frac{q-3}{2}}\right\} = \left\{(-1)^{q^3-2q-4}, \left(\frac{q^2-3}{2}\right)^4, (q^2-q-1)^{\frac{q-1}{2}}, (q^2+q-1)^{\frac{q-3}{2}}\right\}$.

Theorem 3.9: The energy of $\Gamma^{c}_{SL_2(q)}$ denoted by $\varepsilon(\Gamma^{c}_{SL_2(q)})$ is non hyperenergetic as well as non hypoenergetic.

Proof: We have that $\Gamma^{c}_{SL_2(q)}$ is a graph with q^3-q-2 vertices and its spectrum is $\left\{(-1)^{q^3-2q-4}, \left(\frac{q^2-3}{2}\right)^4, (q^2-q-1)^{\frac{q-1}{2}}, (q^2+q-1)^{\frac{q-3}{2}}\right\}$. Thus, $\varepsilon\left(\Gamma^{c}_{SL_2(q)}\right) = (1)(q^3-2q-4) + 4\left(\frac{q^2-3}{2}\right) + \left(\frac{q-1}{2}\right)(q^2-q-1) + \left(\frac{q-3}{2}\right)(q^2+q-1) = 2q^3-4q-8$.

Now, $2(n-1) = 2(q^3-q-2) = 2q^3-2q-4$, since n here represents the number of vertices in the graph.

Clearly, $2q^3-4q-8 < 2q^3-2q-4$ for all q. Hence, $\varepsilon(\Gamma^{c}_{SL_2(q)})$ is non hyperenergetic.

Also, $n = q^3-q-2$ and $\varepsilon\left(\Gamma^{c}_{SL_2(q)}\right) = 2q^3-4q-8$.

If we take the difference between these two expressions, we see that,

$2q^3-4q-8-q^3+q+2 = q^3-3q-6 > 0$ for all q.

Hence, $2q^3-4q-8 > q^3-q-2$ and $\varepsilon\left(\Gamma^{c}_{SL_2(q)}\right)$ is non hypoenergetic.

Also, it implies, $q^3-q-2 < \varepsilon\left(\Gamma^{c}_{SL_2(q)}\right) < 2q^3-2q-4$.

Theorem 3.10: The common neighbourhood matrix of $\Gamma^{c}_{SL_2(q)}$ is a $(q^3-q-2)\times(q^3-q-2)$ matrix and is given by,

$$CN\left(\Gamma^{c}_{SL_2(q)}\right) = \begin{cases} \frac{q^2-5}{2}, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(1) \\ q^2-q-2, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(2) \\ q^2+q-2, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(3) \\ 0, & \text{otherwise} \end{cases},$$

where, $C.C(1)$ represents the conjugacy classes of size $\frac{q^2-1}{2}$, $C.C(2)$ represents the conjugacy classes of size $q(q-1)$ and $C.C(3)$ represents the conjugacy classes of size $q(q+1)$.

Proof: The common neighbourhood matrix of a graph G is given as,

$$CN(G) = \begin{cases} |\Gamma\{v_i, v_j\}|, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}.$$

So we first need to calculate the value of $\Gamma(v_i, v_j)$. We know that there are $\frac{q^2-1}{2}$ vertices in each of the 4 $C.C(1)$. Since all conjugacy classes form a complete graph, each vertex is adjacent to every other vertex. Hence, $\Gamma(v_i, v_j) = \frac{q^2-1}{2} - 2 = \frac{q^2-5}{2}$, when $i \neq j$ and $v_i, v_j \in C.C(1)$. Similarly, we know that there are $q(q-1)$ vertices in each of the $\frac{q-1}{2}$ $C.C(2)$ and thus $\Gamma(v_i, v_j) = q(q-1) - 2 = q^2 - q - 2$, when $i \neq j$ and $v_i, v_j \in C.C(2)$. Also, we know that there are $q(q+1)$ vertices in each of the $\frac{q-3}{2}$ $C.C(3)$ and thus $\Gamma(v_i, v_j) = q(q+1) - 2 = q^2 + q - 2$, when $i \neq j$ and $v_i, v_j \in C.C(3)$. Hence the theorem.

Theorem 3.11: The Laplacian of $\Gamma^{c}_{SL_2(q)}$ is given as:

$$L_{ij} = \begin{cases} \frac{q^3-3}{2}, & \text{if } i = j \text{ and } v_i \in C.C(1) \\ q^2 - q - 1, & \text{if } i = j \text{ and } v_i \in C.C(2) \\ q^2 + q - 1, & \text{if } i = j \text{ and } v_i \in C.C(3) \\ -1, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(1) \text{or } v_i, v_j \in C.C(2) \text{or } v_i, v_j \in C.C(3) \\ 0, & \text{otherwise} \end{cases}$$

where, $C.C(1)$ represents the conjugacy classes of size $\frac{q^2-1}{2}$, $C.C(2)$ represents the conjugacy classes of size $q(q-1)$ and $C.C(3)$ represents the conjugacy classes of size $q(q+1)$.

Proof: By definition, the Laplacian of a graph is given by,

$$L_{i,j} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } i \neq j, v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}.$$

For vertices in v_1,

Each conjugacy class in $C.C(1)$ has $\frac{q^3-1}{2}$ vertices. Hence the degree of each vertex is $\frac{q^3-1}{2} - 1 = \frac{q^3-3}{2}$.

For vertices in v_2,

Each conjugacy class in $C.C(2)$ has $q(q-1)$ vertices. Hence the degree of each vertex is $q^2 - q - 1$.

For vertices in v_3,

Each conjugacy class in $C.C(3)$ has $q(q+1)$ vertices. Hence the degree of each vertex is $q^2 + q - 1$.

Adjacency condition:

Two vertices are adjacent if and only if they belong to the same conjugacy class. Therefore $L_{ij} = -1$ for $i \neq j$ and both $v_i, v_j \in C.C(1)$or $C.C(2)$or $C.C(3)$.

Combining the degree values and the adjacency conditions with the Laplacian definition, we obtain the stated result.

Conclusion

In this paper, we examined the conjugate graph of $SL_2(q)$ where q is an odd prime and showed that the graph is a disjoint union of complete graphs which is non planar. We also computed key graph-theoretic properties such as the chromatic number, clique number, independence number, and dominating number. Additionally, we derived its spectral properties, Laplacian matrix, and common neighborhood matrix, revealing a close connection between the group's algebraic structure and the graph's combinatorial characteristics. Our findings also open avenues for further exploration. For instance, examining conjugate graphs of higher-dimensional special linear groups or other classical groups could provide new insights into the relationship between group properties and graph invariants. Additionally, the spectral and energetic characteristics of these graphs could have implications for applications in algebraic combinatorics and related fields. In conclusion, this work highlights the significance of conjugate graphs as a tool for visualizing and analyzing the structural intricacies of groups. We hope this study inspires future research at the intersection of algebra and graph theory, fostering a deeper understanding of these interconnected domains.

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