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The Riemann Hypothesis and Polar Coordinates

Darrell Cox[1], **Pedro Caceres** [2]

[1,2] *Grayson County College, Universidad Europea de Valencia, Spain*

Abstract

The angles of polar coordinates of a function involving the Riemann zeta function (defined in the critical strip) are used to analyze properties of zeta function zeros.
Keywords: Riemann zeta function, Riemann hypothesis

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{1}$$

A plot of the real components of $\zeta(s)$ for the first non-trivial zeta functon zero ($s = (0.5, 14.1347251417)$) and $n \leq 200$ is

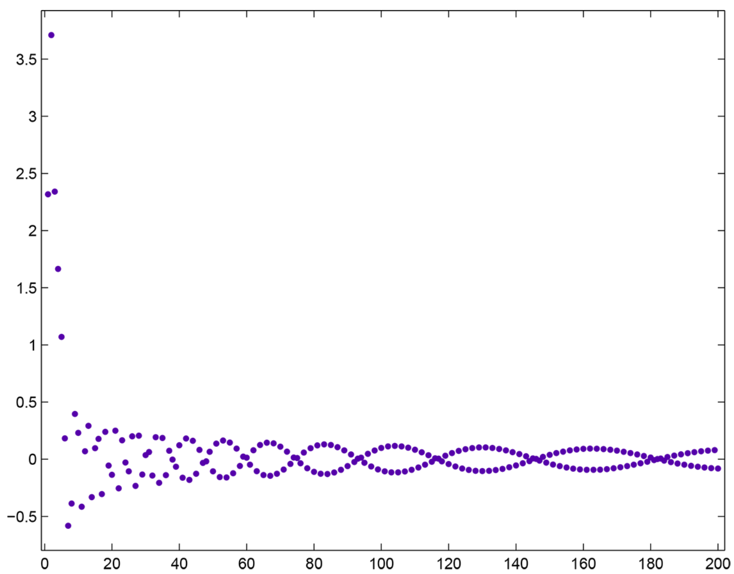

Figure 1

A plot of the values from $n = 147$ to 184 where successive values are connected is

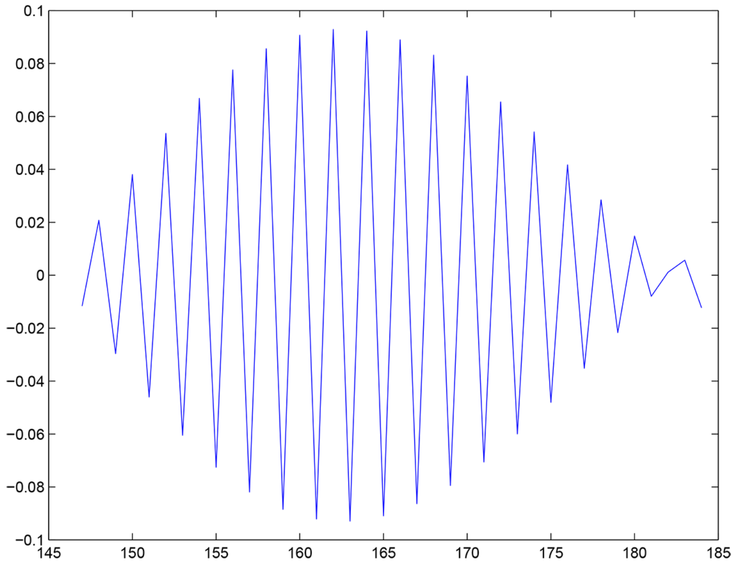

Figure 2

A plot of the n values of the inflection points (where the curve crosses the x-axis from above) for $n \leq 200$ is

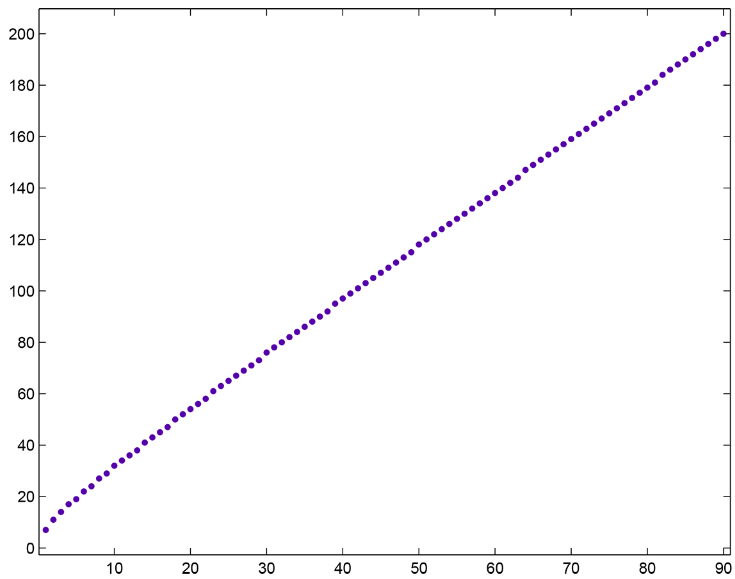

Figure 3

The n values of the inflection points for $n \leq 95$ are 7, 11, 14, 17, 19, 22, 24, 27, 29, 32, 34, 36, 38, 41, 43, 45, 47, 50, 52, 54, 56, 58, 61, 63, 65, 67, 69, 71, 73, 76, 78, 80, 82, 84, 86, 88, 90, 92, and 95. The n values that are at least three greater than the previous n values are 11, 14, 17, 22, 27, 32, 41, 50, 61, 76, and 95. Except for 11, the n values are three greater than the previous n value. The number of n values between 76 and 95 for example is $\frac{95-76-3}{2}$. A plot of the logarithms of the n values that are at least three greater than the previous n values for $n \leq 100000$ is

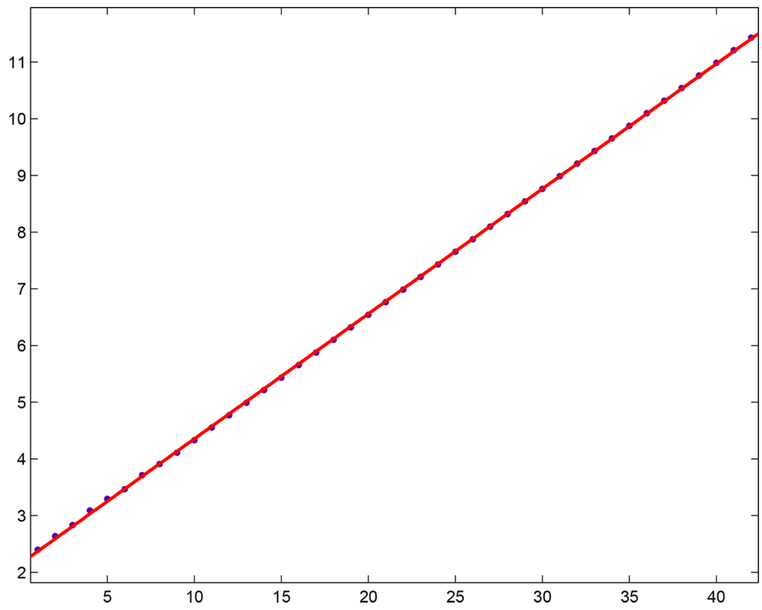

Figure 4

For a linear least-squares fit of the curve, $p_1 = 0.2205$ with a 95% confidence interval of (0.22, 0.2211), $p_2 = 2.15$ with a 95% confidence interval of (2.136, 2.163), SSE=0.01785, R-squared=0.9999, and RMSE=0.02113.

2. A FUNCTION INVOLVING THE RIEMANN ZETA FUNCTION

Let $C(n, a, b)$ denote

$$\frac{2 \cdot n^{-a}}{1 - 2^{1-s}} \cdot (\sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j^s} \cdot \cos(b \cdot (\ln(\frac{n}{j})))) \tag{2}$$

where $s = (a, b)$. See the Methods section for the C code. In polar coordinates, $r = \sqrt{x^2 + y^2}$ and $\theta =$atan2(y, x).

3. A REAL PART OF 0.50

A plot of the θ values of the $C(n, a, b)$ function for j values of inflection points that are at least three greater than the previous j value, $n = 3000001$, and $z = (0.50, 101.3178510057138)$ (a Riemann zeta function zero) versus the corresponding j values of inflection points is

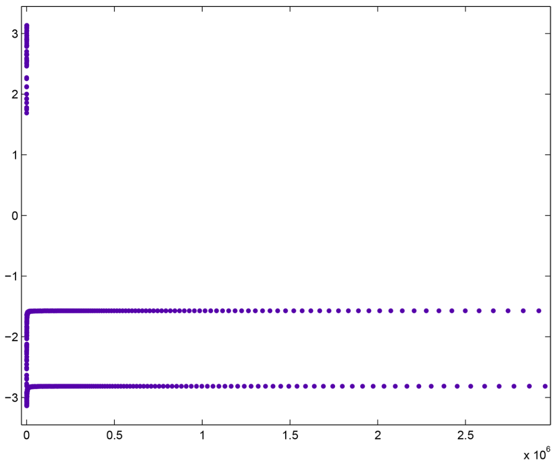

Figure 5

4. EVEN j VALUES

A plot of the even j values of the r values of the above plot versus the logarithms of the j values of the inflection points (neglecting spurious points at the beginning of the curves) is

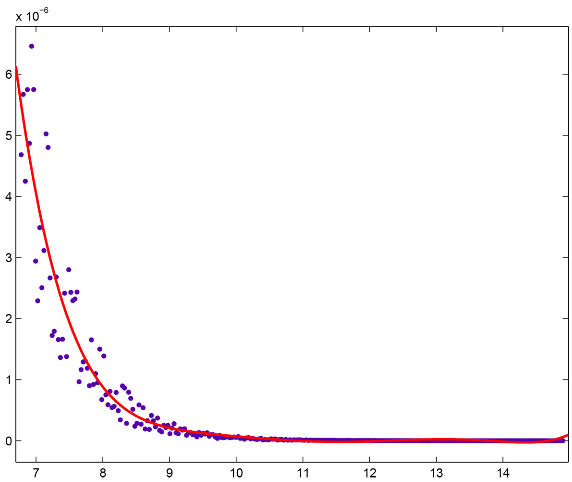

Figure 6

For a sextic least-squares fit of the curve, SSE=$2.94 \cdot 10^{-11}$, R-squared=0.9102, and RMSE=$3.389 \cdot 10^{-7}$.

A plot of the corresponding θ values versus the logarithms of the j values of the inflection points is

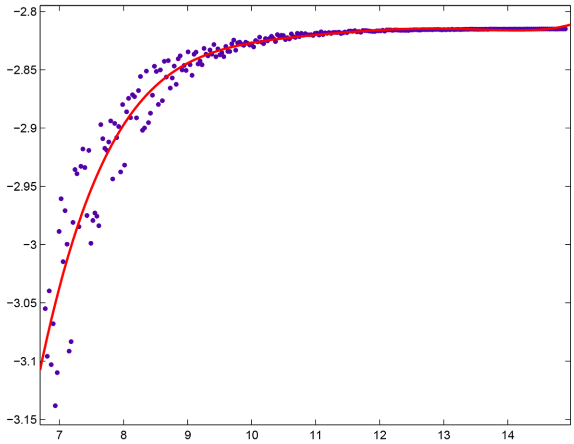

Figure 7

For quintic least-squares fit of the curve, SSE=0.7355, R-squared=0.9291, and RMSE=0.01692.

The differences between the j value of the current inflection point and that of the previous inflection point is three. A plot of the logarithms of the j values of the current inflection points is

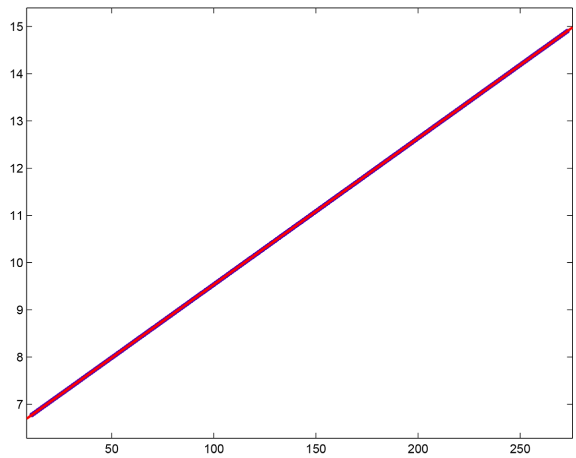

Figure 8

For a linear least-squares fit of the curve, $p_1 = 0.031$ with a 95% confidence interval of (0.031, 0.031), $p_2 = 6.435$ with a 95% confidence interval of (6.434, 6.435), SSE=$3.294 \cdot 10^{-5}$, R-squared=1, and RMSE=0.3552.

5. ODD j VALUES

A plot of the odd j values of the r values of the above plot versus the logarithms of the j values of the inflection points (neglecting spurious points at the beginning of the curves) is

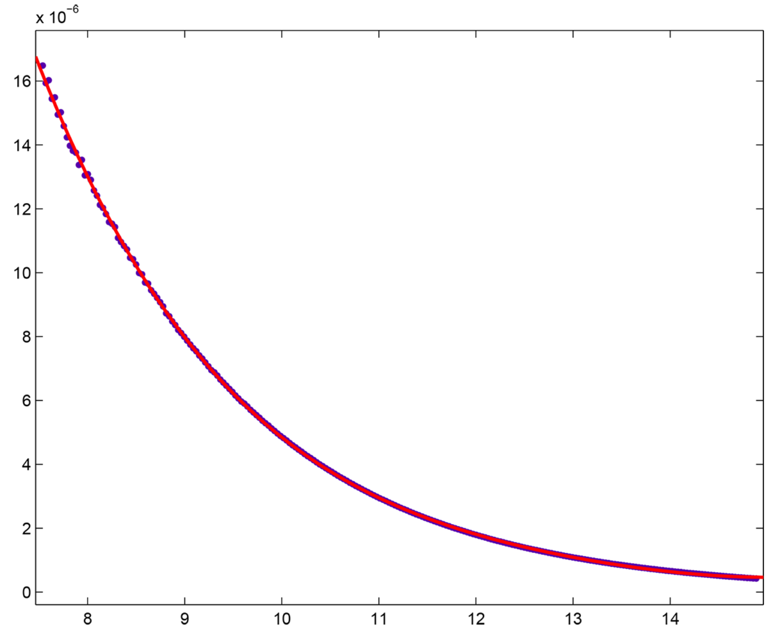

Figure 9

For a quartic least-squares fit of the curve, $p_1 = 6.936 \cdot 10^{-9}$ with a 95% confidence interval of $(6.484 \cdot 10^{-9}, 7.389 \cdot 10^{-9})$, $p_2 = -3.731 \cdot 10^{-7}$ with a 95% confidence interval of $(-3.934 \cdot 10^{-7}, -3.528 \cdot 10^{-7})$, $p_3 = 7.646 \cdot 10^{-6}$ with a 95% confidence interval of $(7.309 \cdot 10^{-6}, 7.983 \cdot 10^{-6})$, $p_4 = -7.114 \cdot 10^{-5}$ with a 95% confidence interval of $(-7.359 \cdot 10^{-5}, -6.869 \cdot 10^{-5})$, $p_5 = 0.0002554$ with a 95% confidence interval of (0.0002488, 0.000262), SSE=$5.828 \cdot 10^{-13}$, SSE=0.9999, and RMSE=$5.001 \cdot 10^{-8}$.

A plot of the corresponding θ values versus the logarithms of the j values of the inflection points is

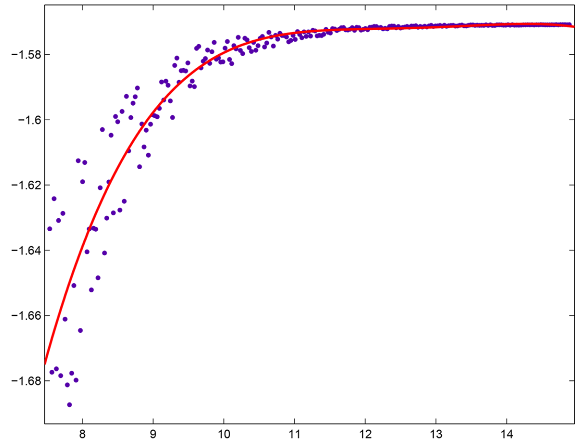

Figure 10

For a quartic least-squares fit of the curve, $p_1 = -0.000133$ with a 95% confidence interval of $(-0.0002108, -5.523 \cdot 10^{-5})$, $p_2 = 0.00682$ with a 95% confidence interval of (0.003328, 0.01031), $p_3 = -0.1308$ with a 95% confidence interval of $(-0.1888, -0.07279)$, $p_4 = 1.112$ with a 95% confidence interval of (0.6905, 1.534), $p_5 = -5.114$ with a 95% confidence interval of $(-6.248, -3.981)$, SSE=0.01723, R-squared=0.8876, and RMSE=0.008599.

The differences between the j value of the current inflection point and that of the previous inflection point is three. A plot of the logarithms of the j values of the current inflection points is

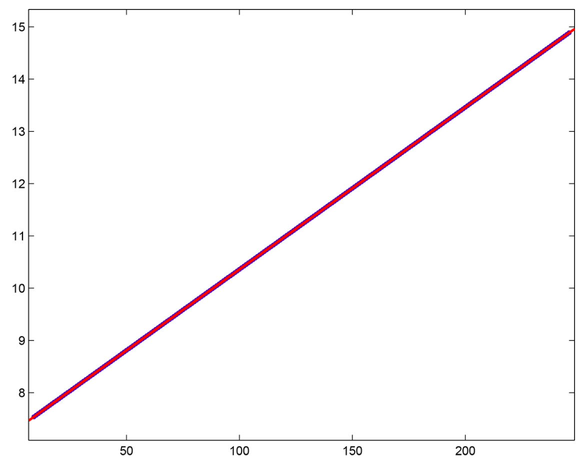

Figure 11

For a linear least-squares fit of the curve, $p_1 = 0.03101$ with a 95% confidence interval of (0.03101, 0.03101), $p_2 = 7.259$ with a 95% confidence interval of (7.259, 7.259), SSE=$4.505 \cdot 10^{-6}$, R-squared=1.0, and RMSE=0.0001382.

6. REAL PART OF 0.5001

A plot of the θ values of the $C(n, a, b)$ function for j values of inflection points that are at least three greater than the previous j value, $n = 3000001$, and $z = (0.5001, 101.3178510057138)$ versus the corresponding j values of inflection points is

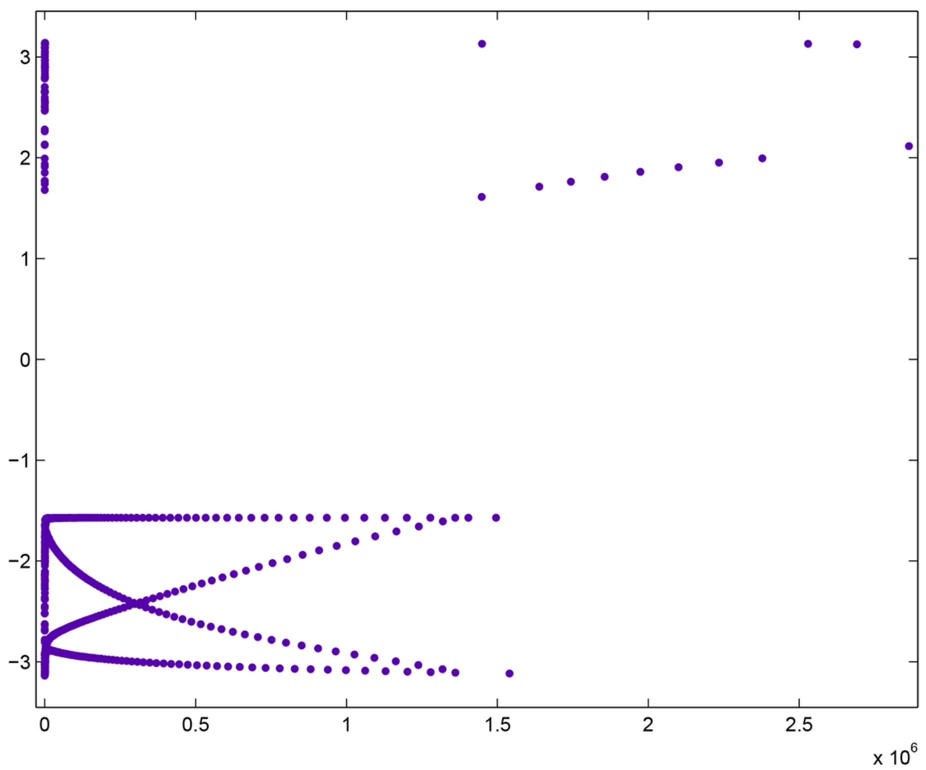

Figure 12

7. EVEN j VALUES

In the above graph, the two bottom curves (originating from about -2.87) corresponding to even j values and the two top curves (originating from less than $-\pi/2$) correspond to the odd j values.

A plot of the ascending curve of θ values versus the logarithms of the j values of the inflection points is

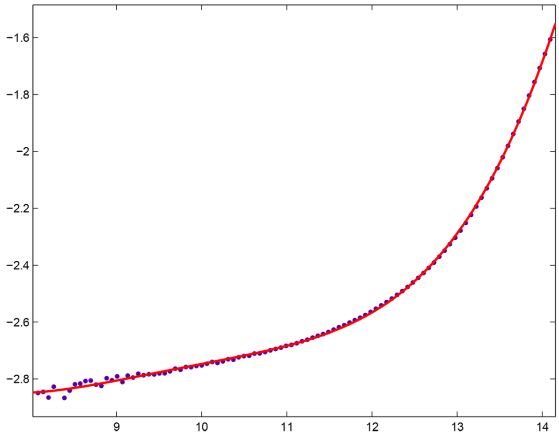

Figure 13

For a quartic least-squares fit of the curve, $p_1 = 0.002445$ with a 95% confidence interval of (0.00223, 0.002659), $p_2 = -0.09465$ with a 95% confidence interval of $(-0.1042, -0.08513)$, $p_3 = 1.373$ with a 95% confidence interval of (1.216, 1.53), $p_4 = -8.787$ with a 95% confidence interval of $(-9.926, -7.647)$, $p_5 = 18.02$ with a 95% confidence interval of (14.95, 21.09), SSE=0.004397, R-squared=0.9995, and RMSE=0.006914.

A plot of the corresponding r values versus the logarithms of the j values of the inflection points is

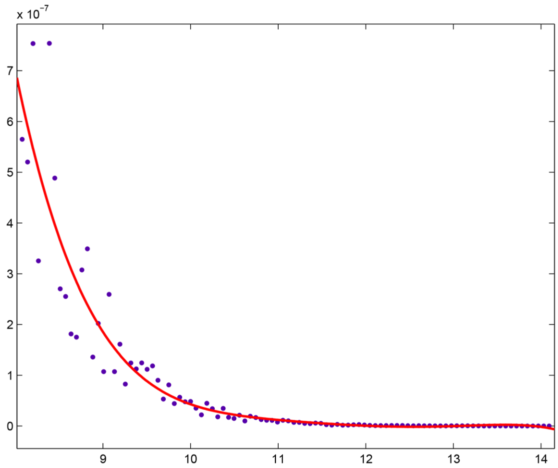

Figure 14

For a quintic least-squares fit of the curve, SSE=$2.874 \cdot 10^{-13}$, R-squared=0.8698, and RMSE=$5.62 \cdot 10^{-8}$.

The difference between the j value of an inflection point and that of the previous inflection point is 3. A plot of the logarithms of the current j values of inflection points is

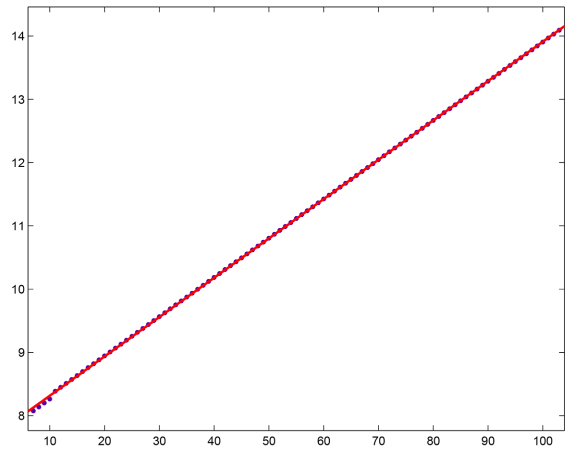

Figure 15

For a linear least-squares fit of the curve, $p_1 = 0.06216$ with a 95% confidence interval of (0.06208, 0.06225), $p_2 = 7.694$ with a 95% confidence interval of (7.689, 7.7), SSE=0.01292, R-squared=1, and RMSE=0.01166.

A plot of the descending curve of θ values versus the logarithms of the j values of the inflection points is

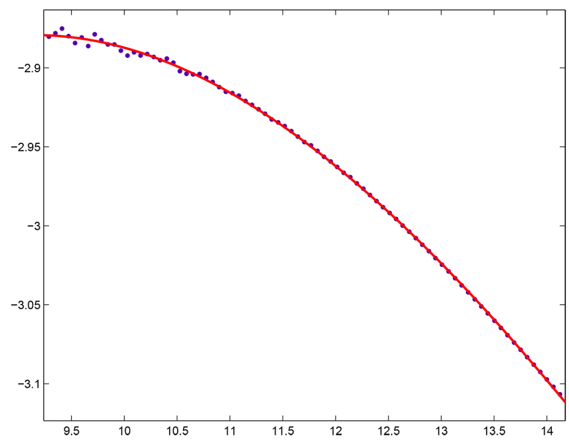

Figure 16

For a cubic least-squares fit of the curve, $p_1 = 0.0004646$ with a 95% confidence interval of (0.0003278, 0006014), $p_2 = -0.02432$ with a 95% confidence interval of $(-0.02913, -0.01952)$, $p_3 = 0.3286$ with a 95% confidence interval of $(0.2727, 0.3845)$, $p_4 = -4.205$ with a 95% confidence interval of $(-4.42, -3.99)$, SSE=0.0001377, R-squared=0.9996, and RMSE=0.001355.

A plot of the corresponding r values versus the logarithms of the j values of the inflection points is

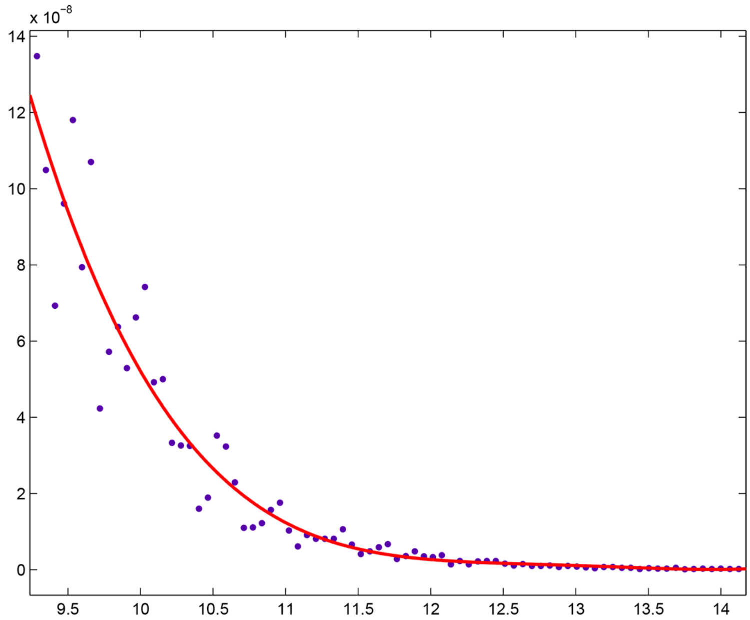

Figure 17

For a quartic least-squares fit of the curve, $p_1 = 6.001 \cdot 10^{-10}$ with a 95% confidence interval of $(-1.187 \cdot 10^{-10}, 1.319 \cdot 10^{-9})$, $p_2 = -3.127 \cdot 10^{-8}$ with a 95% confidence interval of $(-6.493 \cdot 10^{-8}, 2.397 \cdot 10^{-9})$, $p_3 = 6.106 \cdot 10^{-7}$ with a 95% confidence interval of $(2.261 \cdot 10^{-8}, 1.199 \cdot 10^{-6})$, $p_4 = -5.298 \cdot 10^{-6}$ with a 95% confidence interval of $(-9.836 \cdot 10^{-6}, -7.592 \cdot 10^{-7})$, $p_5 = 1.724 \cdot 10^{-5}$ with a 95% confidence interval of $(4.175 \cdot 10^{-6}, 3.03 \cdot 10^{-5})$, SSE=$5.696 \cdot 10^{-15}$, R-squared=0.9268, and RMSE=$8.773 \cdot 10^{-4}$.

The difference between the j value of an inflection point and that of the previous inflection point is 3. A plot of the logarithms of the current j values of inflection points is

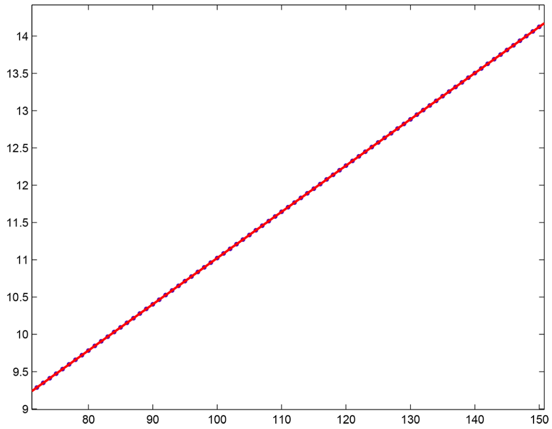

Figure 18

For a linear least-squares fit of the curve, $p_1 = 0.06201$ with a 95% confidence interval of (0.06201, 0.06201), $p_2 = 4.821$ with a 95% confidence interval of (4.821, 4.821), SSE=$4.419 \cdot 10^{-8}$, R-squared=1, and RMSE=$2.396 \cdot 10^{-5}$.

8. ODD j VALUES

A plot of the descending curve of θ values versus the logarithms of the j values of the inflection points is

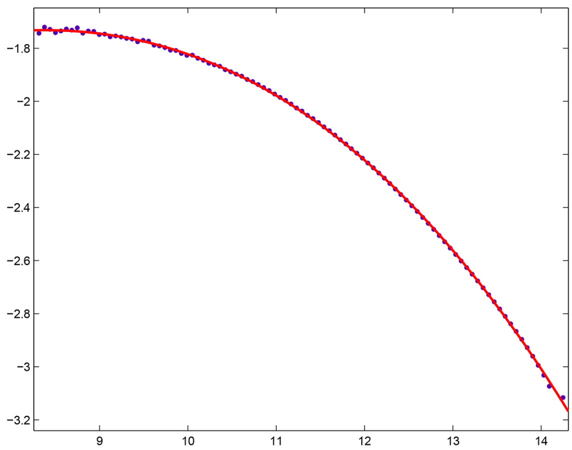

Figure 19

For a cubic least-squares fit of the curve, $p_1 = -0.001544$ with a 95% confidence interval of $(-0.001793, -0.001295)$, $p_2 = 0.007167$ with a 95% confidence interval of $(-0.001255, 0.01559)$, $p_3 = 0.2045$ with a 95% confidence interval of (0.111, 0.2983), $p_4 = -3.042$ with a 95% confidence interval of $(-3.385, -2.699)$, SSE=0.002115, R-squared=0.9999, and RMSE=0.004821.

A plot of the corresponding curve of r values versus the logarithms of the j values of the inflection points is

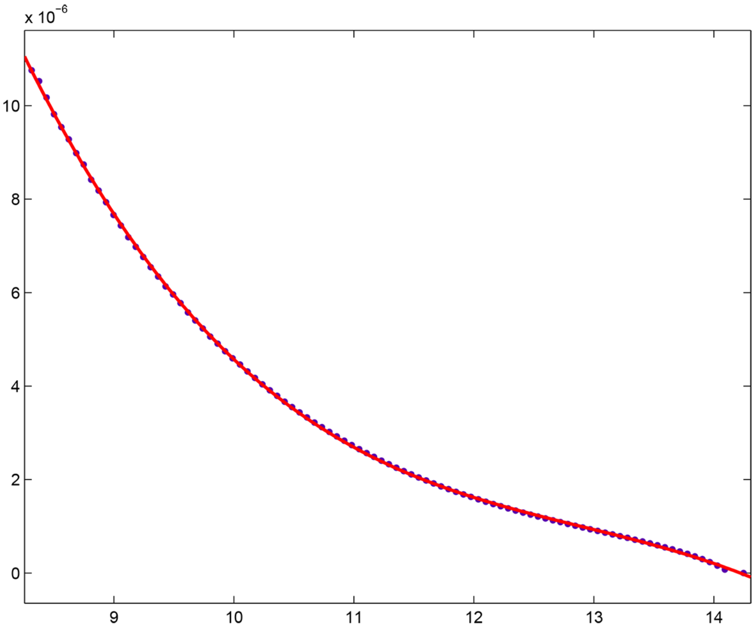

Figure 20

For a quartic least-squares fit of the curve, $p_1 = 6.001 \cdot 10^{-10}$ with a 95% confidence interval of $(-1.187 \cdot 10^{-10}, 1.319 \cdot 10^{-9})$, $p_2 = -3.127 \cdot 10^{-8}$ with a 95% confidence interval of $(-6.493 \cdot 10^{-8}, 2.397 \cdot 10^{-9})$, $p_3 = 6.106 \cdot 10^{-7}$ with a 95% confidence interval of $(2.261 \cdot 10^{-8}, 1.199 \cdot 10^{-6})$, $p_4 = -5.298 \cdot 10^{-6}$ with a 95% confidence interval of $(-9.836 \cdot 10^{-6}, -7.592 \cdot 10^{-7})$, $p_5 = 1.724 \cdot 10^{-5}$ with a 95% confidence interval of $(4.175 \cdot 10^{-6}, 3.03 \cdot 10^{-5})$, SSE=$5.696 \cdot 10^{-15}$, R-squared=0.9268, and RMSE=$8.773 \cdot 10^{-9}$.

In general, the difference between the j value of an inflection point and that of the previous inflection point is not three. A plot of the logarithms of the j values of the current inflection points is

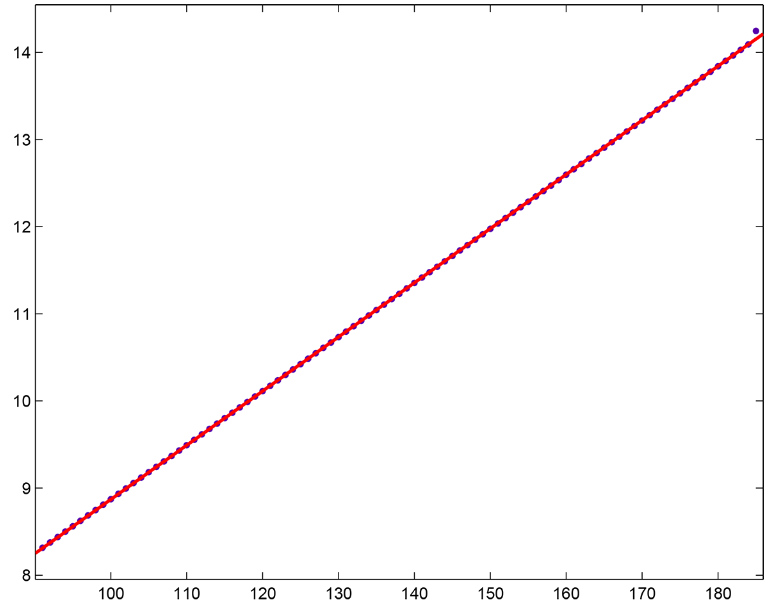

Figure 21

For a linear least-squares fit of the curve, $p_1 = 0.06217$ with a 95% confidence interval of (0.0621, 0.066224), $p_2 = 2.652$ with a 95% confidence interval of (2.642, 2.663), SSE=0.009282, R-squared=1, and RMSE=0.0099.

A plot of the logarithms of the j values of the previous inflection points is

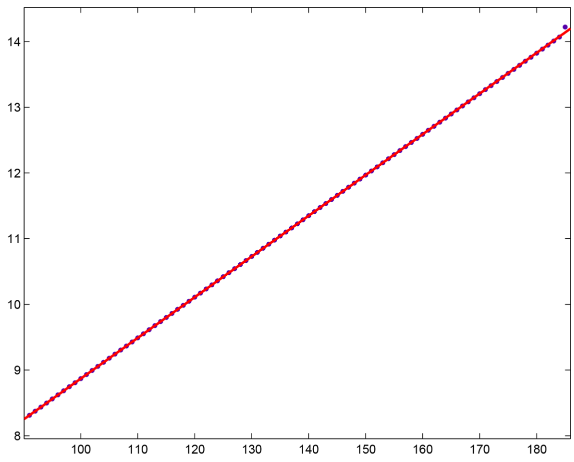

Figure 22

For a linear least-squares fit of the curve, $p_1 = 0.06198$ with a 95% confidence interval of (0.06191, 0.06205), $p_2 = 2.672$ with a 95% confidence interval of (2.663, 2.681), SSE=0.007311, R-squared=1. and RMSE=0.008866.

A plot of the θ values of the upper curve (almost linear) versus the logarithms of the j values of the inflection points is

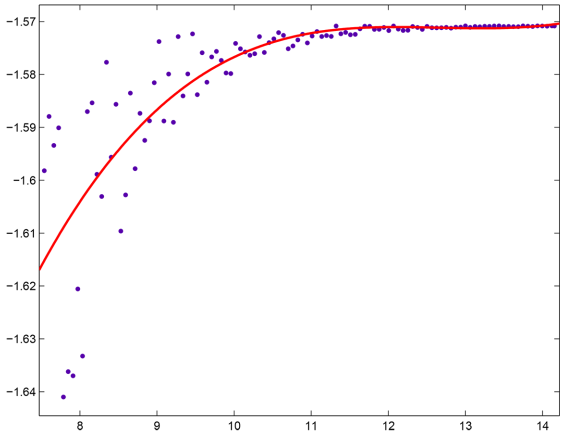

Figure 23

The values approach $-\pi/2$. For a cubic least-squares fit of the curve, $p_1 = 0.0003388$ with a 95% confidence interval of ($5.115 \cdot 10^{-5}$, 0.0006264), $p_2 = -0.01287$ with a 95% confidence interval of (-0.0226, , -0.003491), $p_3 = 0.1627$ with a 95% confidence interval of (0.06236, 0.2631), $p_4 = -2.256$ with a 95% confidence interval of (-2.607, -1.904), SSE=0.007638, R-squared=0.674, and RMSE=0.00857.
A plot of the corresponding r values of the curve versus the logarithms of the j values of the inflection point is

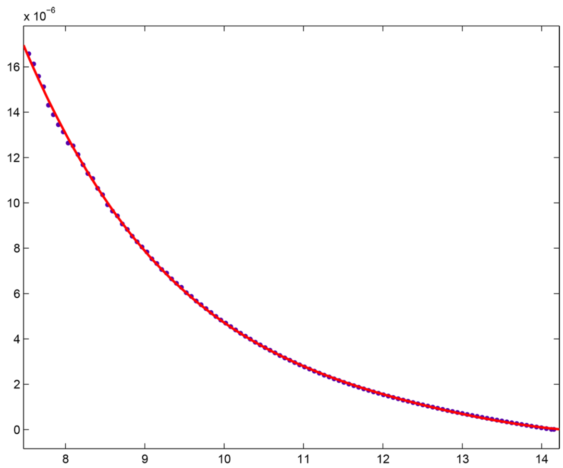

Figure 24

For a cubic least-squares fit of the curve, $p_1 = 9.699 \cdot 10^{-9}$ with a 95% confidence interval of $8.527 \cdot 10^{-9}$, $1.087 \cdot 10^{-8}$), $p_2 = -4.995 \cdot 10^{-7}$ with a 95% confidence interval of ($-5.505 \cdot 10^{-7}$, $-4.486 \cdot 10^{-7}$), $p_3 = 9.777 \cdot 10^{-6}$ with a 95% confidence interval of ($8.956 \cdot 10^{-6}$, $1.06 \cdot 10^{-5}$), $p_4 = -8.69 \cdot 10^{-5}$ with a 95% confidence interval of ($-9.27 \cdot 10^{-5}$, $-8.11 \cdot 10^{-5}$), SSE=$3.565 \cdot 10^{-13}$, R-squared=0.9998, and RMSE=$5.884 \cdot 10^{-8}$.
In general, the difference between the j value of an inflection point and that of the previous inflection point is not three. A plot of the logarithms of the j values of the current inflection points is

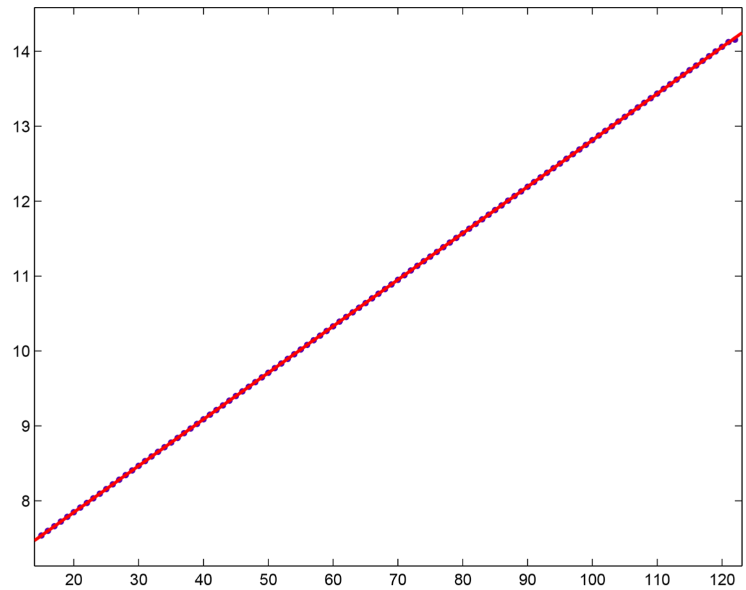

Figure 25

For a linear least-squares fit of the curve, $p_1 = 0.06208$ with a 95% confidence interval of (0.06206, 0.0621), $p_2 = 6.606$ with a 95% confidence interval of (6.604, 6.607), SSE=0.0008204, R-squared=1, and RMSE=0.002782.

A plot of the logarithms of the j values of the previous inflection points is

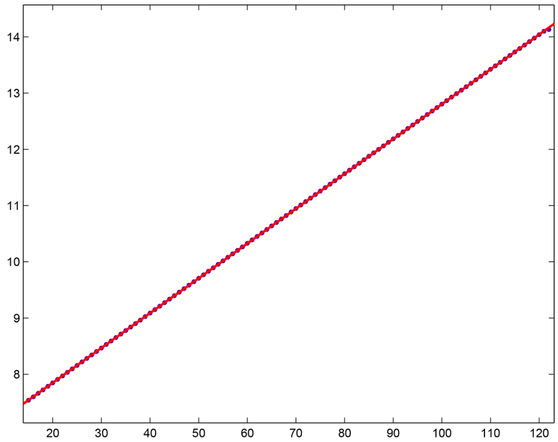

Figure 26

For a linear least-squares fit of the curve, $p_1 = 0.06192$ with a 95% confidence interval of (0.0619, 0.06194), $p_(2) = 6.61$ with a 95% confidence interval of (6.609, 6.612), SSE=0.001439, R-squared=1, and RMSE=0.003685.

9. A DIFFERENT ZETA FUNCTION ZERO

A plot of the θ values of the $C(n, a, b)$ function for j values of inflection points that are at least three greater than the previous j value, $n = 4000001$, and $z = (0.5001, 48.00515088116716)$ (a Riemann zeta function zero when $\Re(z) = 0.50$) versus the corresponding j values of inflection points is

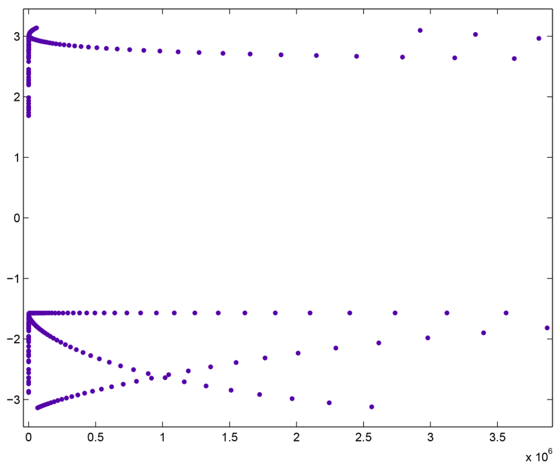

Figure 27

In this plot, the θ values of the top ascending curve approach π and then "wrap-around" to the bottom ascending curve. Similarly, the θ values of the bottom descending curve approach $-\pi$ and then "wrap-around" to the top descending curve (consisting of three points).

A plot of the r values of the bottom descending curve versus the logarithms of the j values (odd) of the inflection points is

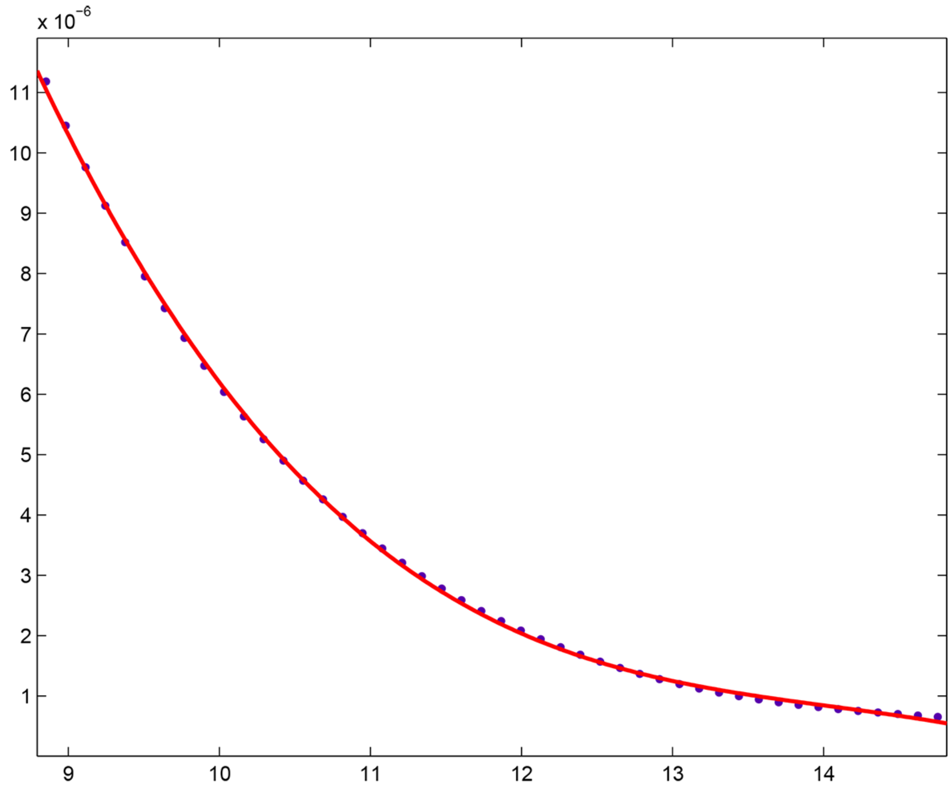

Figure 28

For a cubic least-squares fit of the curve, $p_1 = -6.031 \cdot 10^{-8}$ with a 95% confidence interval of $(-6.381 \cdot 10^{-8}$, $-5.681 \cdot 10^{-8})$, $p_2 = 2.543 \cdot 10^{-6}$ with a 95% confidence interval of $(2.419 \cdot 10^{-6}$, $2.667 \cdot 10^{-6})$, $p_3 = -3.607 \cdot 10^{-5}$ with a 95% confidence interval of $(-3.752 \cdot 10^{-5}$, $-3.462 \cdot 10^{-5})$, $p_4 = 0.0001792$ with a 95% confidence interval of (0.0001673, 0.0001785), SSE=$9.965 \cdot 10^{-14}$, R-squared=0.9998, and RMSE=$4.871 \cdot 10^{-8}$.

A plot of the corresponding θ values is

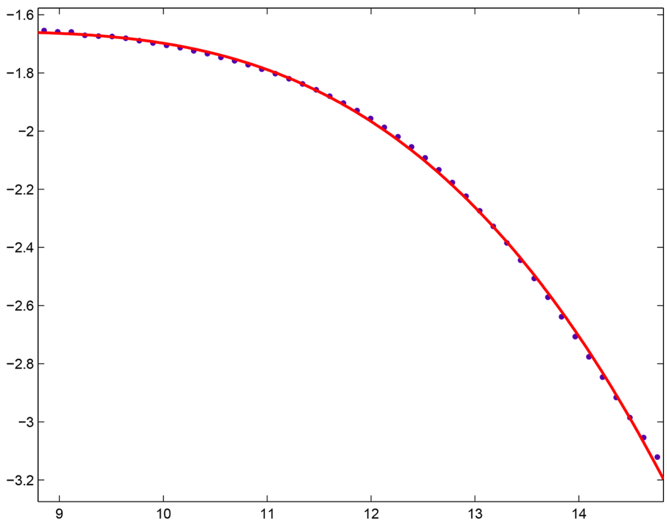

Figure 29

For a cubic least-squares fit of the curve, $p_1 = -0.005059$ with a 95% confidence interval of $(-0.005832, -.004278)$, $p_2 = 0.1234$ with a 95% confidence interval of (0.09599, 0.1508), $p_3 = -1.007$ with a 95% confidence interval of $(-1.327, -0.6871)$, $p_4 = 1.095$ with a 95% confidence interval of $(-0.1371$, 2.327), SSE=0.004854, R-squared=0.9995, and RMSE=0.01075.

A plot of the logarithms of the current j values of the inflection points is

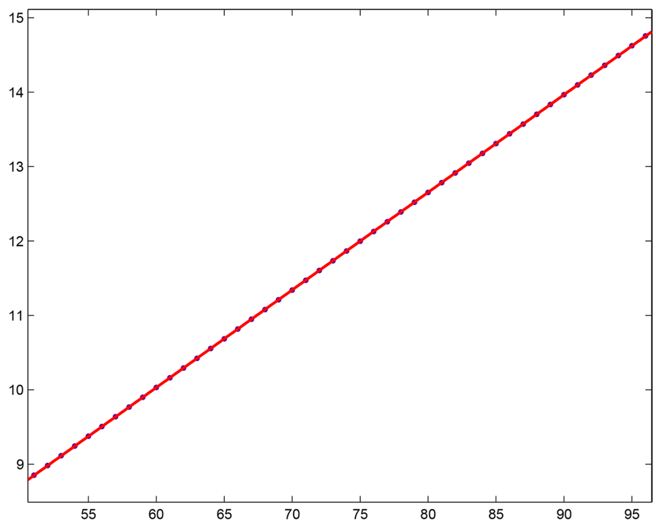

Figure 30

For a linear least-squares fit of the curve, $p_1 = 0.1311$ with a 95% confidence interval of (0.1311, 0.1312), $p_2 = 2.162$ with a 95% confidence interval of (2.159, 2.164), SSE=$9.762 \cdot 10^{-5}$, R-squared=1, and RMSE=0.00149. The difference between the j value of the current inflection point and that of the previous inflection point is not three.

A plot of the logarithms of the previous j values of the inflection points is

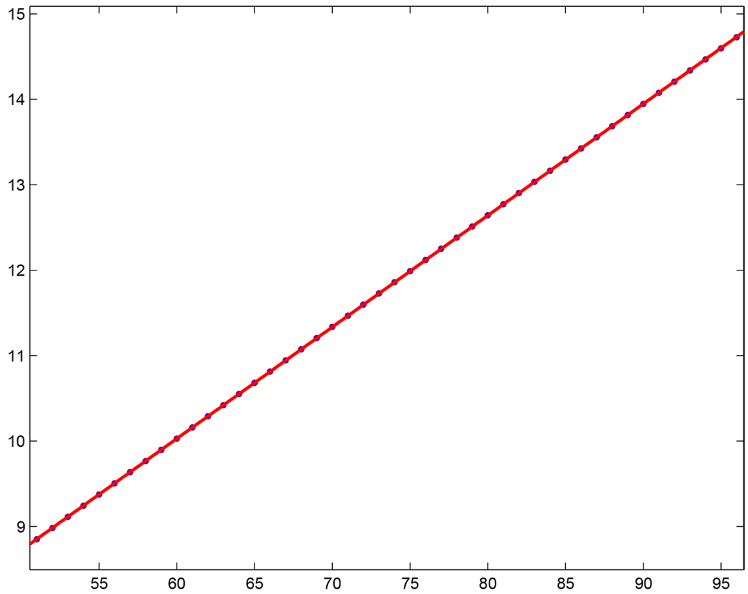

Figure 31

For a linear least-squares fit of the curve, $p_1 = 0.1306$ with a 95% confidence interval of (0.1306, 0.1307), $p_2 = 2.191$ with a 95% confidence interval of (2.189, 2.193), SSE=$8.96 \cdot 10^{-5}$, R-squared=1, and RMSE=0.001427.

10. CONCLUSION

For zeta function zeros, the difference in j values of successive inflection points eventually becomes 3 (excluding differences less than 3). This phenomenon only occurs for zeta function zeros (an empirical result).

The θ and r values of curves generated when the real part of s is not 0.5 and the j values are odd can be described using cubic or quartic curves. In this case, the difference between the j value of the current inflection point and that of the previous inflection point is not three.

11. METHODS

```
#include <math.h>
#include <stdio.h>
//
// compute C4(n,a,b), 09/04/2024 (dkc)
//
unsigned int max=2001;
double a=0.51;
//double b=14.13472514173470;
double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;
//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
//double b=52.97032147771446;
//double b=56.44624769706339;
//double b=59.34704400260235;
//double b=60.83177852460981;
//double b=65.11254404808160;
//double b=67.07981052949417;
//double b=69.54640171117399;
//double b=72.06715767448191;
//double b=75.70469069908393;
//double b=77.14484006887480;
//double b=79.33737502024937;
//double b=84.73549298051705;
//double b=87.42527461312523;
//double b=88.80911120763446;
//double b=92.49189927055849;
//double b=94.65134404051989;
//double b=95.87063422824531;
//double b=98.83119421819369;
//double b=101.31785100573138;
```

```
unsigned int xmin=0; // usually set to 0
unsigned int out=2; // usually 1, 2 for inflection points
unsigned int out3p=1; // usually 0, 1 for differences in j values >=2
unsigned int polar=1; // set to use polar coordinates
void main() {
unsigned int x,oldx;
double sumr,sumi,R,I,temp1,oldsumr,oldsumi,temp,tempa,tempb,y,e,f,g;
FILE *Outfp;
Outfp = fopen("c2nab2f.dat","w");
y=1.0-a;
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(b*log(2)));
f=temp1*(sin(b*log(2)));
e=1.0-e;
f=-f;
y=-a;
if (y>=0.0)
    temp1=pow((double)max,y);
else {
    temp1=pow((double)max,-y);
    temp1=1.0/temp1;
    }
y=2.0*temp1;
oldx=0;
sumr=0.0;
sumi=0.0;
oldsumi=0.0;
oldsumr=0.0;
for (x=1; x<=(max-1); x++) {
    temp1=pow((double)x,a);
    R=temp1*cos(b*log((double)x));
    I=temp1*sin(b*log((double)x));
    temp1=R*R+I*I;
```

```
if (x!=(x/2)*2) {
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    }
else {
    sumr=sumr-R/temp1;
    sumi=sumi+I/temp1;
    }
temp=cos(b*log((double)max/(double)x));
tempa=sumr*temp;
tempb=sumi*temp;
tempa=tempa*y;
tempb=tempb*y;
g=tempa*e-tempb*f;
tempb=tempa*f+tempb*e;
tempa=g;
if (x>xmin) {
    if (out==1)
        fprintf(Outfp,” %.10llf, %.10llf \n”,tempa,tempb);
    if ((out==2)&&((oldsumr¿0.0)&&(tempa¡0.0))) {
        if (out3p==0)
            fprintf(Outfp,” %d %.10llf %.10llf \n”,x,tempa,tempb);
        if ((x-oldx)>2) {
            printf(” %d %d %.10llf %.10llf \n”,x,oldx,oldsumr,tempa);
            if (out3p!=0) {
                if (polar==0)
                    fprintf(Outfp,” %d %d \n”,x,oldx);
                else
                    fprintf(Outfp,” %d %d %.10llf %.10llf \n”,x,oldx,
                        sqrt(tempa*tempa+tempb*tempb),atan2(tempb,tempa));
                }
            }
        oldx=x;
        }
    oldsumr=tempa;
    oldsumi=tempb;
    }
}
```

```
fclose(Outfp);
return;
}
```