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Fermat'z Fuzzy Topological Structures over Bck-Algebra's (Bck-Ideal's) And Their Applications in Decision Making –A New Approach

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Abstract

In this paper, we study the concept of Fermat's fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of Fermat's fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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1. Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (See [22]) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0; 1]. As a generalization of fuzzy set, Atanassov [1] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all fields (See [4, 5, 12, 18] for applications in algebraic structures). In 2013, Yager [19, 20, 21] introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [2]), UP-algebras (See [15]) and topological spaces (See [14]). Senapati et al. [16] introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [17]). Ibrahim et al. [9] introduced (3, 2)-fuzzy sets and applied it to topological spaces. In this paper, we study the concept of Fermat's fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of (m, n)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

2. Preliminaries and Various Basic Concept of BCC-algebras (BCK-algebras)

In this section, we first review some definitions and properties which will be used in the sequel.

A non-empty set G with a constant 0 and binary operation $*$ is called a

BCC-algebra if it satisfies the following conditions:

a) $\left(\left((x * y) * (z * y)\right) * (x * y) = 0\right)$

b) $x * x = 0$

c) $0 * x = 0$

d) $x * 0 = 0$

e) $x * y = 0\,, y * x = 0 \;\Rightarrow x = y$

for all $x, y, z \in G$. In BCC-algebra, the following equality holds $(x * y) * x = 0$.

Obviously, any BCK-algebra is BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebra. We note that a BCC-algebra is BCK-algebra if and if only, it satisfies the equality $(x * y) * z = (x * z) * y$.

A non-empty subset 'S' of a BCK-algebra 'G' is called a sub algebra of G if it is closed under the BCC-operation. Such algebra contains the constant 0 and it is clearly a BCC-algebra, but some sub algebras may be also BCK-algebras. Moreover, there exit BCC-algebras which all sub algebras are BCK-algebras.

A mapping $\varphi: G_1 \to G_2$ of BCC-algebras is called a homomorphism if

$\varphi(x * y) = \varphi(x) * \varphi(y)$ holds, for all $x, y \in G_1$.

For a non-empty given set G, let I be the closed unit interval [0, 1]. Then, an Fermat's fuzzy set is an object of the form $A = \{\langle x, {\delta_A}^n(x), {\lambda_A}^n(x)\rangle / \; x \in G\}$, when the mappings

${\delta_A}^n: G \to I$ and ${\lambda_A}^n: G \to I$ denote the degree of membership (namely, $\delta_A(x)$) and the degree of non-membership (namely, $\lambda_A(x)$) of each element $x \in G$ to the object 'A' respectively satisfying $0 \le {\delta_A}^n(x) + {\lambda_A}^n(x) \le 1$ for all $x \in G$.

The complement of the (m, n)-fuzzy set is $A^C = \{\langle x, {\lambda_A}^n(x), {\delta_A}^n(x)\rangle / \; x \in G\}$. Obviously, every fuzzy A on a non-empty G is an Fermat's fuzzy set of the form

$A = \{\langle x, \delta_A{}^n(x), 1 - \lambda_A{}^n(x)\rangle / x \in G\}$. For the sake of simplicity, we just write $A = \langle \delta_A{}^n, \lambda_A{}^n \rangle$ instead of $A = \{\langle x, \delta_A(x), \lambda_A(x)\rangle / x \in G\}$.

The Fermat's fuzzy sets $0\sim$ and $1\sim$ in G are defined by

$0\sim = \{\langle x, 0, 1\rangle : x \in G\}$ and $1\sim = \{\langle x, 1, 0\rangle : x \in G\}$, respectively.

If φ is a mapping which maps a set G_1 into another set G_2, then the following statement hold:

(a) If $B = \{\langle y, \delta_B{}^n(y), \lambda_B{}^n(y)\rangle / y \in G_2\}$ is an Fermat's fuzzy set in G_2, then the pre image of B under φ, denoted by $\varphi^{-1}(B)$, is still an (m, n)-fuzzy set in G_1, we now write

$\varphi^{-1}(B) = \{\langle x,\ \varphi^{-1}(\delta_B)(x),\ \varphi^{-1}(\lambda_B)(x)\rangle / x \in G_1\}$.

(b) If $A = \{\langle x, \delta_A{}^n(x), \lambda_A{}^n(x)\rangle / x \in G_1\}$ in an Fermat's fuzzy set in G_1, then the image of A under φ, denoted by φ(A), is also an (m, n)-fuzzy set in G_2, which is defined by

$\varphi(A) = \left\{\langle y, \varphi_{\sup}(\delta_A)(y), \varphi_{\inf}(\lambda_A)(y)\ \rangle : y \in G_2\right\}$, where

$$\varphi_{\sup}(\delta_A)(y) = \begin{cases} \sup\limits_{x \in \varphi^{-1}(y)} \delta_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

$$\varphi_{\inf}(\lambda_A)(y) = \begin{cases} \inf\limits_{x \in \varphi^{-1}(y)} \lambda_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

for each $y \in G_2$.

Proposition-2.1: Let A, $A_i (i \in I)$be (m, n)-fuzzy set in G_1 and B an Fermat's fuzzy set in G_2.

If $\varphi: G_1 \to G_2$ is a function, then the following properties hold for the function φ:

(a) If φ is surjective, then $\varphi\left(\varphi^{-1}(B)\right) = B$.

(b) $\varphi^{-1}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \varphi^{-1}(A_i)$.

(c) $\varphi^{-1}(1\sim) = 1\sim$.

(d) $\varphi^{-1}(0\sim) = 0\sim$.

(e) $\varphi(1\sim) = 1\sim$, if φ is surjective

(f) $\varphi(0\sim) = 0\sim$.

Definition-2.2: An Fermat's fuzzy topology on a non-empty set G is a family τ of Fermat's fuzzy sets in G which satisfies the following conditions:

(i) $0\sim, 1\sim \in \tau$.

(ii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2$.

(iii) If $G_j \in \tau$ for all $j \in J$, then $\bigcup_{i\in I} G_i \in \tau$.

The pair (G, τ) is called an Fermat's fuzzy topological space and any Fermat's fuzzy set in τ is called an Fermat's fuzzy open sets in G. The topology τ on a Fermat's fuzzy topological space is said to be an indiscrete Fermat's fuzzy topology if it's only element are $0\sim$ and $1\sim$. On the other hand, Fermat's fuzzy topology τ on a space G is said to be discrete Fermat's fuzzy topology if the topology Fermat's fuzzy topology τ contains all Fermat's fuzzy subsets of G.

If A is an Fermat's fuzzy set in an Fermat's fuzzy topological space (G, τ), then the induced Fermat's fuzzy topological space on A is the family of Fermat's fuzzy sets in A which are the intersection with A of Fermat's fuzzy sets in G. The induced Fermat's fuzzy topology is denoted by τ_A, and the pair (A, τ_A) is called an fuzzy subspace of (G, τ).

Let (G_1, τ_1) and (G_2, τ_2) be two Fermat's fuzzy topological spaces and

$\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be Fermat's fuzzy continuous function if and only if the pre image of each Fermat's fuzzy set in τ_2 is an Fermat's fuzzy set in τ_1. Let

(G_1, τ_1) and (G_2, τ_2) be two Fermat's fuzzy topological spaces and $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be (m, n)-fuzzy open if and only if the image of each Fermat's fuzzy set in τ_1 is an Fermat's fuzzy set in τ_2.

3. (m. n)-fuzzy topological sub algebras

Definition-3.1: An (m, n)-fuzzy set $A = \langle {\delta_A}^m, {\lambda_A}^n \rangle$ in G is called (m, n)- fuzzy sub algebra of G if it satisfies the following conditions;

(m, n)FS1 : ${\delta_A}^m(x * y) \geq \min\{{\delta_A}^m(x), {\delta_A}^m(y)\}$

(m, n)FS2 : ${\lambda_A}^n(x * y) \leq \max\{{\lambda_A}^n(x), {\lambda_A}^n(y)\}$, for all $x, y \in G$.

Example-3.2: Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table.

$+$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let $A = \langle \delta_A{}^m, \lambda_A{}^n \rangle$ be an (m, n)-fuzzy set in G defined by $\delta_A{}^m(4) = 0.07$, $\delta_A{}^m(x) = 0.6$,

$\lambda_A{}^n(x) = 0.5$ and $\lambda_A{}^n(4) = 0.06$ for all $x \neq d$. Then A is (m, n)- fuzzy sub algebra of G.

Definition-3.3: Let τ_1 and τ_2 be an (m, n)-fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is called an (m, n)-fuzzy continuous function which maps (G_1, τ_1) and (G_2, τ_2) if φ satisfies the following conditions:

(i) For every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

(ii) For every (m, n)- fuzzy sub algebra A (of G_2) in τ_2, $\varphi^{-1}(A)$ is (m, n)- fuzzy sub algebra (of G_1) in τ_1.

Proposition-3.4: If in τ_1 is an (m, n)- fuzzy topology on a BCC-algebra G_1 and τ_2 is an

(m, n)- fuzzy topology on a BCC-algebra G_2, then every function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is

a (m, n)-fuzzy continuous function.

Proof: Since τ_2 is an indiscrete (m, n)- fuzzy topology, $\tau_2 = (0\sim, 1\sim)$..

Let $\varphi: G_1 \rightarrow G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is an (m, n)- fuzzy topology on a BCC-algebra G_2.

We now show that φ is (m, n)- fuzzy continuous function. We only need to prove that for every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

For this purpose, we let $0\sim \in \tau_2$. Then for any $x \in G_1$, we have

$\varphi^{-1}(0\sim)(x) = 0\sim(\varphi(x)) = 0 = 0\sim(x)$. This show that $\left(\varphi^{-1}(0\sim)\right) = 0\sim \in \tau_1$.

On the other hand, if $1\sim \in \tau_2$ and $x \in \mathrm{G}_1$, then

$\varphi^{-1}(1\sim)(x) = 1\sim(\varphi(x)) = 1 = 1\sim(x)$. Thus $\left(\varphi^{-1}(1\sim)\right) = 1\sim \in \tau_1$.

This show that φ is indeed an (m, n)- fuzzy continuous function of G_1 to G_2.

Theorem-3.5: Let τ_1 and $\tau_2$be any two discrete (m, n)- fuzzy topologies defined on the BCC-algebras G_1 and G_2 respectively. Then every homomorphism $\varphi: (\mathrm{G}_1, \tau_1) \to (\mathrm{G}_2, \tau_2)$ is an (m, n)- fuzzy continuous function.

Proof: Since τ_1 and τ_2 are discrete (m, n)- fuzzy topologies on the BCC-algebras G_1 and G_2 respectively, we have $\varphi^{-1}(A) \in \tau_1$ for every $A \in \tau_2$.

We note that φ is not the usual inverse homomorphism from G_2 to G_1.

Let $A = \langle \delta_A{}^m, \lambda_A{}^n \rangle$ be an (m, n)-fuzzy sub algebra (of G_2) in τ_2. Then for $x, y \in \mathrm{G}_1$, we have,$\left(\varphi^{-1}(\delta_A{}^m)\right)(x * y) = \delta_A{}^m\left(\varphi(x * y)\right)$

$$= \delta_A{}^m\left(\varphi(x) * \varphi(y)\right)$$

$$\geq \min\left\{\delta_A{}^m\left(\varphi(x)\right), \delta_A{}^m\left(\varphi(y)\right)\right\}$$

$$= \min\left\{\left(\varphi^{-1}(\delta_A{}^m)\right)(x), \left(\varphi^{-1}(\delta_A{}^m)\right)(y)\right\} \text{ and}$$

$\left(\varphi^{-1}(\lambda_A{}^n)\right)(x * y) = \lambda_A{}^n\left(\varphi(x * y)\right)$

$$= \lambda_A{}^n\left(\varphi(x) * \varphi(y)\right)$$

$$\leq \max\left\{\lambda_A{}^n\left(\varphi(x)\right), \lambda_A{}^n\left(\varphi(y)\right)\right\}$$

$$= \max\left\{\left(\varphi^{-1}(\lambda_A{}^n)\right)(x), \left(\varphi^{-1}(\lambda_A{}^n)\right)(y)\right\}$$

Hence $\varphi^{-1}(A)$ is an (m, n)-fuzzy sub algebra (of G_1) in τ_1 and consequently, φ is an (m, n)- fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2).

Definition-3.6: Let (G_1, τ_1) and (G_2, τ_2) be (m, n)- fuzzy topology sub algebras. A function

$\varphi: (\mathrm{G}_1, \tau_1) \to (\mathrm{G}_2, \tau_2)$ is said to be an (m, n)-fuzzy homomorphism if it satisfies the following conditions:

- φ is an injective and surjective function.
- φ is fuzzy continues function which maps G_1 to G_2.
- φ^{-1} is fuzzy continues function which maps G_2 to G_1.

Definition-3.7: Let τ be an (m, n)- fuzzy topology of BCC-algebra G. An (m, n)- fuzzy topology (G, τ) is an (m, n)- fuzzy Hausdorff space if and only if for any discrete (m, n)- fuzzy point $x_1, x_2 \in G$, there exits (m, n)- fuzzy topology $F_1 = \langle \delta^m_{F_1}, \lambda^n_{F_1} \rangle$ and $F_2 = \langle \delta^m_{F_2}, \lambda^n_{F_2} \rangle$ such that $\delta^m_{F_1}(x_1) = 1$, $\lambda^n_{F_1}(x_1) = 0$, $\delta^m_{F_2}(x_2) = 1$, $\lambda^n_{F_2}(x_2) = 0$ and $F_1 \cap F_2 = 0\sim$.

Theorem-3.8: Let τ_1 and τ_2 be (m, n)- fuzzy topologies on the BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be an (m, n)- fuzzy homomorphism. Then G_1 is

an (m, n)- fuzzy Hausdorff space if and only if G_2 is an (m, n)- fuzzy Hausdorff space.

Proof: Suppose that G_1 is a (m, n)- fuzzy Hausdorff space.

Let x_1, x_2 be the (m, n)- fuzzy point in τ_2 with x ≠ y where x, y ∈ G_1.

Then $\varphi^{-1}(x) \neq \varphi^{-1}(y)$ because φ is injective function.

For z ∈ G_1, $\left(\varphi^{-1}(x_1)\right)(z) = x_1\left(\varphi(z)\right)$

$$= \begin{cases} s \in [0,1], & \text{if } \varphi(z) = x \\ 0, & \text{if } \varphi(z) \neq x \end{cases} = \begin{cases} s \in [0,1], & \text{if } z = \varphi^{-1}(x) \\ 0, & \text{if } z \neq \varphi^{-1}(x) \end{cases}$$

$$= \left(\varphi^{-1}(x)\right)_1(z).$$

That is, $\left(\varphi^{-1}(x_1)\right)(z) = \left(\varphi^{-1}(x)\right)_1(z)$ for all z ∈ G. Hence $\varphi^{-1}(x_1)=\left(\varphi^{-1}(x)\right)_1$.

Similarly we can also prove that $\varphi^{-1}(x_2)=\left(\varphi^{-1}(x)\right)_2$. Now by the definition of an

(m, n)- fuzzy Hausdorff space, there exist (m, n)- fuzzy order F_1 and F_2 of $\varphi^{-1}(x_1)$ and $\varphi^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0\sim$. Since φ is an (m, n)- fuzzy continuous map from G_2 to G_1, there exist (m, n)- fuzzy orders $\varphi(F_1)$ and $\varphi(F_2)$ of x_1 and x_2 respectively such that $\varphi(F_1) \cap \varphi(F_2) = \varphi(F_1 \cap F_2) = \varphi(0\sim) = 0\sim$. This implies that G_2 is a (m, n)- fuzzy Hausdorff space.

Conversely, if (G_2, τ_2) is a (m, n)- fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both φ and φ^{-1} are (m, n)- fuzzy continuous functions, we can easily prove that (G_1, τ_1) is an (m, n)- fuzzy Hausdorff space. Hence the proof.

Definition-3.9: Let τ be an (m, n)- fuzzy topology on a BCC-algebra G. Then (G, τ) is called

an (m, n)- fuzzy C_5-disconnected space if there exists an (m, n)- fuzzy open and closed set F

such that $F \neq 0\sim$ and $F \neq 1\sim$.

Theorem-3.10: Let τ_1 and τ_2 be the (m, n)- fuzzy topology sub algebras G_1 and G_2 respectively and let $\varphi: G_1 \to G_2$ be an (m, n)- fuzzy continuous surjective function. If (G_1, τ_1) is an (m, n)- fuzzy C_5-connected space then (G_2, τ_2) is also an (m, n)- fuzzy C_5-connected space.

Proof: Assume that (G_2, τ_2) is a (m, n)- fuzzy C_5-disconnected. Then there exist an (m, n)- fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Since φ is an (m, n)- fuzzy continuous function $\varphi^{-1}(F)$ is both (m, n)- fuzzy open and (m, n)- fuzzy closed set. In this case $\varphi^{-1}(F) \neq 0\sim$ or $\varphi^{-1}(F) \neq 1\sim$.

Since, $F = \varphi(\varphi^{-1}(F)) = \varphi(0\sim) = 0\sim$ and $F = \varphi(\varphi^{-1}(F)) = \varphi(1\sim) = 1\sim$.

We see that these results contradict to our assumption.

Hence the space (G_2, τ_2) must be (m, n)- fuzzy C_5-connected space.

Definition-3.11: Let τ be an (m, n)- fuzzy topology on a BCC-algebra G. An (m, n)- fuzzy topology (G, τ) is called an (m, n)- fuzzy disconnected space if there exist (m, n)- fuzzy open sets $A \neq 0\sim$ and $B \neq 0\sim$ such that $A \cup B = 0\sim$. Naturally, we call the set (G, τ) an (m, n)- fuzzy connected if (G, τ) is not (m, n)- fuzzy disconnected.

Theorem-3.12: Let τ_1 and τ_2 be (m, n)- fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an (m, n)- fuzzy continuous and surjective function. If G_1 is an (m, n)- fuzzy connected space, then so is G_2.

Proof: Suppose that G_2 is an (m, n)- fuzzy disconnected, then there exists (m, n)- fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ in G_2 such that $C \cup D = 1\sim$ and $C \cap D = 0\sim$.

Since φ is (m, n)- fuzzy continuous function, $A = \varphi^{-1}(C)$ and $B = \varphi^{-1}(D)$ are (m, n)- fuzzy open sets in G_1.

Clearly, $C \neq 0\sim$ implies that $A = \varphi^{-1}(C) \neq 0\sim$, and $D \neq 0\sim$ implies that $B = \varphi^{-1}(D) \neq 0\sim$.

Now $C \cup D = 1\sim$.

$\Rightarrow \varphi^{-1}(C \cup D) = \varphi^{-1}(1\sim)$.

$\Rightarrow \varphi^{-1}(C) \cup \varphi^{-1}(D) = 1\sim$ implies $A \cup B = 1\sim$ and

$C \cap D = 0\sim \Rightarrow \varphi^{-1}(C \cap D) = \varphi^{-1}(0\sim)$

$\Rightarrow \varphi^{-1}(C) \cap \varphi^{-1}(D) = 0\sim$ implies $A \cap B = 0\sim$.

This clearly contradicts our hypothesis.

Hence G_2 is an (m, n)- fuzzy connected space.

Definition-3.13: An (m, n)- fuzzy topology space (G, τ) is said to be an (m, n)-fuzzy strongly connected, if there exists no non-zero (m, n)-fuzzy closed sets A and B in G such that $\delta_A^m + \delta_B^m \leq 1$ and $\lambda_A^n + \lambda_B^n \geq 1$.

The following fact follows immediately from the definition.

Propositon-3.14: G is (m, n)-fuzzy strongly connected if and only if there exist an (m, n)-fuzzy open sets A and B in G such that $A \neq 1\sim \neq B$ and $\delta_A^m + \delta_B^m \geq 1$, $\lambda_A^n + \lambda_B^n \leq 1$.

We now formulate the following theorem.

Theorem-3.15: Let τ_1 and τ_2 be (m, n)- fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an (m, n)- fuzzy continuous and surjective mapping. If G_1 is an (m, n)- fuzzy strongly connected, then so is G_2.

Proof: Suppose that G_2 is not an (m, n)- fuzzy strongly connected. Then there exists (m, n)- fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ so that $\delta_C^m + \delta_D^m \leq 1$ and $\lambda_C^n + \lambda_D^n \geq 1$. Since φ is

an (m, n)- fuzzy continuous function, $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are (m, n)-fuzzy closed sets in G_1. Now we can deduce the following equalities;

$$\delta^m_{\varphi^{-1}(C)} + \delta^m_{\varphi^{-1}(D)} = \varphi^{-1}(\delta_C^m) + \varphi^{-1}(\delta_D^m)$$

$$= \delta_C^m \text{ o } \varphi + \delta_D^m \text{ o } \varphi \leq 1 \text{ (Since } \delta_C^m + \delta_D^m \leq 1),$$

$$\lambda^n_{\varphi^{-1}(C)} + \lambda^n_{\varphi^{-1}(D)} = \varphi^{-1}(\lambda_C^n) + \varphi^{-1}(\lambda_D^n)$$

$$= \lambda_C^n \text{ o } \varphi + \lambda_D^n \text{ o } \varphi \geq 1 \text{ (Since } \lambda_C^n + \lambda_D^n \geq 1).$$

$\varphi^{-1}(C) \neq 0\sim$ and $\varphi^{-1}(D) \neq 0\sim$. This contradicts our hypothesis. Hence G_2 is an (m, n)- fuzzy strongly connected space.

Definition-3.16: Let τ be an (m, n)- fuzzy topology on a BCC-algebra G and A be an (m, n)- fuzzy BCC-algebra with (m, n)- fuzzy topology τ_A. Then A is called an (m, n)-fuzzy topological BCC-sub algebra if the self-mapping γ_a: $(A, \tau_A) \to (A, \tau_A)$ defined by

$\gamma_a(x) = x * a$ for all $a \in G$, is a Relatively (m, n)- fuzzy continuous function.

Theorem-3.17: Let $\varphi: G_1 \to G_2$ be a homomorphism of BCC-algebras and τ and τ^* be (m, n)- fuzzy topologies on G_1 and G_2 respectively such that $\tau = \varphi^{-1}(\tau^*)$. If B is an (m, n)- fuzzy topological BCC-sub algebra in G_2, then $\varphi^{-1}(B)$ is an (m, n)- fuzzy topological BCC-sub algebra in G_1.

Theorem-3.18: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively (m, n)- fuzzy topologies on the spaces G_1 and G_2 such that $\tau = \varphi^{-1}(\tau^*)$. If A is an (m, n)- fuzzy topological BCC-sub algebra in G_1, then $\varphi^{-1}(A)$ is an (m, n)- fuzzy topological BCC-sub algebra in G_2.

4. (m, n)-Fuzzy Topological BCC-ideals

Definition-4.1: An (m, n)- fuzzy set $A = \{\langle \delta_A, \lambda_A \rangle\}$ in a BCK-algebra G is called an (m, n)- fuzzy BCK-ideal of G if the following conditions are satisfied;

(i) $\delta_A^m(0) \geq \delta_A^m(x)$ and $\lambda_A^n(0) \leq \lambda_A^n(x)$,

(ii) $\delta_A^m(x) \geq \min\{\delta_A^m(x * y), \delta_A^m(y)\}$

(iii) $\lambda_A^n(x) \leq \max\{\lambda_A^n(x * y), \lambda_A^n(y)\}$ for all $x, y \in G$.

Definition-4.2: An (m, n)- fuzzy set $A = \langle \delta_A, \lambda_A \rangle$ in G is called an (m, n)- fuzzy BCC-ideal of G if it satisfies the following conditions;

(m, n) F_1: $\delta_A^m(0) \geq \delta_A^m(x)$ and $\lambda_A^n(0) \leq \lambda_A^n(x)$

(m, n) F_2: $\delta_A^m(x * z) \geq \min\{\delta_A^m((x * y) * z), \delta_A^m(y)\}$

(m, n) F_3: $\lambda_A^n(x * z) \leq \max\{\lambda_A^n((x * y) * z), \lambda_A^n(y)\}$ for all $x, y, z \in G$.

Putting $z = 0$ in (m, n) F_2 and (m, n) F_3, then we can easily see that an (m, n)- fuzzy BCC-ideal is an (m, n)- fuzzy BCK-ideal. However, the converse does not hold.

Example-4.3: Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table;

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A, \lambda_A \rangle$ be an (m, n)- fuzzy set in G defined by $\delta_A^m(5) = 0.02$, $\delta_A^m(x) = 0.4$, $\lambda_A^n(5) = 0.2$ and $\lambda_A^n(x) = 0.04$ for all $x \neq 5$, then A is an (m, n)- fuzzy BCC-ideal of a BCC-algebra G.

Theorem-4.4: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (m, n)- fuzzy BCC-ideal of G_2. Then $\varphi^{-1}(B)$ is an (m, n)- fuzzy BCC-ideal of G_1.

Proof: It can be easily seen that

$\delta^m_{\varphi^{-1}(B)}(0) \geq \delta^m_{\varphi^{-1}(B)}(x)$ and $\lambda^n_{\varphi^{-1}(B)}(0) \leq \lambda^n_{\varphi^{-1}(B)}(x)$, for all $x \in G_1$.

For any $x, y, z \in G_1$, we can deduce the following

$$\delta^m_{\varphi^{-1}(B)}(x * z) = \delta^m_B(\varphi(x * z))$$

$$\geq \min\left\{\delta^m_B\left(\varphi\big((x * y) * z\big)\right), \delta^m_B(\varphi(y))\right\}$$

$$= \min\left\{\delta^m_B\left(\overset{5}{\big(\varphi(x) * \varphi(y)\big) * \varphi(z)}\right), \delta^m_B(\varphi(y))\right\}$$

$$= \min\{\delta^m_{\varphi^{-1}(B)}\big((x * y) * z\big), \delta^m_{\varphi^{-1}(B)}(y)\}.$$

Also

$$\lambda^n_{\varphi^{-1}(B)}(x * z) = \lambda^n_B(\varphi(x * z))$$

$$\leq \max\left\{\lambda^n_B\left(\varphi\big((x * y) * z\big)\right), \lambda^n_B(\varphi(y))\right\}$$

$$= \max\left\{\lambda^n_B\left(\begin{pmatrix}\varphi(x)5\\ * \varphi(y)\end{pmatrix} * \varphi(z)\right), \lambda^n_B(\varphi(y))\right\}$$

$$= \max\{\lambda^n_{\varphi^{-1}(B)}\big((x * y) * z\big), \lambda^n_{\varphi^{-1}(B)}(y)\}$$

Hence $\varphi^{-1}(B)$ is an (m, n)- fuzzy BCC-ideal of G_1.

Corollarly-4.5: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (m, n)- fuzzy BCK-ideal of G_2. Then $\varphi^{-1}(B)$ is an (m, n)- fuzzy BCK-ideal of G_1.

Since an (m, n)- fuzzy BCC-ideal / BCK-ideal is an (m, n)- fuzzy sub algebra, as a consequence of the above results and theorem-3.17, we obtain the following corollary.

Corollarly-4.6: Let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be a homomorphism of the BCC-algebras. Let τ_1 and τ_2 be the (m, n)-fuzzy topologies on G_1 and G_2 respectively such that $\tau_2 = \varphi^{-1}(\tau_1)$. If B is (m, n)-fuzzy topological BCC-ideal / BCK-ideal of G_2 with the membership function δ^m_B, then $\varphi^{-1}(B)$ is a (m, n)-fuzzy topological BCC-ideal / BCK-ideal of G_1 with the membership function $\delta^{m}_{\varphi^{-1}(B)}$.

Theorem-4.7: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is an (m, n)-fuzzy BCC-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still an (m, n)-fuzzy BCC-ideal of G_2.

Proof: Let A be an (m, n)-fuzzy topological BCC-ideal of G_1. Then, it is trivial that

$\delta^{m}_{\varphi(A)}(0) \geq \delta^{m}_{\varphi(A)}(x)$ and $\lambda^{n}_{\varphi(A)}(0) \leq \lambda^{n}_{\varphi(A)}(x)$, for all $x \in G_2$.

Take $x, y, z \in G_2$, and let $x_0 \in \varphi^{-1}(x)$, $y_0 \in \varphi^{-1}(y)$, $z_0 \in \varphi^{-1}(z)$ such that

$\delta^{m}_{A}(x_0) = \sup\limits_{t\in\, \varphi^{-1}(x)} t$, $\delta^{m}_{A}(y_0) = \sup\limits_{t\in\, \varphi^{-1}(y)} t$ and $\delta^{m}_{A}(z_0) = \sup\limits_{t\in\, \varphi^{-1}(z)} t$.

Then we can deduce the following,

$$\begin{aligned}
\delta^{m}_{\varphi(A)}(x * z) &= \sup_{t\in\, \varphi^{-1}(x*z)} \left(\delta^{m}_{A}(t)\right) \\
&\geq \delta^{m}_{A}(x_0 * z_0) \\
&\geq \min\left\{\delta^{m}_{A}\left((x_0 * y_0) * z_0\right), \delta^{m}_{A}(y_0)\right\} \\
&= \min\left\{\sup_{t\in\, \varphi^{-1}((x*y)*z)} \left(\delta^{m}_{A}(t)\right), \sup_{t\in\, \varphi^{-1}(y)} \left(\delta^{m}_{A}(t)\right)\right\} \\
&= \min\left\{\delta^{m}_{\varphi(A)}\left((x * y) * z\right), \delta^{m}_{\varphi(A)}(y)\right\}
\end{aligned}$$

and
$$\begin{aligned}
\lambda^{n}_{\varphi(A)}(x * z) &= \inf_{t\in\, \varphi^{-1}(x*z)} \left(\lambda^{n}_{\varphi(A)}(t)\right) \leq \lambda^{n}_{A}(x_0 * z_0) \\
&\leq \max\left\{\lambda^{n}_{A}\left((x_0 * y_0) * z_0\right), \lambda^{n}_{A}(y_0)\right\} \\
&= \max\left\{\inf_{t\in\, \varphi^{-1}((x*y)*z)} \left(\lambda^{n}_{A}(t)\right), \inf_{t\in\, \varphi^{-1}(y)} \left(\lambda^{n}_{A}(t)\right)\right\} \\
&= \max\left\{\lambda^{n}_{\varphi(A)}\left((x * y) * z\right), \lambda^{n}_{\varphi(A)}(y)\right\}
\end{aligned}$$

Hence $\varphi(A) = \langle \varphi_{\sup}(\delta_A), \varphi_{\inf}(\lambda_A)\rangle$ is induced an (m, n)-fuzzy BCC-ideal of G_2.

Putting $z = 0$ in the above theorem, we obtain:

Corollarly-4.8: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If

If A is an (m, n)-fuzzy BCK-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still an (m, n)-fuzzy BCK-ideal of G_2.

Summing up theorem-3.18, theorem-4.7 and corollary-4.8, we conclude the following theorem.

Theorem-4.9: Let $\varphi: G_1 \rightarrow G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively (m, n) - fuzzy topologies on the spaces G_1 and G_2 such that $\varphi(\tau) = \tau^*$. If A is an (m, n) - fuzzy topological BCC-ideal / BCK-ideal in G_1, then $\varphi(A)$ is also an (m, n)- fuzzy topological BCC-ideal / BCK-ideal in G_2.

Conclusion

We study the concept of (m, n)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also discussed the characteristic of the homomorphic image and inverse image of (m, n)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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