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Investigation of a Subclass of Harmonic Multivalent Functions Defined By Differential Operator

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Abstract

This manuscript proposes a novel subclass of harmonic multivalent functions discerned by utilization of the differential operator with in the confines of the unit disc, subsequently deriving requisite coefficient bounds, convex combinations, extreme points and convolution conditions for this specific class .

Key words and phrases: Harmonic functions, Multivalent functions, Differential Operator.

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1. INTRODUCTION

A complex-valued function $f = u + iv$ defined on a simply connected domain $D \subset C$ in the complex plane is harmonic in D (Complex plane) if both its real and imaginary components, u and v are satisfy Laplace's equation in D. Clunie and Shiel-Small [12] demonstrated that a harmonic function f in any simply connected domain D can be decomposed into $f = h + \overline{g}$, where h and g are both analytic functions in D, denoted as the analytic and co-analytic parts of f.

An essential condition for f to be locally univalent and orientation-preserving in D is that the modulus of the derivative of its analytic part exceeds the modulus of the derivative of its co-analytic part, i.e., $\left|h'(z)\right| > \left|g'(z)\right|$ in D [see [12]]. A class of harmonic functions H, denoted by $f = h + \overline{g}$, is defined as those functions that are harmonic, univalent, and orientation-preserving in the open unit disc $U = \{z \in C : |z| < 1\}$, with the normalization condition that $f(0) = h(0) = f_z(0) - 1 = 0$. Duren et al [14] initially proposed the concept of multivalent harmonic complex valued functions through the application of the argument principle. Subsequently, Ahuja and Jahangiri [1],[2], formulated a class $H(p)$ of p-valent harmonic functions that preserve orientation, characterized by the expression

$$f(z) = h(z) + \overline{g(z)} \tag{1.1}$$

wherein $h(z)$ and $g(z)$ are analytic, p-valent functions defined within the unit disc $U = \{z \in C : |z| < 1\}$. Specifically, $h(z)$ and $g(z)$ adhere to the forms

$$h(z) = z^p + \sum_{l=2}^{\infty} a_{l+p-1} z^{l+p-1}, \quad g(z) = \sum_{l=1}^{\infty} b_{l+p-1} z^{l+p-1}, \quad |b_p| < 1. \tag{1.2}$$

Investigations initiated by Clunie and Sheil-Small [12] in 1984 into the class S_H and its corresponding geometric subclasses resulted in coefficient bounds. Subsequent research by Sheil-Small [12], Silverman [18], Silverman and Silvia [19] and Jahangiri [15] examined various subclasses of harmonic univalent functions. This research area has been further developed by an extensive number of authors [4],[9],[10],[18]. Furthermore, we cite Duren [13] and Ponnusamy [17] provided foundational results and references pertinent to the subject.

In 2016, Makinde [16] introduced the differential operator $F^\nu : A \to A$ and defined as

$$F^\nu f(z) = z^p + \sum_{l=2}^{\infty} C_{(l+p-1)\nu}\, a_{l+p-1}\, z^{l+p-1}, \quad C_{(l+p-1)\nu} = \frac{(l+p-1)!}{|(l+p-1)-\nu|!}, \quad \nu \in N_0 = N \cup \{0\}.$$

Drawing upon this existing research, we investigated the subclass $G_{H_p}(\nu, \rho, \beta)$ characterised by harmonic multivalent functions $f(z)$ expressed as (1.1) and satisfying the condition

$$Re\left\{(1 + \rho e^{i\alpha})\frac{F^{\nu+1} f(z)}{F^\nu f(z)} - \rho e^{i\alpha}\right\} > \beta \tag{1.3}$$

for $\nu \in N_0,\ 0 \leq \beta < 1,\ \rho \geq 0,\ \alpha \in R,\ (z \in U)$, where $F^\nu f(z)$ is defined as

$$F^\nu f(z) = F^\nu h(z) + (-1)^\nu \overline{F^\nu g(z)},\ \nu \in N_0 \tag{1.4}$$

Let $G_{\overline{H_p}}(\nu,\rho,\beta)$ denote the subclass of $G_{H_p}(\nu,\rho,\beta)$ consisting of harmonic functions of the form

$$f_\nu(z)=h(z)+\overline{g_\nu(z)} \tag{1.5}$$

where

$$h(z)=z^p-\sum_{l=2}^{\infty}|a_{l+p-1}|\ z^{l+p-1}$$

and

$$g_\nu(z)=(-1)^\nu\sum_{l=1}^{\infty}|b_{l+p-1}|\ z^{l+p-1}.$$

The primary objective of this study entails deriving sufficient conditions for functions $f(z)\in G_{H_p}(\nu,\rho,\beta)$ expressed in the form (1.1). Furthermore, the investigation seeks to establish necessary and sufficient conditions for functions $f_\nu(z)\in G_{\overline{H_p}}(\nu,\rho,\beta)$ expressed in the form (1.5). Additionally, the research aims to identify convolution, convex combination, and extreme points for functions $f_\nu(z)\in G_{\overline{H_p}}(\nu,\rho,\beta)$ expressed in the form (1.5).

2. MAIN RESULTS

Theorem 2.1 Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.1). If

$$\begin{aligned}&\textstyle\sum_{l=2}^{\infty}\left[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho\right]C_{(l+p-1)\nu}\,|a_{l+p-1}|\\&\quad+\textstyle\sum_{l=1}^{\infty}\left[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho\right]C_{(l+p-1)\nu}\,|b_{l+p-1}|\le(1-\beta)\end{aligned} \tag{2.1}$$

where $0\le\beta<1$, $\nu\in N_0=N\cup\{0\}$, $\alpha\in R$, $\rho\ge0$ and $C_{(l+p-1)\nu}=\frac{(l+p-1)!}{|l+p-1-\nu|!}$. Then f is orientation-preserving, harmonic in U and $f\in G_{H_p}(\nu,\rho,\beta)$.

Proof :$f(z)$ is sense-preserving in U because

$$\left|h'(z)\right|\ge p\,|z|^{p-1}-\sum_{l=2}^{\infty}(l+p-1)\,|a_{l+p-1}|\,|z|^{l+p-2}$$

$$\ge p\,|z|^{p-1}\left[1-\sum_{l=2}^{\infty}\frac{(l+p-1)}{p}\,|a_{l+p-1}|\,|z|^{l-1}\right]$$

$$\geq p\,|z|^{p-1}\left[1-\sum_{l=2}^{\infty}\frac{[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]\,C_{(l+p-1)\nu}\,|a_{l+p-1}|}{p(1-\beta)}\right]$$

$$\geq p\,|z|^{p-1}\left[\sum_{l=1}^{\infty}\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]\,C_{(l+p-1)\nu}\,|b_{l+p-1}|}{p(1-\beta)}\right]$$

$$\geq p\,|z|^{p-1}\left[\sum_{l=1}^{\infty}\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]\,C_{(l+p-1)\nu}\,|b_{l+p-1}|\,|z|^{l-1}}{p(1-\beta)}\right]$$

$$\geq \sum_{l=1}^{\infty}(l+p-1)\,|b_{l+p-1}|\,|z|^{l+p-2}$$

$$\geq \left|g^{'}(z)\right|$$

Now we show that $f(z)\in G_{H_p}(\nu,\rho,\beta)$ using the fact that $Re(\alpha)>\beta$ if and only if $|1-\beta+\alpha|\geq|1+\beta-\alpha|$. It is suffices to show that

$$\left|1-\beta+(1+\rho e^{i\alpha})\frac{F^{\nu+1}f(z)}{F^{\nu}f(z)}-\rho e^{i\alpha}\right|-\left|1+\beta-(1+\rho e^{i\alpha})\frac{F^{\nu+1}f(z)}{F^{\nu}f(z)}+\rho e^{i\alpha}\right|\geq 0$$

Now,

$$\left|(1-\beta-\rho e^{i\alpha})F^{\nu}f(z)+(1+\rho e^{i\alpha})F^{\nu+1}f(z)\right|-\left|(1+\beta+\rho e^{i\alpha})F^{\nu}f(z)-(1+\rho e^{i\alpha})F^{\nu+1}f(z)\right|$$

$$=\left|\begin{array}{c}(2-\beta)z^p+\sum_{l=2}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho e^{i\alpha}+(1-\beta-\rho e^{i\alpha})]C_{(l+p-1)\nu}a_{l+p-1}z^{l+p-1}\\ -(-1)^{\nu}\sum_{l=1}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho e^{i\alpha}-(1-\beta-\rho e^{i\alpha})]C_{(l+p-1)\nu}\overline{b_{l+p-1}z^{l+p-1}}\end{array}\right|$$

$$-\left|\begin{array}{c}\beta z^p+\sum_{l=2}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho e^{i\alpha}-(1+\beta+\rho e^{i\alpha})]C_{(l+p-1)\nu}a_{l+p-1}z^{l+p-1}\\ +(-1)^{\nu}\sum_{l=1}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho e^{i\alpha}+(1+\beta+\rho e^{i\alpha})]C_{(l+p-1)\nu}\overline{b_{l+p-1}z^{l+p-1}}\end{array}\right|$$

$$\begin{aligned}\geq (2-\beta)\,|z|^p-\sum_{l=2}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho+(1-\beta-\rho)]C_{(l+p-1)\nu}\,|a_{l+p-1}|\,|z|^{l+p-1}\\ -\sum_{l=1}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho-(1-\beta-\rho)]C_{(l+p-1)\nu}\,|b_{l+p-1}|\,|z|^{l+p-1}\\ -\beta\,|z|^p-\sum_{l=2}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho-(1+\beta+\rho)]C_{(l+p-1)\nu}\,|a_{l+p-1}|\,|z|^{l+p-1}\end{aligned}$$

$$-\sum_{l=1}^{\infty}[|l+p-1-\nu|+|l+p-1-\nu|\,\rho+(1+\beta+\rho)]C_{(l+p-1)\nu}\,|b_{l+p-1}|\,|z|^{l+p-1}$$

$$=2(1-\beta)\,|z|^{p}-2\sum_{l=2}^{\infty}[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\,|a_{l+p-1}|\,|z|^{l+p-1}$$
$$-2\sum_{l=1}^{\infty}[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\,|b_{l+p-1}|\,|z|^{l+p-1}$$

$$=2(1-\beta)\,|z|^{p}\left[\begin{array}{c}1-\sum_{l=2}^{\infty}\frac{[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\left|a_{l+p-1}\right||z|^{l-1}}{(1-\beta)}\\ -\sum_{l=1}^{\infty}\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\left|b_{l+p-1}\right||z|^{l-1}}{(1-\beta)}\end{array}\right]$$

(since $z\in U,\ \ |z|<1$)

$$=2(1-\beta)\left[\begin{array}{c}1-\sum_{l=2}^{\infty}\frac{[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\left|a_{l+p-1}\right|}{(1-\beta)}\\ -\sum_{l=1}^{\infty}\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\left|b_{l+p-1}\right|}{(1-\beta)}\end{array}\right]$$

The last expression is non-negative by (2.1), therefore the proof is complete.
If we put $\rho=0$ in theorem 2.1 then corollary 2.2 is obtained.

Corollary 2.2 : [8] Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.1). If

$$\sum_{l=2}^{\infty}[|l+p-1-\nu|-\beta]\,C_{(l+p-1)\nu}\,|a_{l+p-1}|+\sum_{l=1}^{\infty}[|l+p-1-\nu|+\beta]\,C_{(l+p-1)\nu}\,|b_{l+p-1}|\le(1-\beta)$$

where
$0\le\beta<1,\quad \nu\in N_0=N\cup\{0\}$ and $C_{(l+p-1)\nu}=\frac{(l+p-1)!}{|l+p-1-\nu|!}$. Then f is orientation-preserving harmonic multivalent in U and $f\in B_{H_p}(\nu,\beta)$.

The harmonic function given bellow shows that the coefficient bound given by (2.1) is sharp.

$$f(z) = z^p + \sum_{l=2}^{\infty} \frac{(1-\beta)}{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}} u_{l+p-1} z^{l+p-1}$$
$$+ \sum_{l=1}^{\infty} \frac{(1-\beta)}{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}} \overline{\nu_{l+p-1} z^{l+p-1}}$$

where
$\nu \in N_0 = N \cup \{0\}$, $\rho \geq 0$ and

$$\sum_{l=2} |u_{l+p-1}| + \sum_{l=1} |v_{l+p-1}| = 1$$

The above defined harmonic function is in $G_{H_p}(\nu, \rho, \beta)$.
We have,

$$\sum_{l=i}^{\infty} \left[\begin{array}{c} \frac{[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\left|a_{l+p-1}\right|}{(1-\beta)} \\ +\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\left|b_{l+p-1}\right|}{(1-\beta)} \end{array} \right]$$
$$= 1 + \sum_{l=2} |u_{l+p-1}| + \sum_{l=1} |v_{l+p-1}| = 2$$

The following theorem shows that the necessary condition for the function $f_\nu(z) = h(z) + \overline{g_\nu(z)}$ of the form (1.5) is the condition (2.1).

Theorem 2.2 Let function $f_\nu(z) = h(z) + \overline{g_\nu(z)}$ be given by (1.5). Then $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$ if and only if

$$\begin{aligned} &\textstyle\sum_{l=2}^{\infty}[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\,|a_{l+p-1}| \\ &\quad + \textstyle\sum_{l=1}^{\infty}[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\,|b_{l+p-1}| \\ &\leq (1-\beta) \end{aligned} \tag{2.2}$$

Proof : It is easy to prove the "if part" , since $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta) \subset G_{H_p}(\nu, \rho, \beta)$. Now we prove that only if part of the theorem(2.2) . Let $f_\nu(z) = h(z) + \overline{g_\nu(z)} \in G_{\overline{H_p}}(\nu, \rho, \beta)$. Then the condition (1.3) is equivalent to

$$Re\left\{(1+\rho e^{i\alpha})\frac{F^{\nu+1}f(z)}{F^\nu f(z)} - (\rho e^{i\alpha} + \beta)\right\} \geq 0$$

implies that

$$Re\left\{\frac{(1+\rho e^{i\alpha})F^{\nu+1}f(z) - (\beta+\rho e^{i\alpha})F^\nu f(z)}{F^\nu f(z)}\right\} \geq 0$$

Therefore

$$Re\left\{\frac{(1+\rho e^{i\alpha})\left[z^p-\sum_{l=2}^{\infty}C_{(l+p-1)(\nu+1)}\left|a_{l+p-1}\right|z^{l+p-1}+(-1)^{(2\nu+1)}\sum_{l=1}^{\infty}C_{(l+p-1)(\nu+1)}\left|b_{l+p-1}\right|\overline{z^{l+p-1}}\right]}{z^p-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|z^{l+p-1}+\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|\overline{z^{l+p-1}}}\right.$$
$$\left.-\frac{(\beta+\rho e^{i\alpha})\left[z^p-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|z^{l+p-1}+(-1)^{(2\nu)}\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|\overline{z^{l+p-1}}\right]}{z^p-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|z^{l+p-1}+\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|\overline{z^{l+p-1}}}\right\}\geq 0$$

Let

$$H=(1-\beta)-\sum_{l=2}^{\infty}[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho e^{i\alpha}]C_{(l+p-1)\nu}\left|a_{l+p-1}\right|z^{l-1}.$$

After simplification, we get

$$Re\left\{\frac{\begin{array}{c}H-\frac{\overline{z^p}}{z^p}(-1)^{2\nu}\sum_{l=1}^{\infty}[|l+p-1-\nu|+\beta\\ +(|l+p-1-\nu|+1)\rho e^{i\alpha}]C_{(l+p-1)\nu}\left|b_{l+p-1}\right|\overline{z^{l-1}}\end{array}}{1-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|z^{l+p-2}+\frac{\overline{z^p}}{z^p}\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|\overline{z^{l-1}}}\right\}\geq 0 \tag{2.3}$$

The condition (2.3) must hold for all values of z^p on the positive real axis $0\leq|z|^p=r<1$. Choose the values of z^p on positive real axis, where $0\leq|z|^p=r<1$.
Now we have

$$Re\frac{\left\{\begin{array}{c}H-e^{i\alpha}[\sum_{l=2}^{\infty}[|l+p-1-\nu|-1]\rho C_{(l+p-1)\nu}\left|a_{l+p-1}\right|r^{l-1}\\ +\sum_{l=1}^{\infty}[|l+p-1-\nu|+1]\rho C_{(l+p-1)\nu}\left|b_{l+p-1}\right|r^{l-1}]\end{array}\right\}}{1-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|r^{l-1}+\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|r^{l-1}}\geq 0$$

Since $Re(-e^{i\alpha})\geq-\left|e^{i\alpha}\right|=-1$, the above inequality become

$$Re\frac{\left[\begin{array}{c}(1-\beta)-\sum_{l=2}^{\infty}[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\left|a_{l+p-1}\right|r^{l-1}\\ -\sum_{l=1}^{\infty}[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\left|b_{l+p-1}\right|r^{l-1}\end{array}\right]}{1-\sum_{l=2}^{\infty}C_{(l+p-1)\nu}\left|a_{l+p-1}\right|r^{l-1}+\sum_{l=1}^{\infty}C_{(l+p-1)\nu}\left|b_{l+p-1}\right|r^{l-1}}\geq 0 \tag{2.4}$$

If the condition (2.2) is not satisfied, then numerator in (2.4) is negative for r sufficient close to 1. This contradicts the condition for $f_\nu(z)\in G_{\overline{H_p}}(\nu,\rho,\beta)$. Therefore proof is complete.

3. CONVOLUTION

For harmonic functions

$$f_\nu(z) = z^p - \sum_{l=2}^{\infty} |a_{l+p-1}|\, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |b_{l+p-1}|\, \overline{z^{l+p-1}}$$

and

$$F_\nu(z) = z^p - \sum_{l=2}^{\infty} |A_{l+p-1}|\, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |B_{l+p-1}|\, \overline{z^{l+p-1}}$$

We define the convolution of two harmonic function $f_\nu(z) \ \ and \ \ F_\nu(z)$ as

$$\begin{aligned}(f_\nu * F_\nu)(z) = f_\nu(z) * F_\nu(z) = \\ z^p - \sum_{l=2}^{\infty} |a_{l+p-1}A_{l+p-1}|\, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |b_{l+p-1}B_{l+p-1}|\, \overline{z^{l+p-1}}\end{aligned} \tag{3.1}$$

Theorem 3.1 For $0 \le \beta_1 \le \beta_2 < 1$, let $f_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta_2)$ and $F_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta_1)$. Then

$$f_\nu(z) * F_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta_2) \subset G_{\overline{H_p}}(\nu,\rho,\beta_1)$$

Proof: Let

$$f_\nu(z) = z^p - \sum_{l=2}^{\infty} |a_{l+p-1}|\, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |b_{l+p-1}|\, \overline{z^{l+p-1}} \in G_{\overline{H_p}}(\nu,\rho,\beta_2)$$

and

$$F_\nu(z) = z^p - \sum_{l=2}^{\infty} |A_{l+p-1}|\, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |B_{l+p-1}|\, \overline{z^{l+p-1}} \in G_{\overline{H_p}}(\nu,\rho,\beta_1)$$

then the convolution $(f_\nu * F_\nu)(z)$ is given by (3.1). We wish to show that the coefficient of $(f_\nu * F_\nu)(z)$ satisfies the required condition given in theorem 2.3 for $F_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta_1)$ we note that $|A_{l+p-1}| \le 1 \quad and\, |B_{l+p-1}| \le 1$. Now for the coefficient of $(f_\nu * F_\nu)(z)$, we have

$$\sum_{l=2}^{\infty}\left[\frac{[|l+p-1-\nu|-\beta_1+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}{(1-\beta_1)}\,|a_{l+p-1}A_{l+p-1}|\right]$$
$$+\sum_{l=1}^{\infty}\left[\frac{[|l+p-1-\nu|+\beta_1+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}{(1-\beta_1)}\,|b_{l+p-1}B_{l+p-1}|\right]$$

$$\leq\sum_{l=2}^{\infty}\left[\frac{[|l+p-1-\nu|-\beta_1+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}{(1-\beta_1)}\,|a_{l+p-1}|\right]$$
$$+\sum_{l=1}^{\infty}\left[\frac{[|l+p-1-\nu|+\beta_1+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}{(1-\beta_1)}\,|b_{l+p-1}|\right]$$

$$\leq\sum_{l=2}^{\infty}\left[\frac{[|l+p-1-\nu|-\beta_2+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}{(1-\beta_2)}\,|a_{l+p-1}|\right]$$
$$+\sum_{l=1}^{\infty}\left[\frac{[|l+p-1-\nu|+\beta_2+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}{(1-\beta_2)}\,|b_{l+p-1}|\right]\leq 1$$

Since

$$0\leq\beta_1\leq\beta_2<1, and\ \ f_\nu(z)\in G_{\overline{H_p}}(\nu,\rho,\beta_2).$$

$$Thus\ \ f_\nu(z)*F_\nu(z)\in G_{\overline{H_p}}(\nu,\rho,\beta_2)\subset G_{\overline{H_p}}(\nu,\rho,\beta_1).$$

Therefore proof of the theorem is complete.

4. CONVEX COMBINATION

Let the functions $f_{\nu_i}(z)$ be defined, for $i=1,2,3,......,j,....$ by

$$f_{\nu_i}(z)=z^p-\sum_{l=2}^{\infty}\left|a_{(l+p-1),i}\right|z^{l+p-1}+(-1)^\nu\sum_{l=1}^{\infty}\left|b_{(l+p-1),i}\right|\overline{z^{l+p-1}} \qquad (4.1)$$

Theorem 4.1 Let the function $f_{\nu_i}(z)$ of the form (4.1) belong to the class $G_{\overline{H_p}}(\nu,\rho,\beta)$ for every $i=1,2,3,....j,...$ Then the functions $t_i(z)$ defined by

$$t_i(z)=\sum_{i=1}^{j}d_if_{\nu_i}(z)$$

$0 \leq d_i \leq 1$ are also in the class $G_{\overline{H_p}}(\nu, \rho, \beta)$.
Where

$$\sum_{i=1}^{j} d_i = 1.$$

Proof: By the definition of $t_i(z)$, we can write

$$\begin{aligned} t_i(z) = z^p - \sum_{l=2}^{\infty} \left\{ \sum_{i=1}^{j} d_i \left| a_{(l+p-1),i} \right| \right\} z^{l+p-1} \\ +(-1)^{\nu} \sum_{l=1}^{\infty} \left\{ \sum_{i=1}^{j} d_i \left| b_{(l+p-1),i} \right| \right\} \overline{z^{l+p-1}} \end{aligned}$$

On comparing above equation with (1.5), we obtain $|a_{l+p-1}| = \left(\sum_{i=1}^{j} d_i \left| a_{(l+p-1),i} \right| \right)$ and $|b_{l+p-1}| = \left(\sum_{i=1}^{j} d_i \left| b_{(l+p-1),i} \right| \right)$. In order to prove $t_i(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$ we show that the condition (2.2) satisfies. Consider

$$\sum_{l=2}^{\infty} \left[|l+p-1-\nu| - \beta + (|l+p-1-\nu| - 1)\rho \right] C_{(l+p-1)\nu} \left| a_{l+p-1} \right| \\ + \sum_{l=1}^{\infty} \left[|l+p-1-\nu| + \beta + (|l+p-1-\nu| + 1)\rho \right] C_{(l+p-1)\nu} \left| b_{l+p-1} \right|$$

$$= \sum_{l=2}^{\infty} \left[|l+p-1-\nu| - \beta + (|l+p-1-\nu| - 1)\rho \right] C_{(l+p-1)\nu} \left(\sum_{i=1}^{j} d_i \left| a_{(l+p-1),i} \right| \right) \\ + \sum_{l=1}^{\infty} \left[|l+p-1-\nu| + \beta + (|l+p-1-\nu| + 1)\rho \right] C_{(l+p-1)\nu} \left(\sum_{i=1}^{j} d_i \left| b_{(l+p-1),i} \right| \right)$$

$$= \sum_{i=1}^{j} d_i \left(\begin{array}{c} \sum_{l=2}^{\infty} \left[|l+p-1-\nu| - \beta + (|l+p-1-\nu| - 1)\rho \right] C_{(l+p-1)\nu} \left| a_{(l+p-1),i} \right| \\ + \sum_{l=1}^{\infty} \left[|l+p-1-\nu| + \beta + (|l+p-1-\nu| + 1)\rho \right] C_{(l+p-1)\nu} \left| b_{(l+p-1),i} \right| \end{array} \right)$$

$$\leq \sum_{i=1}^{j} d_i p(1-\beta) \leq (1-\beta).$$

Since $f_{\nu_i}(z)$ are in $G_{\overline{H}}(\nu, \rho, \beta)$ for every $i = 1, 2, 3, \ldots$.
Therefore by Theorem 2.3, $t_i(z) \in G_{\overline{H}}(\nu, \rho, \beta)$ and so the proof is complete.

5. EXTREME POINTS

Theorem 5.1 Let $f_\nu(z)$ be given by (1.5). Then $f_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta)$ if and only if

$$f_\nu(z) = \sum_{l=1}^{\infty} \left[\mu_{l+p-1} h_{l+p-1}(z) + \eta_{l+p-1} g_{(l+p-1)\nu}(z)\right] \tag{5.1}$$

Where

$$h_p(z) = z^p,$$

$$h_{l+p-1}(z) = z^p - \left(\frac{(1-\beta)}{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}\right) z^{l+p-1},$$
$$l = 2,3,....$$
$$g_{(l+p-1)\nu}(z) = z^p + (-1)^\nu \left(\frac{(1-\beta)}{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}\right) \overline{z^{l+p-1}},$$
$$l = 1,2,3,....$$

and

$$\sum_{l=1}^{\infty} [\mu_{l+p-1} + \eta_{l+p-1}] = 1, \quad \mu_{l+p-1} \geq 0, \quad \eta_{l+p-1} \geq 0.$$

In particular, the extreme points of $G_{\overline{H_p}}(\nu,\rho,\beta)$ are $\{h_{l+p-1}(z)\}$ and $\left\{g_{(l+p-1)\nu}(z)\right\}$.

Proof: For functions $f_\nu(z)$ of the form (5.1), we have

$$\begin{aligned} f_\nu(z) &= \sum_{l=1}^{\infty} \left[\mu_{l+p-1} h_{l+p-1}(z) + \eta_{l+p-1} g_{(l+p-1)\nu}(z)\right] \\ &= \sum_{l=1}^{\infty} (\mu_{l+p-1} + \eta_{l+p-1}) z^p \end{aligned}$$
$$- \sum_{l=2}^{\infty} \left(\frac{(1-\beta)}{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}\right) \mu_{l+p-1} z^{l+p-1}$$
$$+(-1)^\nu \sum_{l=1}^{\infty} \left(\frac{(1-\beta)}{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}\right) \eta_{l+p-1} \overline{z^{l+p-1}}$$

Now on comparing above equation with (1.5) we obtain

$$|a_{l+p-1}| = \left(\frac{(1-\beta)}{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}\right) \mu_{l+p-1}$$

and

$$|b_{l+p-1}| = \left(\frac{(1-\beta)}{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}} \right) \eta_{l+p-1}$$

for proving $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$ we show that condition (2.2) satisfies. Consider,

$$\sum_{l=2}^{\infty} \left(\frac{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}{(1-\beta)} \right) |a_{l+p-1}|$$
$$+ \sum_{l=1}^{\infty} \left(\frac{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}{(1-\beta)} \right) |b_{l+p-1}|$$
$$= \sum_{l=2}^{\infty} \mu_{l+p-1} + \sum_{l=1}^{\infty} \eta_{l+p-1} = (1-\mu_p) \leq 1$$

Therefore by Theorem 2.3, $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$.
Conversely, suppose that $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$. Let

$$\mu_{l+p-1} = \left(\frac{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}}{(1-\beta)} \right) |a_{l+p-1}|$$
$$0 \leq \mu_{l+p-1} \leq 1, \qquad l = 2, 3, ...$$

and

$$\eta_{l+p-1} = \left(\frac{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}}{(1-\beta)} \right) |b_{l+p-1}|$$
$$0 \leq \eta_{l+p-1} \leq 1, \qquad l = 1, 2, 3, ...$$

and

$$\mu_p = 1 - \sum_{l=2}^{\infty} \mu_{l+p-1} - \sum_{l=1}^{\infty} \eta_{l+p-1}.$$

Then $f_\nu(z)$ can be written as

$$f_\nu(z) = z^p - \sum_{l=2}^{\infty} |a_{l+p-1}| \, z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty} |b_{l+p-1}| \, \overline{z^{l+p-1}}$$
$$= z^p - \sum_{l=2}^{\infty} \left(\frac{(1-\beta)}{[|l+p-1-\nu| - \beta + (|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}} \right) \mu_{l+p-1} z^{l+p-1}$$
$$+ (-1)^\nu \sum_{l=1}^{\infty} \left(\frac{(1-\beta)}{[|l+p-1-\nu| + \beta + (|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}} \right) \eta_{l+p-1} \overline{z^{l+p-1}}$$
$$= z^p + \sum_{l=2}^{\infty} (h_{l+p-1}(z) - z^p) \, \mu_{l+p-1} + \sum_{l=1}^{\infty} \left(g_{\nu(l+p-1)}(z) - z^p \right) \eta_{l+p-1}$$

$$= \sum_{l=2}^{\infty} h_{l+p-1}(z)\mu_{l+p-1} + \sum_{l=1}^{\infty} g_{\nu(l+p-1)}(z)\eta_{l+p-1} + z^p\left(1 - \sum_{l=2}^{\infty}\mu_{l+p-1} - \sum_{l=1}^{\infty}\eta_{l+p-1}\right)$$

$$= \sum_{l=1}^{\infty}\left[h_{l+p-1}(z)\mu_{l+p-1} + g_{\nu(l+p-1)}(z)\eta_{l+p-1}\right]$$

So the proof is complete.

Theorem 5.2 Each member of $G_{\overline{H}}(\nu, \rho, \beta), \quad (0 \leq \beta < 1)$ maps U on to a starlike domain.

Proof: We only need to show that if $f_\nu(z) \in G_{\overline{H_p}}(\nu, \rho, \beta)$ then

$$Re\left\{\frac{zh'(z) - \overline{zg'_\nu(z)}}{h(z) + \overline{g_\nu(z)}}\right\} > 0$$

using the fact that $Re(\alpha) > 0$ if and only if $|1+\alpha| \geq |1-\alpha|$. It is suffices to show that

$$\left|h(z) + \overline{g_\nu(z)} + zh'(z) - \overline{zg'_\nu(z)}\right|$$
$$- \left|h(z) + \overline{g_\nu(z)} - zh'(z) + \overline{zg'_\nu(z)}\right|$$

$$= \left|(p+1)z^p - \sum_{l=2}^{\infty}(l+p)\left|a_{l+p-1}\right| z^{l+p-1} - (-1)^\nu \sum_{l=1}^{\infty}(l+p-2)\left|b_{l+p-1}\right| \overline{z^{l+p-1}}\right|$$
$$- \left|(1-p)z^p + \sum_{l=2}^{\infty}(l+p-2)\left|a_{l+p-1}\right| z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty}(l+p)\left|b_{l+p-1}\right| \overline{z^{l+p-1}}\right|$$

$$\geq \left((p+1)\left|z\right|^p - \sum_{l=2}^{\infty}(l+p)\left|a_{l+p-1}\right|\left|z\right|^{l+p-1} - (-1)^\nu \sum_{l=1}^{\infty}(l+p-2)\left|b_{l+p-1}\right|\left|z\right|^{l+p-1}\right)$$
$$- \left((1-p)\left|z\right|^p + \sum_{l=2}^{\infty}(l+p-2)\left|a_{l+p-1}\right|\left|z\right|^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty}(l+p)\left|b_{l+p-1}\right|\left|z\right|^{l+p-1}\right)$$

$$\geq \left(2p\left|z\right|^p - 2\sum_{l=2}^{\infty}(l+p-1)\left|a_{l+p-1}\right|\left|z\right|^{l+p-1} - (-1)^\nu 2\sum_{l=1}^{\infty}(l+p-1)\left|b_{l+p-1}\right|\left|z\right|^{l+p-1}\right)$$

$$= 2p\,|z|^p \left[1 - \left(\sum_{l=2}^{\infty}(\frac{l+p-1}{p})\,|a_{l+p-1}|\,|z|^{l-1} + \sum_{l=1}^{\infty}(\frac{l+p-1}{p})\,|b_{l+p-1}|\,|z|^{l-1}\right)\right]$$

$$\geq 2p\,|z|^p \left\{ \begin{array}{c} 1-\sum_{l=2}^{\infty}[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]C_{(l+p-1)\nu}\,|a_{l+p-1}| \\ -\sum_{l=1}^{\infty}[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]C_{(l+p-1)\nu}\,|b_{l+p-1}| \end{array} \right\}$$

$$\geq 2p\,|z|^p\,[1-p(1-\beta)]$$

$$= 2p\,|z|^p\,[1-p+p\beta] \geq 0.$$

Hence proof is complete.

Theorem 5.3 Let $f_\nu(z)$ be given by (1.5) belong to the class $G_{\overline{H_p}}(\nu,\rho,\beta)$ and c is any real number with $c > -1$. Then the function $L_c(f_\nu(z))$ defined as $L_c(f_\nu(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f_\nu(t)dt \quad, c > -1$, is also belong to the class $G_{\overline{H_p}}(\nu,\rho,\beta)$.

Proof: From definition of $L_c(f_\nu(z))$, it follows that

$$L_c(f_\nu(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f_\nu(t)dt$$

$$= \frac{c+1}{z^c}\int_0^z t^{c-1}\left(t^p - \sum_{l=2}^{\infty}|a_{l+p-1}|\ t^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty}|b_{l+p-1}|\ \overline{t}^{l+p-1}\right)dt$$

$$= \left(\frac{c+1}{c+p}\right)z^p - \sum_{l=2}^{\infty}\left(\frac{c+1}{c+l+p-1}\right)|a_{l+p-1}|\ z^{l+p-1}$$

$$+(-1)^\nu \sum_{l=1}^{\infty}\left(\frac{c+1}{c+l+p-1}\right)|b_{l+p-1}|\ \overline{z}^{l+p-1}$$

$$= \left(\frac{c+1}{c+p}\right)z^p - \sum_{l=2}^{\infty}|A_{l+p-1}|\ z^{l+p-1} + (-1)^\nu \sum_{l=1}^{\infty}|B_{l+p-1}|\ \overline{z}^{l+p-1}$$

Now, on comparing the above equation with (1.5), we obtain $|a_{l+p-1}| = |A_{l+p-1}| = \left(\frac{c+1}{c+l+p-1}\right)|a_{l+p-1}|$ and $|b_{l+p-1}| = |B_{l+p-1}| = \left(\frac{c+1}{c+l+p-1}\right)|b_{l+p-1}|$. In order to prove $L_c(f_\nu(z)) \in G_{\overline{H_p}}(\nu,\rho,\beta)$, we need to show that condition (2.2) satisfies. Consider,

$$\sum_{l=2}^{\infty}\frac{[|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho]\,C_{(l+p-1)\nu}}{(1-\beta)}\left(\frac{c+1}{c+l+p-1}\right)|a_{l+p-1}|$$
$$+\sum_{l=1}^{\infty}\frac{[|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho]\,C_{(l+p-1)\nu}}{(1-\beta)}\left(\frac{c+1}{c+l+p-1}\right)|b_{l+p-1}|$$

$$\begin{aligned}\leq &\sum_{l=2}^{\infty}\left[\frac{|l+p-1-\nu|-\beta+(|l+p-1-\nu|-1)\rho}{(1-\beta)}\right]|a_{l+p-1}|\\ +&\sum_{l=1}^{\infty}\left[\frac{|l+p-1-\nu|+\beta+(|l+p-1-\nu|+1)\rho}{(1-\beta)}\right]|b_{l+p-1}|\\ \leq &\ 1\end{aligned}$$

Since $f_\nu(z) \in G_{\overline{H_p}}(\nu,\rho,\beta)$. Therefore by Theorem 2.3, $L_c(f_\nu(z)) \in G_{\overline{H_p}}(\nu,\rho,\beta)$. Proof is complete.

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