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Transitive and Absorbent Implicative Filters of Lattice Wajsberg Algebras

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Abstract

In this paper, we introduce the notion of transitive implicative filters in lattice Wajsberg algebras. Also, we derive a necessary and sufficient condition that every implicative filter is a transitive implicative filter. Moreover, we introduce the concept of absorbent implicative filters of lattice Wajsberg algebras and investigate some of their properties.

Keywords: Wajsberg algebras; Lattice Wajsberg algebras; Implicative filter; Transitive implicative filter; Absorbent implicative filter.

Mathematical Subject Classification 2020: 03G10, 03G25.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertain information. Mony-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. In order to research the logical system whose propositional value is given in a lattice, Mordewaj Wajsberg[7] proposed the concept of Wajsberg algebras in 1935.to show that the valued $\aleph_0$ Lukariewicz propositional calculus was complete with respect to axiomatic conjectured by Lukariewicz itself. Font, Rodriguez and Torrens[2] introduced the notion of lattice Wajsberg algebras in 1984, which is an algebraic structure that is established by combining a lattice and Wajsberg algebra, and discussed some of their properties, For the general development of lattice Wajsberg algebras, filter theory plays an important role.

Font,et al.[2] introduced the notion of implicative filters in a lattice Wajsberg algebras, and investigated their properties. Basheer Ahamed and Ibrahim [1] introduced the definitions of fuzzy implicative and an anti fuzzy implicative filters of lattice Wajsberg algebras and obtained some properties with illustrations. In this paper, we introduce the concept of transitive implicative filters in lattice Wajsberg algebras, and investigate some of their properties. We give equivalent condition that every implicative filter in transitive implicative filter. We introduce the notion of absorbent implicative filters and derive some properties with useful illustrations.

2. Preliminaries

In this section, we recall some basic definitions and properties that are helpful for developing our main results.

Definition 2.1[1] let $(A,\to,*,1)$ be an algebra with complement $"*"$ and a binary operation $"\to"$ is called a Wajsberg algebra if and only if it satisfied the following axioms for all $x,y,z\in A$.

(i) $1\to x=x$

(ii) $(x\to y)\to((y\to z)\to(x\to z))=1$

(iii) $(x\to y)\to y=(y\to x)\to x$

(iv) $(x^*\to y^*)\to(y\to x)=1$

Proposition 2.2[1] The Wajsberg algebra $(A,\to,*,1)$ satisfied the following axioms for all $x,y,z\in A$.

(i) $x\to x=1$

(ii) If $x\to y=y\to x=1$ then $x=y$

(iii) $x\to 1=1$

(iv) $x\to(y\to x)=1$

(v) If $x\to y=y\to z=1$ then $x\to z=1$

(vi) $(x\to y)\to((z\to x)\to(z\to y))=1$

(vii) $x\to(y\to z)=y\to(x\to z)$

(viii) $x\to 0=x\to 1^*=x^*$

(ix) $(x^*)^*=x$

(x) $x^*\to y^*=y\to x$

Proposition 2.3[1] The Wajsberg algebra $(A, \to, *, 1)$ satisfied the following axioms for all $x, y, z \in A$.

(i) If $x \le y$ then $x \to z \ge y \to z$
(ii) If $x \le y$ then $z \to x \le z \to y$
(iii) $x \le y \to z$ if and only if $y \le x \to z$
(iv) $(x \vee y)^* = (x^* \wedge y^*)$
(v) $(x \wedge y)^* = (x^* \vee y^*)$
(vi) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(vii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
(viii) $(x \to y) \vee (y \to x) = 1$
(ix) $x \to (y \vee z) = (x \to y) \vee (x \to z)$
(x) $(x \wedge y) \to z = (x \to y) \vee (x \to z)$
(xi) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$
(xii) $(x \wedge y) \to z = (x \to y) \to (x \to z)$

Definition 2.4[1] The Wajsberg algebra *A* is called lattice Wajsberg algebra, if it satisfied the following conditions for all $x, y \in A$.

(i) A partial ordering $\le$ on a lattice Wajsbrg algebra *A* such that $x \le y$ if and only if $x \to y = 1$
(ii) $(x \vee y) = (x \to y) \to y$
(iii) $(x \wedge y) = \left((x^* \to y^*) \to y^*\right)^*$

Thus, we have $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower 0 and upper bound 1.

Definition 2.5[1] Let *A* be a lattice Wajsberg algebra. A subset *F* of *A* is called an implicative filter of *A*, if it satisfied the following axioms for all $x, y \in A$.

(i) $1 \in F$
(ii) $x \in F$ and $x \to y \in F$ implies $y \in F$.

3. Main Results

In this section, we introduce the definition of a transitive filter and implication algebra and investigate some properties with illustrations.

3.1 Transitive implicative filter

Definition 3.1.1 A non empty subset F of a lattice Wajsberg algebra A is called a transitive implicative filter, if it satisfies the following for all $x, y, z \in A$:

(i) $1 \in F$
(ii) $x \to y \in F$, $y \to z \in F$ imply $x \to z \in F$.

It is evident that the principle filter {1} of a lattice Wajsberg algebra is a transitive filter. Consider $x, y, z \in A$ be such that $x \to y \in$ {1} and $y \to z \in$ {1}. Then, we have $x \leq y$ and $y \leq z$. Hence, $x \leq z$ which implies $x \to z = 1 \in$ {1}.

Example 3.1.2 Let $A = \{0, a, b, c, 1\}$ be a set. Define the partially ordered relation on A as $0 < a < b < c < 1$ and also define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ for all $x, y \in A$. Define the operations " ′" and $\to$ on A are as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

Table 3.1: Complement

→	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Table 3.2: Implication

Then, clearly A is a lattice Wajsberg algebra. Now consider the set $F = \{a, b, c, 1\}$.

It can be easily verified that F is a transitive filter on A.

Some properties of transitive filters can be observed in the following.

Proposition 3.1.3 Let F be a transitive filter of A. Then we have the following

(i) $x \in F, x \leq y$ implies $y \in F$
(ii) $x \in F, x \to y \in F$ implies $y \in F$

(iii) Set intersection of transitive filters is again a transitive filter

Proof.

(i) Suppose $x \in F$ and $x \leq y$. Then, we get that $x \to y = 1 \in F$. Since $1 \to x \in F$ and $x \to y \in F$,We get that $y = 1 \to y \in F$

(ii) Let $x \in F$ and $x \to y \in F$ .Then we get that $1 \to x \in F$ and $x \to y \in F$ .Since F is transitive, it yields that $y \in F$.

(iii) It is clear. □

Proposition3.1.4 Every transitive filter of *A* is a lattice filter.

Proof **.**Let *F* be a transitive filter of a lattice Wajsberg algebra *A*. Let $x, y \in F$.

We have $y \leq x \to y = 1 \wedge (x \to y) = (x \to x) \wedge (x \to y) = x \to (x \wedge y)$. we get that $x \to (x \wedge y) \in F$.

Since $1 \to x \in F$ and *F* is transitive, it yields that $x \wedge y \in F$.

Therefore, *F* is a lattice filter of *A*.

□

But the converse of the above proposition 3.1.4 is not true . It can be observed in the following example.

Example 3.1.5 Let $A = \{0, a, b, c, d, 1\}$ be a set whose Hasse diagram is given in the following figure. Define a unary operation “ $'$ ” and a binary operation $\to$ on *A* as in the following tables respectively:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 3.3: Complement

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 3.4: Implication

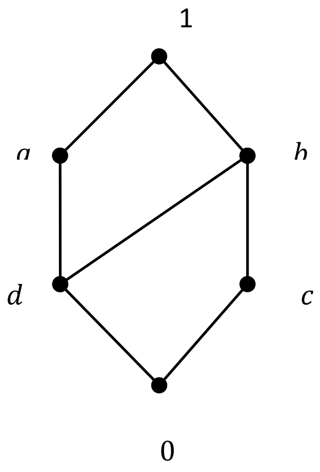

Figure 3.1: Hasse diagram

Define the operations $\vee$ and $\wedge$ on *A* as follows:

$$x \vee y = (x \to y) \to y \;\; ; \; x \wedge y = ((x' \to y') \to y')'$$

For all $x, y \in A$ .Then *A* is a lattice Wajsberg algebra.It is easy to check that $F = \{b, 1\}$ is a lattice filter of *A*, but it is not a transitive filter, since $a \to b = b \in F$ and $b \to c = b \in F$ and $a \to c = c \notin F$. In the following, a sufficient condition is obtained for every lattice filter of a lattice Wajsberg algebra to become a transitive filter.

Proposition 3.1.6 Let *F* be a lattice filter of *A*. If $x \wedge (x \to y) = x \wedge y$ for all $x, y \in A$, then *F* is a transitive filter.

Proof .Let *F* be a filter of *A* .Let $x, y, z \in A$ be such that $x \to y \in F$ and $y \to z \in F$.

Since $y \to z \le x \to (y \to z)$, we get $x \to (y \to z) \in F$.

Hence, we get $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$

$$= x \to [y \wedge (y \to z)]$$

$$= (x \to y) \wedge [x \to (y \to z)] \in F$$

Thus, $x \to z \in F$ .Therefore *F* is a transitive filter.

It can be easily observed that every transitive filter is a filter. However, in the following, a necessary and sufficient condition is derived for every filter of a lattice Wajsberg to become a transitive filter. □

Proposition 3.1.7 Let *F* be a filter of a lattice Wajsberg algebra *A*. Then *F* is *A* Transitive filter if and only if for all $x, y, z \in A$, it satisfies the following condition :

$x \to y \in F, (x \to y) \to (y \to z) \in F$ implies $(x \to y) \to (x \to z) \in F$ (1)

Proof. Let *F* be a filter of *A*.

Assume that *F* is a transitive filter of *A*.

Let $x, y, z \in A$ be such that $x \to y \in F$ and $(x \to y) \to (y \to z) \in F$.

Then we have the following consequence

$$(y \to z) \to [(x \to y) \to (x \to z)] = (x \to y) \to [(y \to z) \to (x \to z)]$$

$$= (x \to y) \to \{x \to [(y \to z) \to z]\}$$

$$=(x\to y)\to\{x\to[(z\to y)\to y]\}$$

$$=(x\to y)\to[(z\to y)\to(x\to y)]$$

$$=(z\to y)\to[(x\to y)\to(x\to y)]$$

$$=(z\to y)\to 1$$

$$=1\in F.$$

Since $(x\to y)\to(y\to z)\in F,(y\to z)\to[(x\to y)\to(x\to z)]\in F$ and F is a transitive filter,

We can conclude that $(x\to y)\to[(x\to y)\to(x\to z)]\in F$.

Since $x\to y\in F$ and *F* is a filter, we get that $(x\to y)\to(x\to z)\in F$.

Conversely , assume the condition(1) . Let $x\to y\in F$ and $y\to z\in F$. Then we get

$y\to z\le(x\to y)\to(y\to z)\in F.$ By the assumed condition, we get$(x\to y)\to(x\to z)\in F$ Now $x\to y\in F,(x\to y)\to(x\to y)\in F$ and *F* is a filter, we get that $x,y\in F$.

Therefore *F* is a transitive filter. □

In the following, some sufficient conditions are obtained for a filter of a lattice Wajsberg algebra to become a transitive filter.

Proposition 3.1.8 Let *F* be a filter of *A*.Then *F* is a transitive filter if it satisfies the following condition: $x\to(y\to z)\in F$ implies$(x\to y)\to z\in F$ for all $x,y,z\in A$.

Proof. Let *F* be a filter of *A* which is satisfying the given condition. Let $x,y,z\in A$ be such that $x\to y\in F$ and $y\to z\in F$ .since $y\to z\le x\to(y\to z)$, we can get $x\to(y\to z)\in F$. By the assumed condition, we get $(x\to y)\to z\in F$. Since *F* is a filter and $x\to y\in F$, it yields that $z\in F$. Since $z\le x\to z,$ we get $x\to z\in F$.

Therefore *F* is a transitive filter. From above proposition, the following proposition is a direct consequence. □

Proposition 3.1.9 Let *F* be a filter of *A*. If *F* is an associative filter, then it is a transitive filter.

3.2 Absorbent implicative filter

In this section, we introduce the notion of absorbent filter of lattice Wajsberg algebras and investigate some properties of with illustrations.

Definition 3.2.1 A non-empty subset F of a lattice Wajsberg algebra A is called an absorbent filter, if it satisfies the following properties, for all $x, y \in A$:

(i) $1 \in F$

(ii) $(x \to y) \to x \in F$ implies $x \in F$.

Proposition 3.2.2 Every associate filter of A is an absorbent filter.

***Proof*.** Let F be an associative filter of A .Let $x, y \in A$ be such that $(x \to y) \to x \in F$.

Then $(x \to y) \to (1 \to x) = 1 \to [(x \to y) \to x] = (x \to y) \to x \in F.$

Since F is associative. Therefore F is an absorbent filter of A . □

Example 3.2.3 Let $A = \{0, a, b, c, 1\}$ be a distributive lattice whose Hasse diagram is given in the following figure. Then clearly $(A, \wedge, \vee, \to, 0, 1)$ is Wajsberg algebra. Define a unary operation " $'$ " and a binary operation $\to$ on A as in the following tables respectively:

x	x'
0	1
a	b
b	a
c	c
1	0

Table 3.5: Complement

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Table3.6: Implication

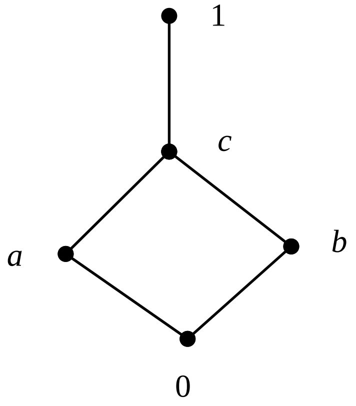

Figure 3.2: Hasse diagram

Then $F = \{1, b\}$ is an absorbent filter of A.

In general, every filter of a lattice Wajsberg algebra need not be an absorbent filter. It can be seen from the example 3.5. Consider the set $F = \{c, 1\}$ .Then clearly F is a filter of A. but not an absorbent filter. For consider $c, d \in A$.Then it is clear that $(d \to c) \to d = b \to d = c \in F$ and $d \notin F$.

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