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The Dirichlet Inverses of Mertens Function and the Sum of the Reciprocals of the Primes

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Abstract

Chebyshev's second function, the Möbius function, and Mersenne primes are used to derive a function similar to the log integral function. This function is compared to the Dirichlet inverse of the Mertens function. The Dirichlet inverse of the sum of reciprocals of primes is compared to Mertens function. An analogue of the Mertens function is derived from the Dirichlet inverse of the sum of reciprocals of primes.

Keywords: Chebyshev's second function, Möbius function, Mersenne primes, log integral function, Dirichlet inverse, Mertens function, sum of reciprocals of primes.

1. INTRODUCTION

Chebyshev's second function is the summatory Mangoldt function, that is,

$$\psi(x) = \sum_{n \le x} \Lambda(n), x > 0. \tag{1}$$

$\Lambda(n)$ equals $\log(p)$ if $n = p^m$ for some prime p and some $m \ge 1$ or 0 otherwise. The prime number theorem is equivalent to the asymptotic formula

$$\sum_{n \le x} \Lambda(n) \sim x, x \to \infty \tag{2}$$

This asymptotic formula states that

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{3}$$

The log integral function is

$$li(x) = \lim_{\delta \to +0} \left(\int_0^{1-\delta} + \int_{1+\delta}^{x} \right) \frac{dt}{\log t}, (x > 1). \tag{4}$$

Let v_j denote $\sum_{i|j}(\psi(i+1) - \psi(i))\mu(i)$ where $\mu(i)$ denotes the Möbius function and j is odd. The Möbius function is defined as follows. $\mu(1)$ is set to 1. For $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\mu(n) = (-1)^k$ if $a_1 = a_2 = \ldots = a_k = 1$ or 0 otherwise. ψ_1 is set to 0. For prime j other than Mersenne primes (3, 7, 31, 127, 8191,...), v_j then equals $\log(2)$. If j is a Mersenne prime, then $v_j = 0$ since $j+1$ is a prime power and the ψ values increase by $\log(2)$ at this point (cancelling out the difference between ψ_2 and ψ_1). In general, v_j equals an integer multiple of $\log(2)$ (including a multiple of 0). For $j < 200000$, there are 30 v_j values equal to $2\log 2$, 60146 v_j values equal to $\log 2$, 34639 v_j values equal to 0, 4949 v_j values equal to $-\log 2$, 221 v_j values equal to $-2\log 2$, and 15 v_j values equal to $-3\log 2$. Let w_n denote $\frac{\sum_{i=1}^{n} v_{2i-1}}{\log \sum_{i=1}^{n} v_{2i-1}}$. C code for computing w_n is given in the Methods section.

2. w_n VERSUS LOG INTEGRAL FUNCTION

Let $li_1(x)$ denote $li(2x)$. A C program for computing $li_1(x)$ is given in the Methods section. A plot of w_n versus $li_1(n)$ for $n = 1, 2, 3, \ldots 1000$ is

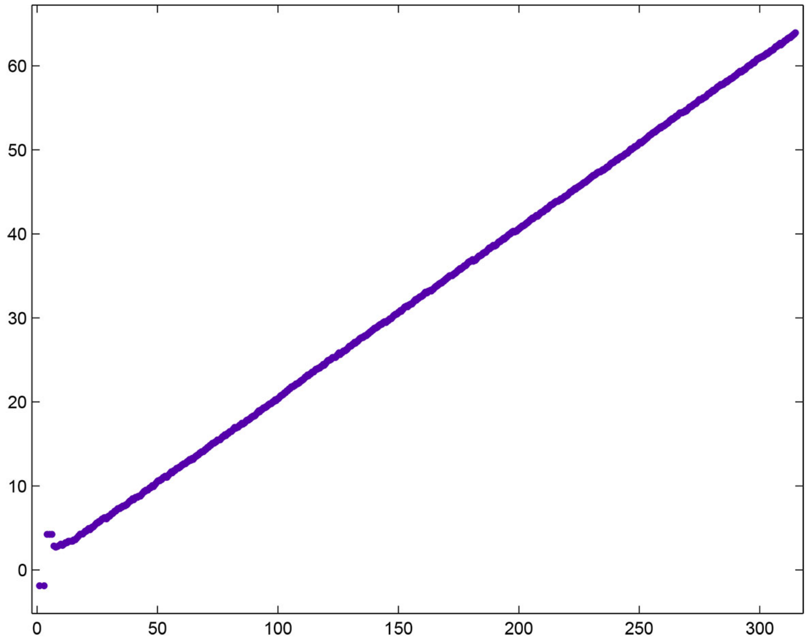

Figure 1

For a linear least-squares fit of the curve, $p_1 = 0.2016$ with a 95% confidence interval of (0.2014, 0.2017), $p_2 = 0.3853$ with a 95% confidence interval of (0.3537, 0.4169), SSE=51.39, R-squared=0.9998, and RMSE=0.2269.

In 1899, de la Vallée [1] proved that

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\log x}}) \tag{5}$$

as $x \to \infty$ where $Li(x) = li(x) - li(2)$ and a is some constant. In 1901, von Koch [2] proved that if the Riemann hypothesis is true, the above error term can be improved to

$$\pi(x) = Li(x) + O(\sqrt{x}\log x) \tag{6}$$

In 1976, Schoenfeld [3] showed, by assuming the Riemann hypothesis, that

$$|\pi(x) - li(x)| < \frac{\sqrt{x}\log x}{8\pi} \tag{7}$$

for $x \geq 2657$.

A plot of $li_1(n) - w_n$ versus $\sqrt{n}$ for $n = 1, 2, 3, \ldots, 10000$ is

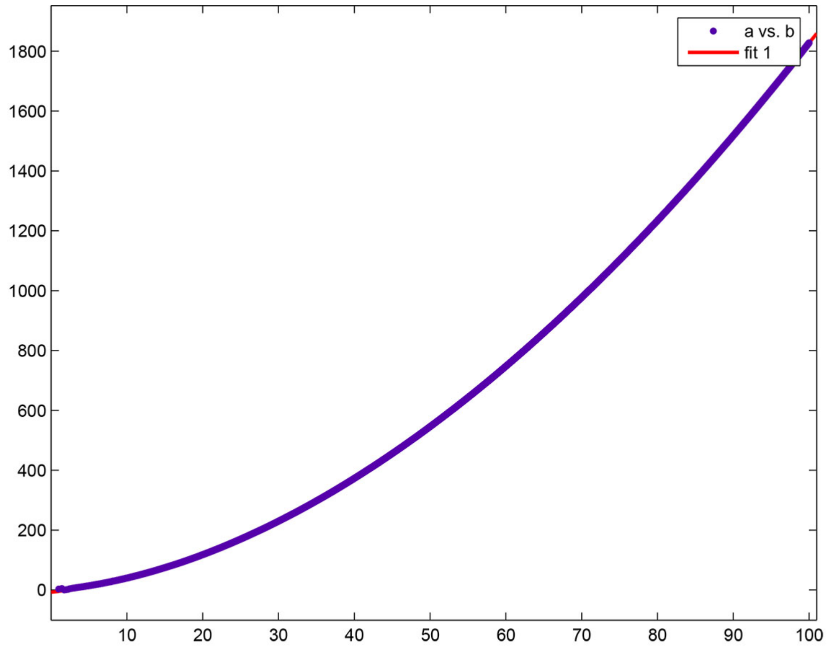

Figure 2

For a cubic least-squares fit of the curve, $p_1 = -0.0001551$ with a 95% confidence interval of (-0.0001556, -0.0001545), $p_2 = 0.169$ with a 95% confidence interval of

(0.1689, 0.1691), $p_3 = 2.999$ with a 95% confidence interval of (2.994, 3.003), $p_4 = -7.65$ with a 95% confidence interval of (−7.727, −7.573), SSE=1534, R-squared=1, and RMSE=64.33. See Cox [7] for similar details.

3. THE DIFFERENCE BETWEEN w_n AND THE AVERAGE OF THE DIRICHLET INVERSE OF THE MERTENS FUNCTION

Theorem 2.8 of Apostol's [6] book is

Theorem 1. *If f is an arithmetical function with $f(1) \neq 0$ there is a unique arithmetical function f^{-1}, called the Dirichlet inverse of f, such that $f * f^{-1} = f^{-1} * f = I$. Moreover, f^{-1} is given by the recursion formulas $f^{-1}(1) = 1/f(1)$, $f^{-1}(n) = \frac{-1}{f(1)} \sum_{d|n, d<n} f(\frac{n}{d}) f^{-1}(d)$ for $n > 1$.*

The Mertens function (denoted by $M(n)$) is the summatory Möbius function. A plot of the Dirichlet inverse of $M(n)$ (denoted by $M'(n)$) for $n = 1, 2, 3, \ldots, 1000$ is

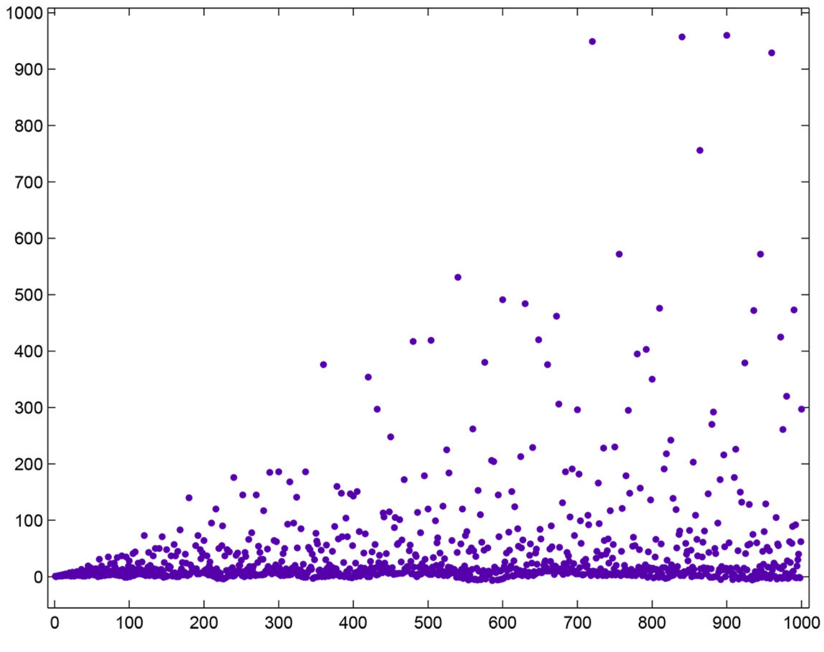

Figure 3

Let $A(n)$ denote the average of the $M'(n)$ values. For example, $A(10)$ is the average of the $M'(n)$ values from $n = 1$ to 10. A plot of $A(n)$ for $n = 1, 2, 3, \ldots, 1000$ is

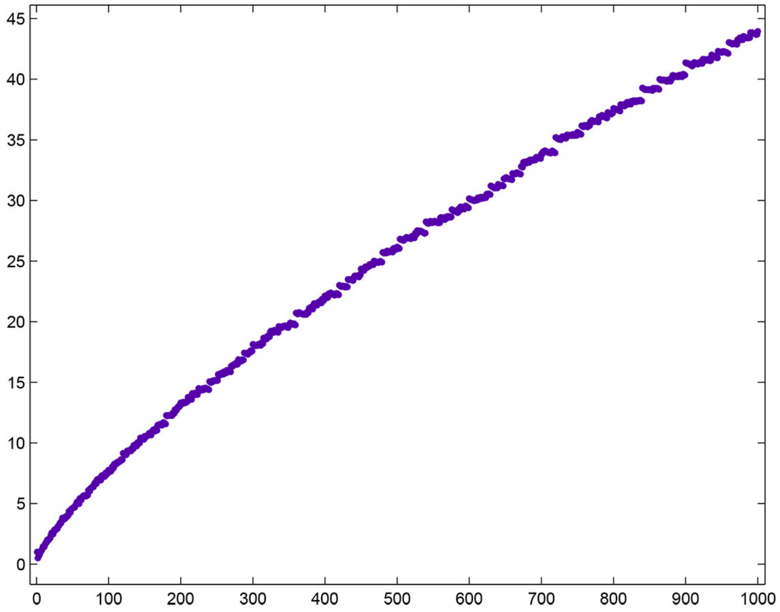

Figure 4

A plot of w_n and $A(n)$ for $n = 1, 2, 3, \ldots, 10000$ is

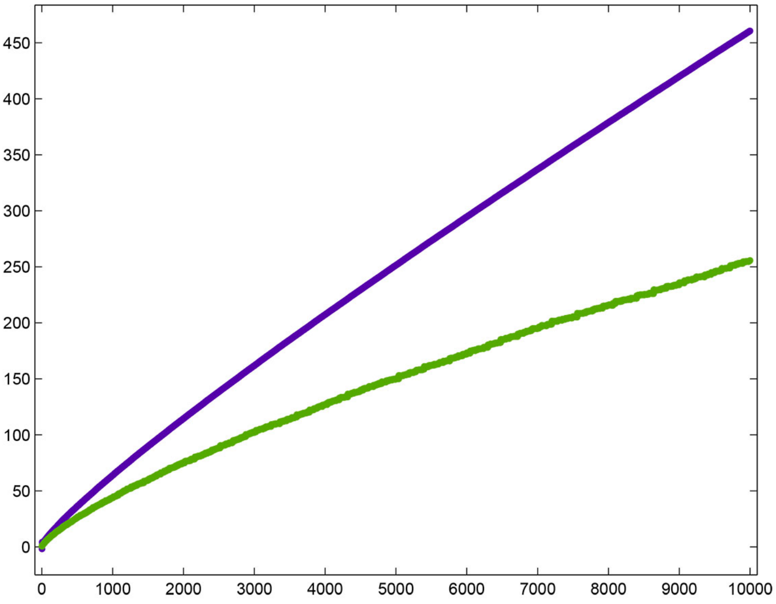

Figure 5

$A(n)$ is the lower curve. A plot of $w_n - A(n)$ for $n = 1, 2, 3, \ldots, 50000$ is

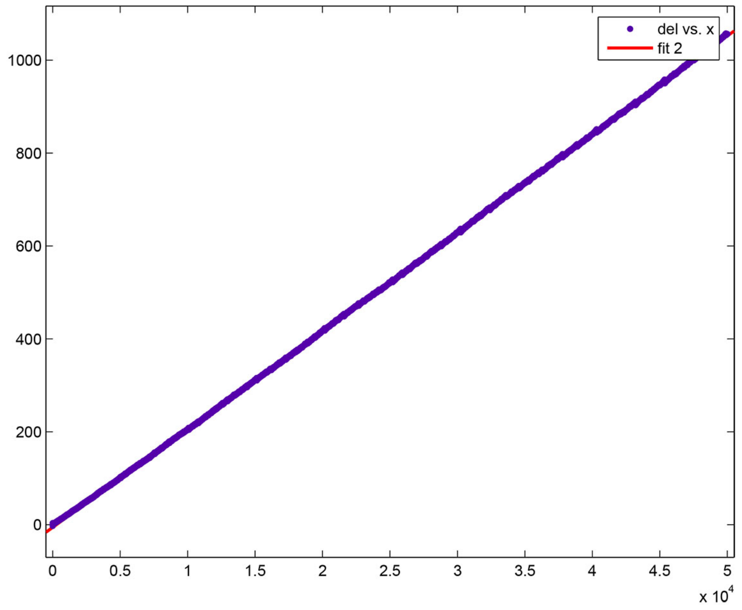

Figure 6

For a linear least-squares fit of the curve, $p_1 = 0.02116$ with a 95% confidence interval of (0.02116, 0.02116), $p_2 = -5.89$ with a 95% confidence interval of (-5.925, -5.856), SSE=$1.964 \cdot 10^5$, R-squared=1, and RMSE=1.982.

A plot of the p_1 parameters of the linear least-squares fits of $w_n - A(n)$ for n upper bounds of 25000, 50000, 75000,. . .,500000 is

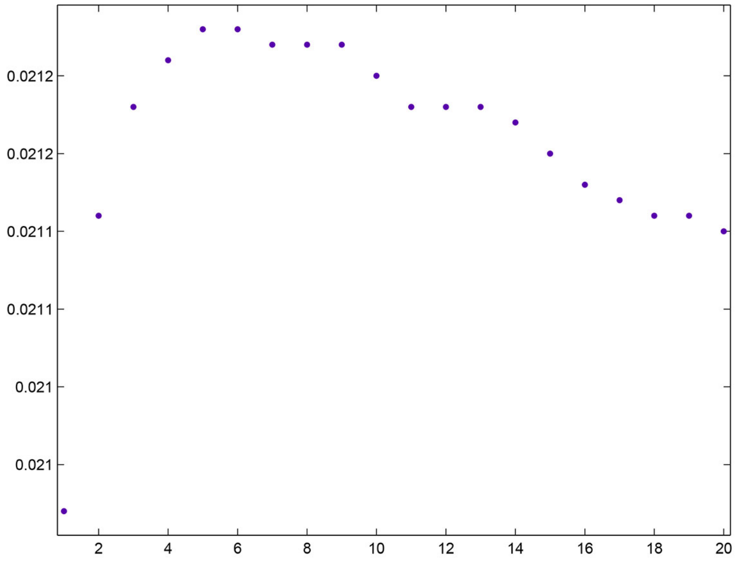

Figure 7

A plot of the p_2 parameters is

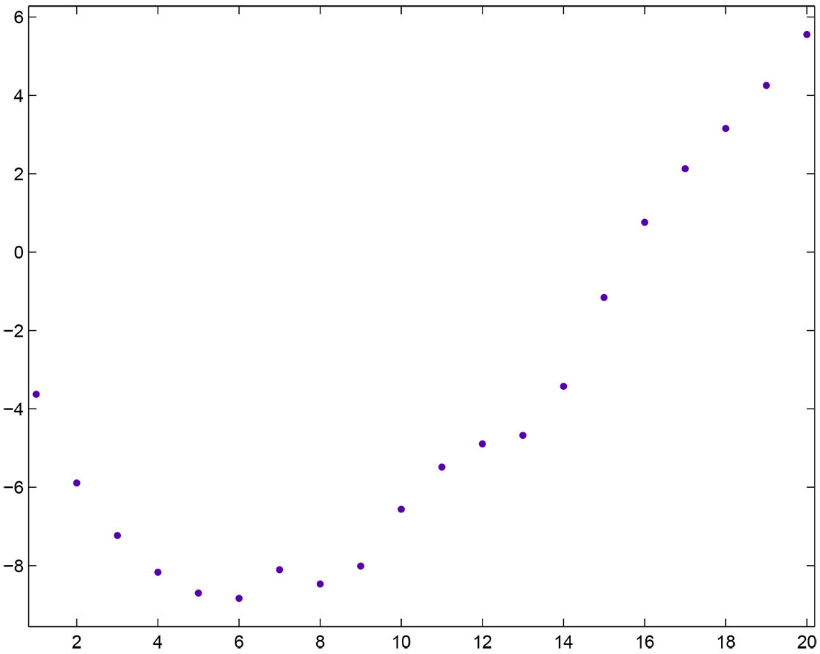

Figure 8

A plot of the SSE values is

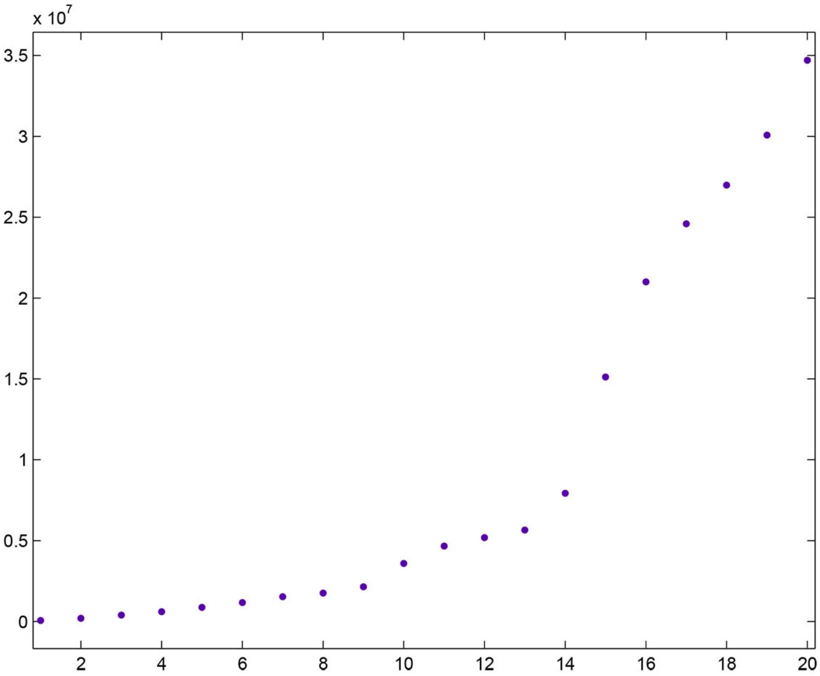

Figure 9

A plot of the RMSE values is

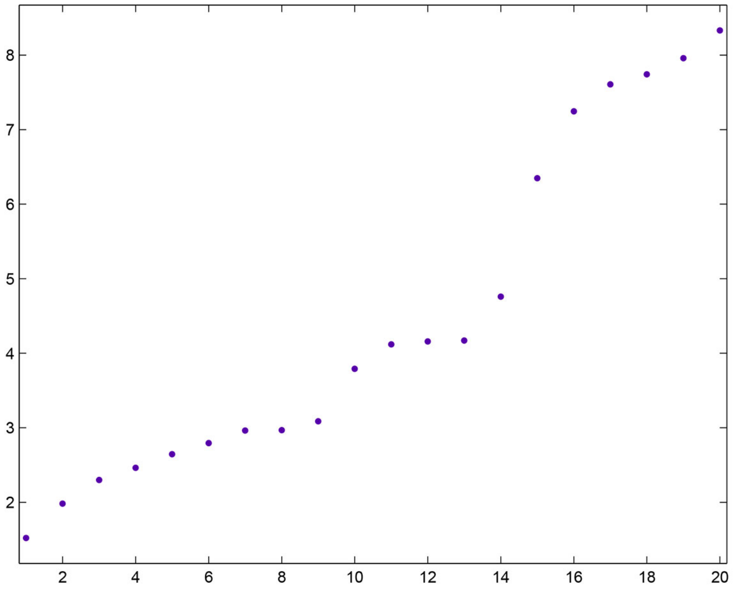

Figure 10

The oscillations of the curves are probably due to the first few non-trivial zeta function zeros. The R-squared values are 1.0. A plot of the SSE values versus the RMSE values is

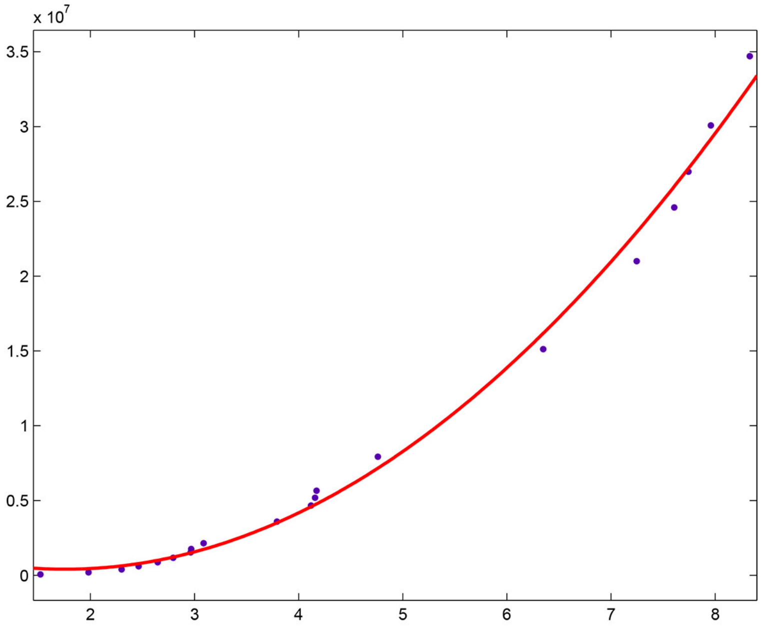

Figure 11

The curve is approximately quadratic.

4. THE DIRICHLET INVERSE OF THE SUM OF THE RECIPROCALS OF THE PRIMES

In computing the sum of the reciprocals of the primes, 1 is considered to be a prime. A plot of the Dirichlet inverse of the sum of the reciprocals of the primes for $n = 1, 2, 3, \ldots, 1000$ is

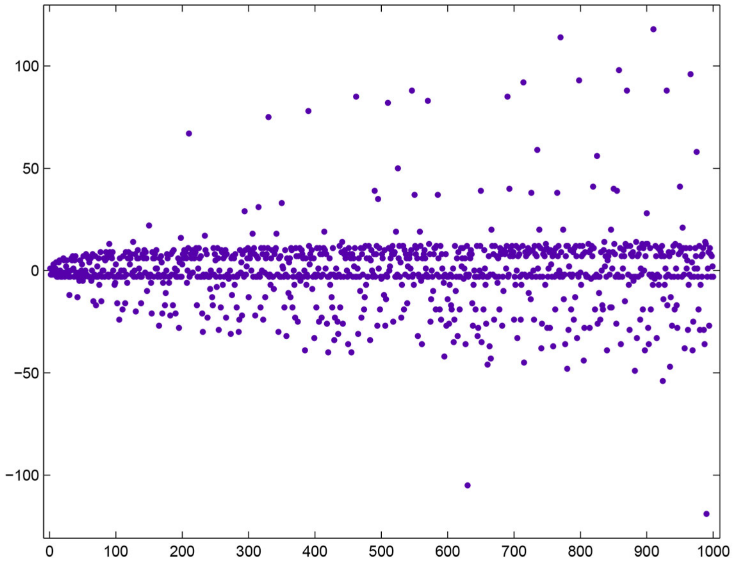

Figure 12

The values are integers. Let $B(x)$ denote the averages of the Dirichlet inverse values. A plot of $B(x)$ for $x = 1, 2, 3, \ldots, 1000$ is

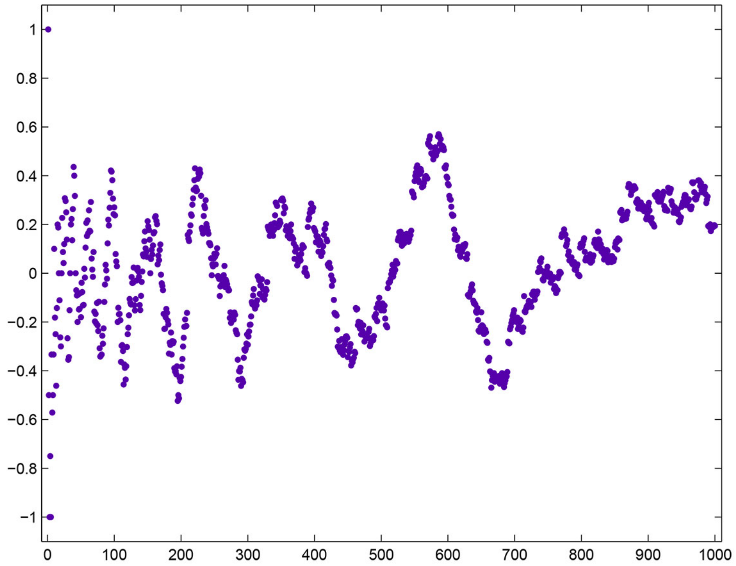

Figure 13

The values are between 1 and -1. A plot of $\sum_{i=1}^{x} |B(\lfloor x/i \rfloor)|i$ for $x = 1, 2, 3, \ldots, 1000$ is

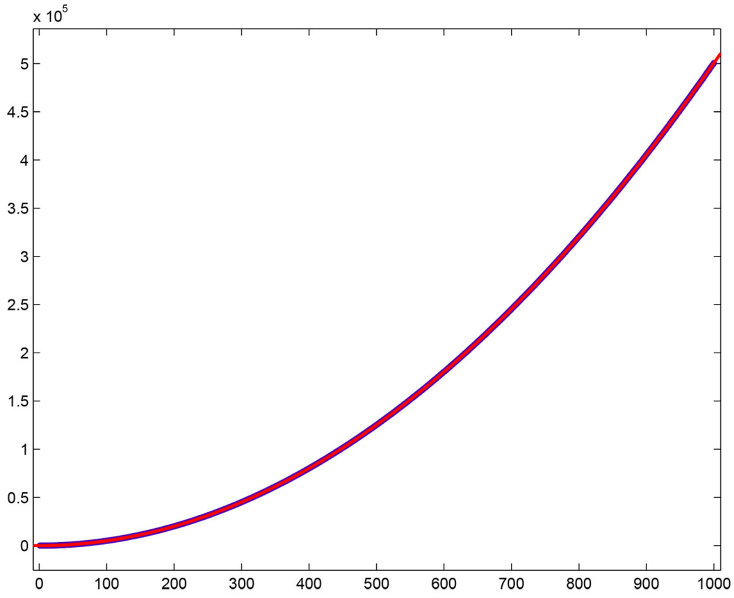

Figure 14

For a quadratic least-squares fit of the curve, $p_1 = 0.5001$ with a 95% confidence interval of (0.5, 0.5001), $p_2 = 0.503$ with a 95% confidence interval of (0.4569, 0.5491), $p_3 = -1.136$ with a 95% confidence interval of $(-11.13, 8.855)$, SSE=$2.86 \cdot 10^6$, R-squared=1, and RMSE=53.56.

A plot of the first 1000 Mertens function values is

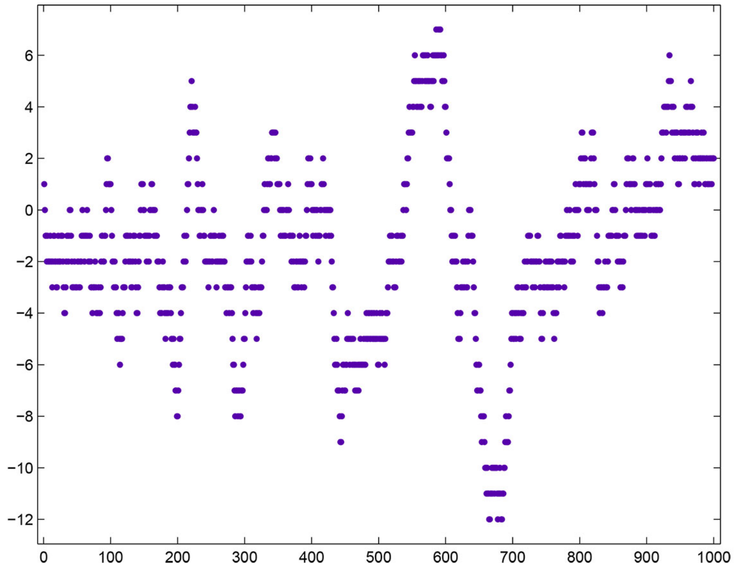

Figure 15

The peaks and valleys are at about the same locations as those of $B(x)$. A plot of $\sum_{i=1}^{x} |M(\lfloor x/i \rfloor)|i$ for $x = 1, 2, 3, \ldots, 1000$ is

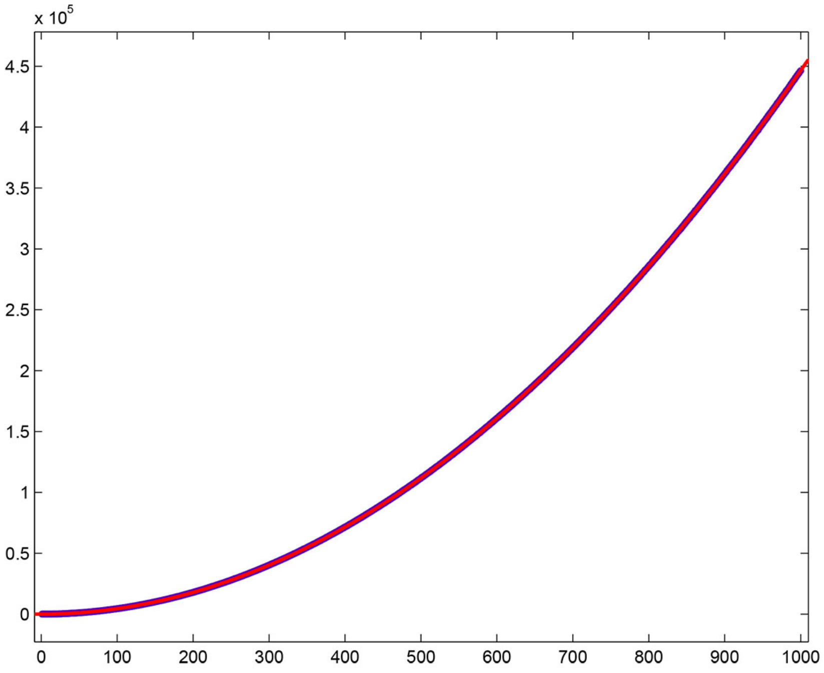

Figure 16

For a quadratic least-squares fit of the curve, $p_1 = 0.4461$ with a 95% confidence interval of (0.446, 0.4462), $p_2 = 0.4536$ with a 95% confidence interval of (0.3648, 0.5425), $p_3 = -1.15$ with a 95% confidence interval of (-20.41, 18.11), SSE=$1.063 \cdot 10^7$, R-squared=1, and RMSE=103.3. The fit is not quite as good as that for $\sum_{i=1}^{x} |B(x/i)|i$.

For a quadratic least-squares fit of the curve $y = x(x+1)/2$, the first and second parameters are 0.5 and the third parameter is 0. Let σ_1 denote the sum of divisors function. A property of the Mertens function is

$$\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\sigma_1(i) = \frac{x(x+1)}{2} \tag{8}$$

See Cox [7] for a proof using Möbius inversion. Let $C(x)$ denote the averages of the sum of reciprocals of the primes. A plot of $\sum_{i=1}^{x} C(\lfloor x/i \rfloor)\sigma_1(i)$ versus $x(x+1)/2$ for $x = 1, 2, 3, \ldots, 1000$ is

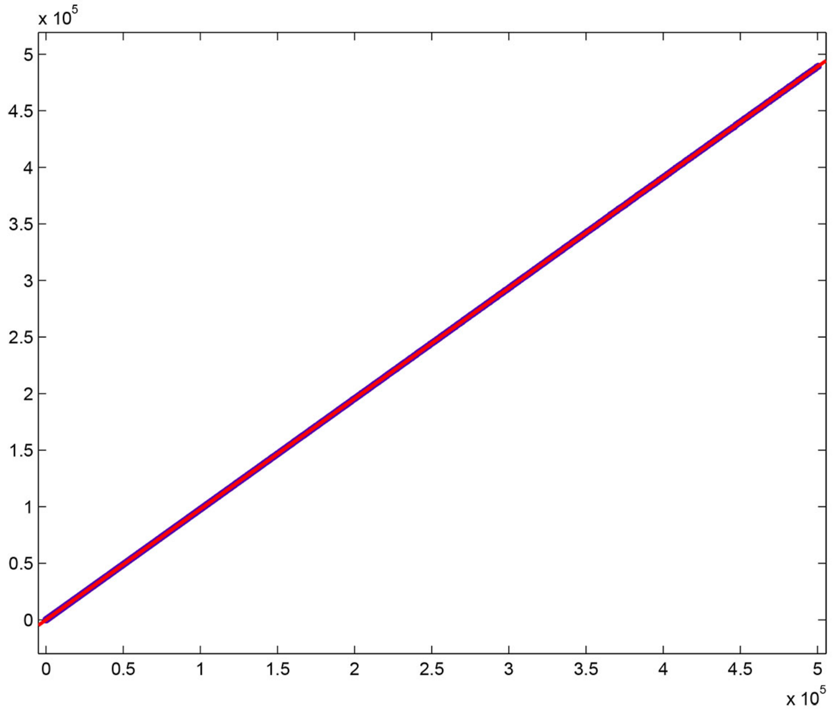

Figure 17

For a linear least-squares fit of the curve, $p_1 = 0.9779$ with a 95% confidence interval of (0.9778, 0.9779), $p_2 = -21.07$ with a 95% confidence interval of (-27.9, -14.23), SSE=$5.373 \cdot 10^6$, R-squared=1, and RMSE=73.38. For a linear least-squares fit of $\sum_{i=1}^{x} B(\lfloor x/i \rfloor)\sigma_1(i)$ versus $x(x+1)/2$ for $x = 1, 2, 3, \ldots, 1000$, $p_1 = 0.7358$ with a 95% confidence interval of (0.7358, 0.7358), $p_2 = 0.3835$ with a 95% confidence interval of (-6.131, 6.898), SSE=$4.882 \cdot 10^6$, R-squared=1, and RMSE=69.94.

Let $\varphi(x)$ denote Euler's totient function. Another property of the Mertens function is

$$\sum_{i=1}^{x} M(\lfloor x/i \rfloor) i = \sum_{i=1}^{x} \varphi(i) \tag{9}$$

Let $\Phi(x)$ denote $\sum_{i=1}^{x} \varphi(i)$. By the Schwarz inequality, $\Phi(x)/\sqrt{x(x+1)(2x+1)/6}$ is a lower bound of $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor) M(\lfloor x/i \rfloor)}$. Theorem 330 in Hardy and Wright's [8] book is

$$\Phi(n) = \frac{3n^2}{\pi^2} + O(n \log n). \tag{10}$$

This will be relevant later. A plot of $\sum_{i=1}^{x} C(\lfloor x/i \rfloor) i$ versus $\sum_{i=1}^{x} \varphi(i)$ for $x = 1, 2, 3, \ldots, 1000$ is

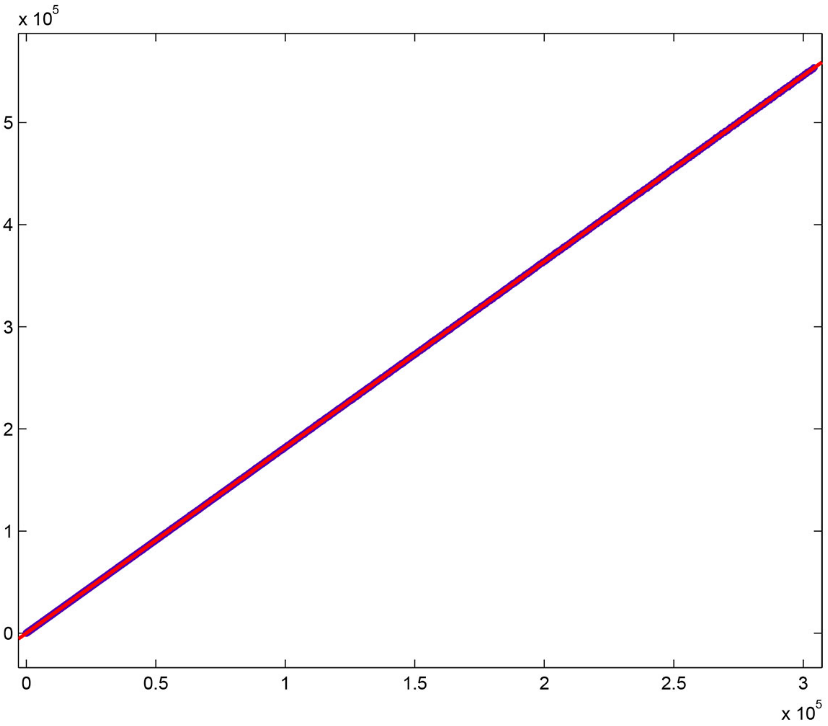

Figure 18

For a linear least-squares fit of the curve, $p_1 = 1.82$ with a 95% confidence interval of (1.82, 1.82), $p_2 = -0.5759$ with a 95% confidence interval of $(-16.46, 15.31)$, SSE=$2.902 \cdot 10^7$, R-squared=1, and RMSE=170.5. For a linear least-squares fit of $\sum_{i=1}^{x} B(\lfloor x/i \rfloor) i$ versus $\sum_{i=1}^{x} \varphi(i)$ for $x = 1, 2, 3, \ldots, 1000$, $p_1 = 1.359$ with a 95% confidence interval of (1.359, 1.359), $p_2 = 0.4897$ with a 95% confidence interval of $(-7.448, 8.427)$, SSE=$7.248 \cdot 10^6$, R-squared=1, and RMSE=85.22.

Let σ_0 denote the number of divisors function. Another property of the Mertens function is

$$\sum_{i=1}^{x} M(\lfloor x/i \rfloor) \log(i) \sigma_0(i)/2 = \log(x!) \tag{11}$$

A plot of $\sum_{i=1}^{x} B(\lfloor x/i \rfloor)\log(i)\sigma_0(i)/2$ versus $\log(x!)$ for $x = 1, 2, 3, \ldots, 1000$ is

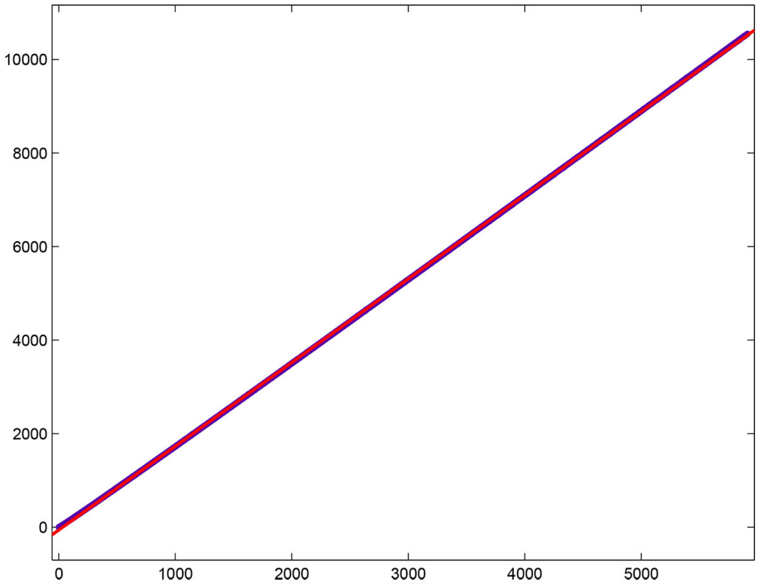

Figure 19

For a linear least-squares fit of the curve, $p_1 = 1.789$ with a 95% confidence interval of (1.789, 1.79), $p_2 = -58.81$ with a 95% confidence interval of (-60.79, -56.82), SSE=$3.025 \cdot 10^5$, R-squared=1, and RMSE=17.41.

A plot of $C(x)$ versus $\log x$ for $x = 1, 2, 3, \ldots, 200000$ is

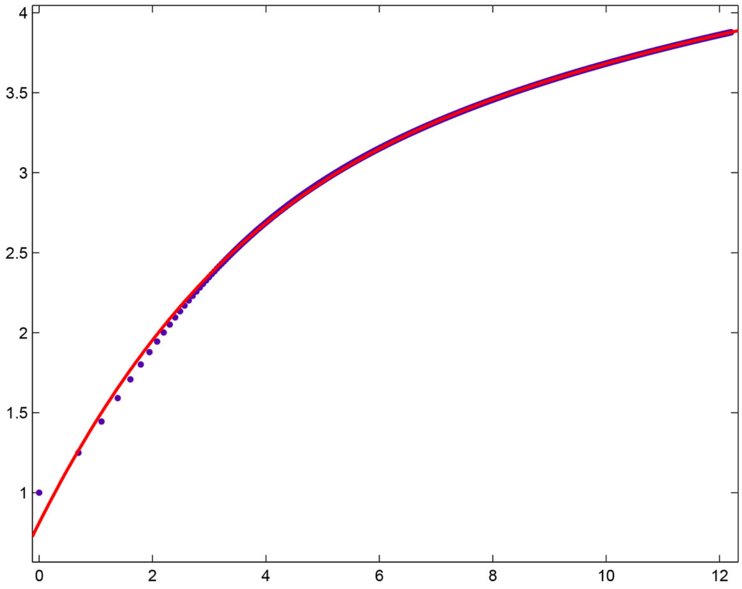

Figure 20

For a quartic least-squares fit of the curve, $p_1 = -9.181 \cdot 10^{-5}$ with a 95% confidence interval of $(-9.195 \cdot 10^{-5}, -9.166 \cdot 16^{-5})$, $p_2 = 0.004109$ with a 95% confidence interval of (0.004104, 0.004114), $p_3 = -0.07393$ with a 95% confidence interval of $(-0.07349, -0.07336)$, $p_4 = 0.7022$ with a 95% confidence interval of (0.0719, 0.7.025), $p_5 = 0.8116$ with a 95% confidence interval of (0.0811, 0.08122), SSE=0.07251, R-squared=1, and RMSE=0.006021.

Let $\lambda(n)$ denote the Liouville function ($\lambda(1) = 1$ or if $n = p^{\alpha_1} \cdots p^{\alpha_k}$, $\lambda(n) = (-1)^{\alpha_1 + \ldots + \alpha_k}$). $\sum_{d|n} \lambda(d)$ equals 1 if n is a perfect square or 0 otherwise. Let $L(x)$ denote $\sum_{n \le x} \lambda(n)$. $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)$ where the summation is over i values that are perfect squares equals $L(x)$. Haselgrove [9] used Ingram's [10] smoothing function to disprove the Pólya conjecture (that $L(x) < 0$) and gives the corresponding smoothing function for the Mertens conjecture (now disproved). A plot of $-L(x)$ and $\sum_{i=1}^{x} C(\lfloor x/i \rfloor)$ where the summation is over i values that are perfect squares for $x = 1, 2, 3, \ldots, 50000$ is

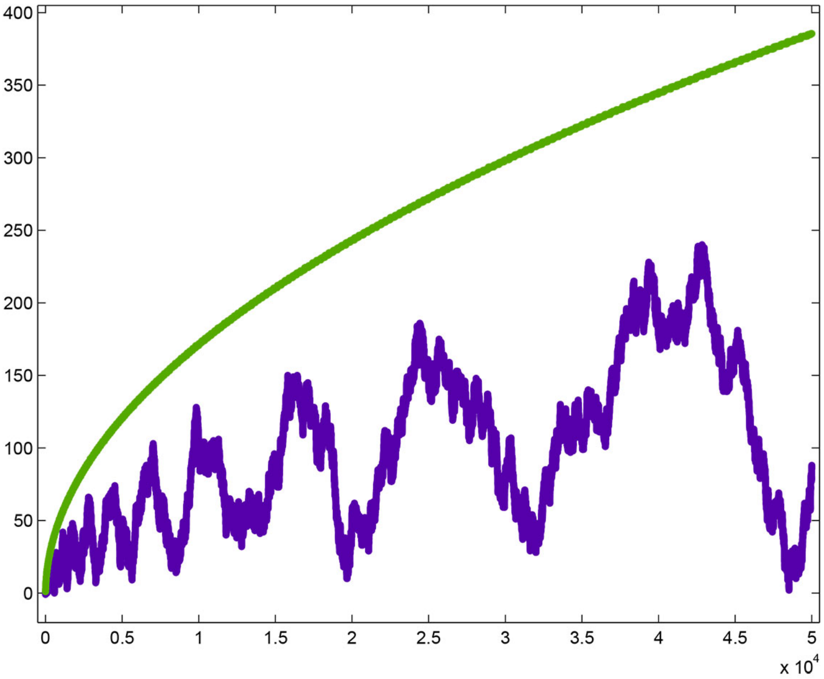

Figure 21

For a linear least-squares fit of $\sum_{i=1}^{x} C(\lfloor x/i \rfloor)$ where the summation is over i values that are perfect squares versus $\sqrt{x}$ for $x = 1, 2, 3, \ldots, 100000$, $p_1 = 1.733$ with a 95% confidence interval of (1.733, 1.733), $p_2 = -1.904$ with a 95% confidence interval of

$(-1.91, -1.897)$, SSE=$1.203 \cdot 10^4$, R-squared=1, and RMSE=0.3469.

A more complex property of the Mertens function is

$$\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\frac{i}{\varphi(i)} = \sum_{i=1}^{x} \frac{\mu^2(i)}{\varphi(i)} \tag{12}$$

A plot of $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\frac{i}{\varphi(i)}$ and $\sum_{i=1}^{x} B(\lfloor x/i \rfloor)\frac{i}{\varphi(i)}$ is

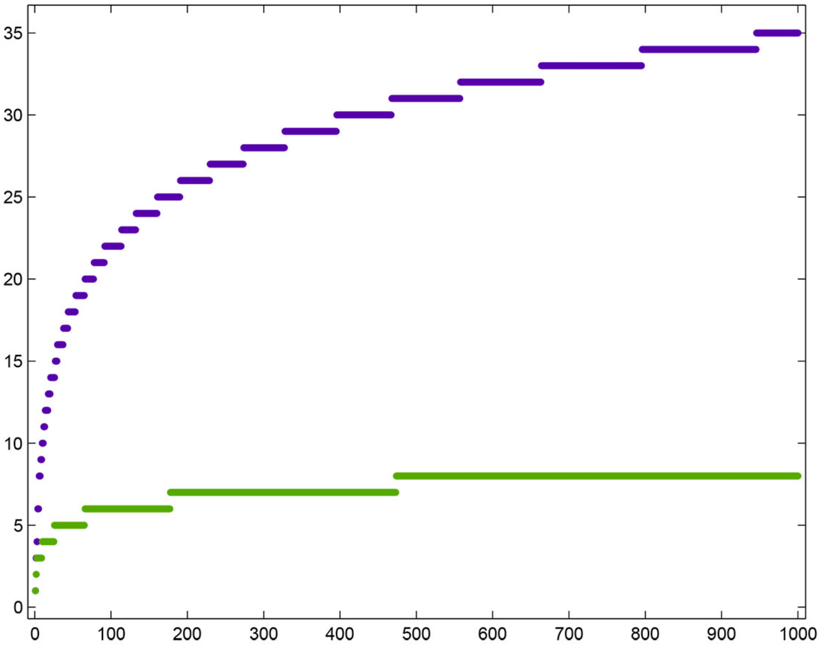

Figure 22

The lower step-function is $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\frac{i}{\varphi(i)}$.

The Mertens function has many such properties. See Cox and Ghosh [11] for details. Apparently there are usually analogues for $B(x)$.

A plot of $\sqrt{\sum_{i=1}^{x} B(\lfloor x/i \rfloor)B(\lfloor x/i \rfloor)}$, $\sqrt{\sum_{i=1}^{x} C(\lfloor x/i \rfloor)C(\lfloor x/i \rfloor)}$, $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)M(\lfloor x/i \rfloor)}$, $\Phi(x)/\sqrt{x(x+1)(2x+1)/6}$, and $|M(x)|$ versus $\sqrt{x}$ for $x = 1, 2, 3, \ldots, 5000$ is

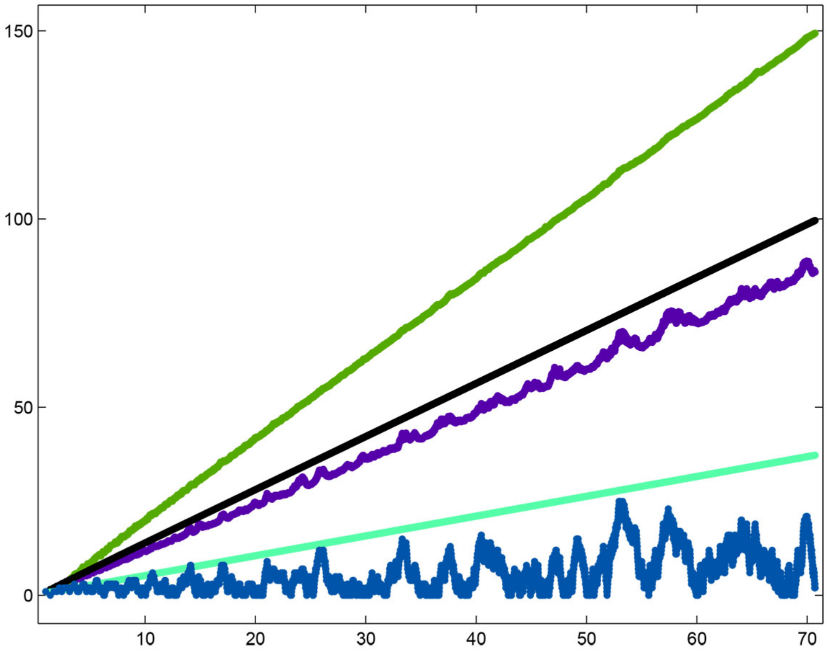

Figure 23

$\sqrt{\sum_{i=1}^{x} B(\lfloor x/i \rfloor) B(\lfloor x/i \rfloor)}$, is the top curve, $\sqrt{\sum_{i=1}^{x} C(\lfloor x/i \rfloor) C(\lfloor x/i \rfloor)}$ is the next-to-top curve and $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor) M(\lfloor x/i \rfloor)}$ is the following curve. For a linear least squares fit of $\sqrt{\sum_{i=1}^{x} B(\lfloor x/i \rfloor) B(\lfloor x/i \rfloor)}$ versus $\sqrt{x}$ for $x = 1, 2, 3, \ldots, 200000$, $p_1 = 2.126$ with a 95% confidence interval of (2.126, 2.126), $p_2 = -0.6108$ with a 95% confidence interval of $(-0.6266, -0.595)$, SSE=$2.891 \cdot 10^5$, R-squared=1, and RMSE=1.202.

In 1912, Littlewood [12] proved the following theorem

The Riemann hypothesis is equivalent to the statement that for every $\epsilon > 0$ the function $M(x) x^{-(1/2)-\epsilon}$ approaches zero as $x \to \infty$.

The Mertens conjecture (that $M(x)$ is bounded by $\pm\sqrt{x}$) was disproved by Odlyzko and te Riele [13]. For a linear least squares fit of $\sqrt{\sum_{i=1}^{x} C(\lfloor x/i \rfloor) C(\lfloor x/i \rfloor)}$ versus $\sqrt{x}$ for $x = 1, 2, 3, \ldots, 200000$, $p_1 = 1.409$ with a 95% confidence interval of (1.409, 1.409), $p_2 = -0.02723$ with a 95% confidence interval of $(-.02732, -0.02713)$, SSE=10.24, R-squared=1, and RMSE=0.007154. How much of the SSE that is due to rounding of floating-point arithmetic is unknown. In 2004, Kotnik and de Lune [14] found that $M(7766842813) = 50286$. In this case, $M(x)/\sqrt{x} =$

.570591. Multiplying the slope of the above linear least-squares fit (1.409) and $\sqrt{7766842813}/50286$ gives about 2.46937.

5. METHODS

compute w(n)

```
#include <math.h>
#include <stdio.h>
#include ”cheby3hk.h” // 300001 maximum
#include ”table5.h”
int mobius(unsigned int a, unsigned int *table, unsigned int tsize);
void main () {
unsigned int h,i,N,count,index;
int r;
unsigned int tsize=114155;
double sum,sum1,temp;
FILE *Outfp;
Outfp = fopen(”sortu.dat”,”w”);
if (Outfp==NULL)
    return;
index=1;
count=0;
sum1=0.0;
N=1;
for (h=1; h<=200000; h++) {
    sum=0.0;
    for (i=1; i<=N; i++) {
        if (N==(N/i)*i)
            sum=sum+(zero[i]-zero[i-1])*mobius(i,table,tsize);
        }
    r=(int)(sum/(log(2)-0.01));
    temp=(double)r*log(2);
    sum1=sum1+(double)r*log(2);
    if (h==(h/1000)*1000)
        printf(” %d %.16llf %d \n”,h,sum1/log(sum1),count);
    N=N+2;
    for (i=index; i<=tsize; i++) {
        if (table[i-1]<2*h)
```

```
            count=count+1;
        else {
            index=i;
            break;
            }
        }
    fprintf(Outfp,” %.16llf, %d \n”,sum1/log(sum1),count);
    }
fclose(Outfp);
return;
}
```

compute li(2x)

```
#include <math.h>
#include <stdio.h>
void main () {
unsigned int h,MAXN;
int j;
double temp,x,f;
FILE *Outfp;
Outfp = fopen(”sortz.dat”,”w”);
if (Outfp==NULL)
    return;
fprintf(Outfp,” %d, %.16llf, \n”,2,1.045164);
for (h=2; h<=75000; h++) {
    MAXN=h*2;
    f=-1e+99;
    x=log(MAXN);
    temp=x-10;
    if (temp<0.0)
        temp=-temp;
    if (temp>=12.0)
        goto L2;
    if (x==0.0)
        goto L4;
    temp=x;
    if (temp<0.0)
```

```
        temp=-temp;
    j=(int)(10.0+2.0*temp);
    f=1.0/(double)((j+1)*(j+1));
    L1: f=(f*(double)j*x+1.0)/(double)(j*j);
    j=j-1;
    if (j!=0.0)
        goto L1;
    temp=x;
    if (temp<0.0)
        temp=-temp;
    f=f*x+log(1.781072418*temp);
    goto L4;
    L2: temp=x;
    if (temp<0.0)
        temp=-temp;
    j=(int)(5.0+20.0/temp);
    f=x;
    L3: f=1.0/(1.0/f-1.0/(double)j)+x;
    j=j-1;
    if (j!=0)
        goto L3;
    f=exp(x)/f;
    L4: printf(” %d %.16llf \n”,MAXN,f);
    fprintf(Outfp,” %d, %.16llf, \n”,MAXN,f);
    }
fclose(Outfp);
return;
}
```

compute Dirichlet inverse

```
void dirinv(double *input, double *output, unsigned int MAXN) {
unsigned int i,N;
double sum;
output[0]=1.0/input[0];
for (N=2; N<=MAXN; N++) {
    sum=0.0;
    for (i=1; i<N; i++) {
```

```
        if (N==(N/i)*i)
            sum=sum+input[N/i-1]*output[i-1];
        }
    output[N-1]=-1.0/input[0]*sum;
    }
return;
}
```

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