Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 2 (2025), pp. 309-317

http://www.ripublication.com/gjpam.htm

More on Bol Loops Together with its Parastrophs

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Abstract

In this paper we investigate Bol loops and produce very important Results on them. We also proved here that the parastrophs of every Bol loop are Bol loops under some well-known conditions.

Keywords: Bol loop; Nucleus, Di-associative, Power-associative, Parastrophs, Exponent two, Automorphic inverse property loop.
Subject Classification Code: **20N05.**

Introduction

Loops satisfying identities (1) or (2) of definition :**1** under here are name after G. Bol, who showed in $[1]$ that the identities(1) and (2) correspond to certain configuration in 3-webs. Thus, a loop which satisfies both (1) and (2) must be a Moufang loop. Since there is a duality between right and left Bol loops, in the sequel we shall consider only right Bol loops and shall call them Bol loops omitting the word 'right'.It has been proven in $[2]$ and $[3]$ that 8 is the smallest possible order of a Bol loop which is not a group, and that there are exactly six non-isomorphic Bol loops of order 8. Thus the loop in example:**1** under here is one of these six smallest Bol loops. Still in $[2]$ R.P.Burn proved that for any prime *p*, Bol loop of order p, p^2 and *2p* are groups. But in our example:2 under here we constructed an example which of course is not true for p^3 as we see there the order been 2^3. Since Bol loops have applications in many different fields of mathematics see in $[4]$, we shall also give some important examples of Bol loops and their classifications. For further examples of finite Bol loops, one may also consult $[6]$ and $[11]$.

A comprehensive study of algebraic properties of Bol loop was first done by D.A. Robinson in [13]. In this work, we shall use many of its results.

Definition:1

A loop satisfying the identical relation

$$((x \cdot y) \cdot z) \cdot y = x \cdot ((y \cdot z) \cdot y) \qquad (1)$$

is called a right Bol loop. A loop satisfying the identical relation

$$\big(x\cdot(y\cdot x)\big)\cdot z = x\cdot(y\cdot(x\cdot z)) \qquad (2)$$

is called a left Bol loop.

Example:1
Let $(\mathcal{R},+,\cdot)$ be the ring of integer modulo 2, the algebraic structure $(B,*)$ such that $B=\mathcal{R}\times\mathcal{R}\times\mathcal{R}$ and the operation $(*)$ is defined by:

$$(a,b,c)*(m,n,p)=(a+m,b+n,c+p+bmn)$$

is a Bol loop of order 8.
One can easily observe that:

a-) $(B,*)$ satisfies identity (1)

b-) The identity element of $(B,*)$ and the inverse element of (a,b,c) are obvious.

c-) After we recall the following definitions of the left nucleus N_λ (middle nucleus N_μ,right nucleus N_ρ) of any groupoid $(B,\cdot)$:

$$N_\lambda=\{a\in B, a\cdot(x\cdot y)=(a\cdot x)\cdot y,\ x,y\in B\}$$
$$N_\mu=\{a\in B,\ (x\cdot a)\cdot y=x\cdot(a\cdot y),\ x,y\in B\}$$
$$N_\rho=\{a\in B,(x\cdot y)\cdot a=x\cdot(y\cdot a),\ x,y\in B\}$$
$$N=N_\lambda\cap N_\mu\cap N_\rho$$

We observed that:

$c_1-)$ for every $x\in B, x^2\in N_\rho=N$

$c_2-)$ $(1,0,0)\in N_\lambda$ but $(1,0,0)\notin N_\rho$

d-) One may also use the element $a=(1,1,0)$; $b=(0,1,0)$ and $c=(1,1,1)$ of *B* to show that the Moufang identity does not hold.

Theorem:1
If $(G,\cdot)$ is a Bol loop with identity element 1, then the following hold:
(i) $(G,\cdot)$ has the right inverse property

$$(x\cdot y)\cdot y^\rho = x \qquad (3)$$

where y^ρ is defined by: $y\cdot y^\rho=1$ for all $y\in G$.
(ii) The left and right element y^λ and $y^\rho$of any $y\in G$, coincide i.e: $y^\lambda=y^\rho=y^{-1}$
(iii) $(G,\cdot)$ satisfies the right alternative law

$$(x\cdot y)\cdot y=x\cdot(y\cdot y) \qquad (4)$$

for all $x,y\in G$.

Proof:
$(\boldsymbol{i})$ From identity (1) with $z=y^\rho$, we have
$\big((x\cdot y)\cdot y^\rho\big)\cdot y = x\cdot\big((y\cdot y^\rho)\cdot y\big)=x\cdot y,\quad \big((x\cdot y)\cdot y^\rho\big)\cdot y=x\cdot y$ or $(x\cdot y)\cdot y^\rho=x$, which is (3).

(ii) Setting $x = y^{\lambda}$ such that $y^{\lambda} \cdot y = 1$ in (3), we have $\left(y^{\lambda} \cdot y\right) \cdot y^{\rho} = y^{\lambda}$, or $y^{\rho} = y^{\lambda}$. We denote $y^{\rho} = y^{\lambda}$ by y^{-1} and we have $y^{-1} \cdot y = y \cdot y^{-1} = 1$.

(iii) The right alternative law $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ follows directly from (1) with $z = 1$.

■

Definition: 2

A quasigroup $(G,\cdot)$ is power-associative means that $(\langle a\rangle,\cdot)$ is associative for each $a \in G$.

Defintion:3

A quasigroup $(G,\cdot)$ is di-associative means that $(\langle a, b\rangle,\cdot)$ is associative for all $a, b \in G$ with a, b not necessary distinct.

Any quasigroup $(G,\cdot)$ which is di-associative is automatically power-associative because $\{a\} = \{a, a\}$ for all $a \in G$. In $[8]$, we have there an example of quasigroups which are power-associative without being di-associative.

As we saw in example:1, a Bol loop does not have to be di-associative. But we shall prove that every Bol loop $(G,\cdot)$ is power-associative. In our case, it suffices to show that for all integers n and m and for all $y \in G$, $y^{m} \cdot y^{n} = y^{m+n}$, where the powers of y are defined as follows: $y^{0} = 1$, $y^{n} = y^{n-1} \cdot y$, and $y^{-n} = (y^{-1})^{n}$.

Lemma:1

If $(G,\cdot)$ is a Bol loop, then

$$x \cdot y^{n} = \left(x \cdot y^{n-1}\right) \cdot y = (x \cdot y) \cdot y^{n-1} \tag{5}$$

for all $x, y \in G$ and all integers n.

Proof:

Since obviously identity (5) holds for $n = 1$, we can assume that identity (5) is holdind for any $n \le k$. Then

$$x \cdot y^{k} = \left(x \cdot y^{k-1}\right) \cdot y = (x \cdot y) \cdot y^{k-1}. \tag{6}$$

Setting $x = 1$ in (6), we also have

$$y^{n} = \left(y^{n-1}\right) \cdot y = y \cdot y^{n-1} \tag{7}$$

Let us now prove that (5) holds for $n = k + 1$. Using relations (1),(6) and (7), we have

$$x \cdot y^{k+1} = x \cdot (y^{k} \cdot y) = x \cdot ((y^{k-1} \cdot y) \cdot y) = x \cdot \left((y \cdot y^{k-1}) \cdot y\right) = \left((x \cdot y) \cdot y^{k-1}\right) \cdot y = \left((x \cdot y^{k}) \cdot y\right)$$

and also

$$x \cdot y^{k+1} = x \cdot (y^{k} \cdot y) = x \cdot \left((y \cdot y^{k-1}) \cdot y\right) = \left((x \cdot y) \cdot y^{k-1}\right) \cdot y = (x \cdot y) \cdot y^{k}.$$

Thus

$$x \cdot y^{k+1} = (x \cdot y^{k}) \cdot y = (x \cdot y) \cdot y^{k} \tag{8}$$

Which is relation (5) with $n = k + 1$. Thus, by induction, (5) holds for all positive n. Identity (5) holds certainly for $n = 0$. Replacing y by y^{-1} in (8), we get

$x \cdot (y^{-1})^{k+1} = x \cdot (y^{-1})^k \cdot (y^{-1}) = (x \cdot y^{-1}) \cdot (y^{-1})^k$, or

$x \cdot y^{-(k+1)} = (x \cdot y^{-k}) \cdot y^{-1} = (x \cdot y^{-1}) \cdot y^{-k}$.

Replacing x by $x \cdot y$, we write

$(x \cdot y) \cdot y^{-(k+1)} = \left((x \cdot y) \cdot y^{-k}\right) \cdot y^{-1} = ((x \cdot y) \cdot y^{-1}) \cdot y^{-k} = x \cdot y^{-k}$, or

$x \cdot y^{-k} = (x \cdot y) \cdot y^{-k-1}$. Multiplying both sides of

$(x \cdot y) \cdot y^{-(k+1)} = (x \cdot y^{-k}) \cdot y^{-1}$ on the right by y, we also get

$x \cdot y^{-k} = (x \cdot y^{-k-1}) \cdot y$. This complete the proof of our lemma. ■

Theorem: A

If $(G,\cdot)$ is a Bol loop then

$$(x \cdot y^m) \cdot y^n = x \cdot y^{m+n} \qquad (9)$$

for all $x, y \in G$ and all integers m and n.

Proof: From lemma:1, we know that identity (9) holds for $n = 1$. We now assume that identity (9) holds for positive $n \leq k$.

$$(x \cdot y^m) \cdot y^k = x \cdot y^{m+k} \qquad (10)$$

Then by lemma:1 and identity (10) we obtain:

$$x \cdot y^{m+k+1} = (x \cdot y^{m+k}) \cdot y = \left((x \cdot y^m) \cdot y^k\right) \cdot y \qquad (11)$$

But from identity (5) with $x \cdot y^m$ instead of x, we have:

$$\left((x \cdot y^m) \cdot y^k\right) \cdot y = (x \cdot y^m) \cdot y^{k+1} \qquad (12)$$

Combining identities (11) and (12), we get:

$$x \cdot y^{m+k+1} = (x \cdot y^m) \cdot y^{k+1}$$

Thus identity (9) holds for $n = k + 1$ and, by induction, for all positive integers n, identity (9) obviously holds for $n = 0$. To get the desired result for negative integers one can replace m by $m - n$ in identity (9) we then have:

$(x \cdot y^{m-n}) \cdot y^n = x \cdot y^m$.

Then we have $x \cdot y^{m-n} = (x \cdot y^m) \cdot (y^n)^{-1} = (x \cdot y^m) \cdot y^{-n}$

Using $x = 1$ in identity (9), we obtain the following: ■

Corollary:1

Bol loops are power-associative

Since Bol loops are less restricted than Moufang loops, it seems to me natural to expect that some of the properties of Moufang loops are not shared by Bol loops. We saw this in the case of di-associativity. Another missing property pertains to the nuclei. In the literature more precisely in $[12]$ it was shown that the nuclei of

Moufang loops coincide. That is $N_\lambda = N_\mu = N_\rho = N$. This is not true for Bol loops. As le counterexample we can use the Bol loops of example:1, here we have $N_\rho \neq N_\lambda$.

Definitin:4
Let $(G,\cdot)$ be a quasigroup, from this quasigroup one can define five other quasigroups denoted $(G,/), (G,\backslash), (G,*), (G,\circ)$ and $(G,\odot)$ and defined by:

1-) $a \odot b = c$ means that $b \cdot a = c$.
2-) $a/b = c$ means that $a = c \cdot b$ i.e for loops $a \cdot b^{-1} = c$
3-) $a \backslash b = c$ means that $b = a \cdot c$ i.e: for loops $a^{-1} \cdot b = c$
4-) $a * b = c$ means that $a = b \cdot c$ i.e: for loops $b^{-1} \cdot a = c$
5-) $a \circ b = c$ means that $b = c \cdot a$ i.e: for loops $b \cdot a^{-1} = c$.

The five newly defined quasigroups are called the parastrophs of $(G,\cdot)$. Some recent work done on parastrophs of quasigroups and some loops include Mengue Mengue D J [10], Sokhatski F N [15],Duplak J [5], Gushan V. V and Sokhatski F N [7], Sade A [14] and Jaiyeola Temitope [8] .

Parastrophs of Bol loops.
In this section we assume that the quasigroup $(G,\cdot)$ is a Bol loop and we are going to show that each of its parastrophs is also a Bol loop under conditions.
Definition:5
A loop $(G,\cdot)$ is called an automorphic inverse property loop (AIPL) if and only if it obeys the identity:

$$(x \cdot y)^\rho = x^\rho \cdot y^\rho \text{ or } (x \cdot y)^\lambda = x^\lambda \cdot y^\lambda \tag{13}$$

for all $x, y \in G$.

Lemma:2
If $(G,\cdot)$ is a Bol loop and $(G,\odot)$ one of its parastroph defined in the above definition:4 then $(G,\odot)$ is also a Bol loop.

Proof:

Recall that $(G,\odot)$ is a quaigroup by definition in Pflufelder,H [12] and in [9] Kunen Kenneth showed that $(G,\odot)$ has an identity element hence it is a loop. To show that $(G,\odot)$ is a Bol loop, we only need to show that it satisfies the identity (1) of definition:1. Recall that $a \odot b = c$ means that $b \cdot a = c$. That is $a \odot b = b \cdot a$ hence we have: For all $a, b, c \in G$,

$$\begin{aligned}((a \odot b) \odot c) \odot b &= ((b \cdot a) \odot c) \odot b \\ &= (c \cdot (b \cdot a)) \odot b \\ &= b \cdot (c \cdot (b \cdot a))\end{aligned}$$

and

$$a\odot\big((b\odot c)\odot b\big)=a\odot\big((c\cdot b)\odot b\big)$$
$$=a\odot\big(b\cdot(c\cdot b)\big)$$
$$=\big(b\cdot(c\cdot b)\big)\cdot a$$

By hypothesis $b\cdot\big(c\cdot(b\cdot a)\big)=\big(b\cdot(c\cdot b)\big)\cdot a$ it then follows that
$\big((a\odot b)\odot c\big)\odot b=a\odot\big((b\odot c)\odot b\big)$ hence $(G,\odot)$ is a Bol loop. ■

Lemma:3

If $(G,\cdot)$ is a Bol loop of exponent two with automorphic inverse property then the it parastroph $(G,/)$ is a Bol loop.

Proof:

We only need to show that $(G,/)$ satisfied identity

$$\big((b\,/\,a)\,/\,c\big)/a=\big((b\cdot a^{-1})\,/\,c\big)\,/\,a \qquad \text{(by the definition)}$$
$$=\big((b\cdot a^{-1})\cdot c^{-1}\big)\,/\,a$$
$$=\big((b\cdot a^{-1})\cdot c^{-1}\big)\cdot a^{-1}$$
$$=\big((b\cdot a)\cdot c\big)\cdot a \qquad \text{(since } x^2=e,\forall x\in G)$$

and

$$b/((a/c)/a)=b\,/\,\big((a\cdot c^{-1})\,/\,a\big)$$
$$=b\,/\,\big((a\cdot c^{-1})\cdot a^{-1}\big)$$
$$=b\cdot\big((a\cdot c^{-1})\cdot a^{-1}\big)^{-1}$$
$$=b\cdot\big((a\cdot c^{-1})^{-1}\cdot a\big)$$

(Automorphic inverse property)

$$=b\cdot\big((a^{-1}\cdot c)\cdot a\big)$$
$$=b\cdot\big((a\cdot c)\cdot a\big) \qquad \text{(because by hypothesis } a^{-1}=a)$$

hence $\big((b\,/\,a)\,/\,c\big)/a=b/((a/c)/a)$ thus $(G,/)$ satisfied identity (1) ■

Lemma:4

If $(G,\cdot)$ is a Bol loop of exponent two with automorphic inverse property then the it parastroph $(G,*)$ is a Bol loop.

Proof:

$$b*\big(((a*c))*a\big)=b*\big((c^{-1}\cdot a)*a\big)$$
$$=b*\big(a^{-1}\cdot(c^{-1}\cdot a)\big)$$
$$=\big(a^{-1}\cdot(c^{-1}\cdot a)\big)^{-1}\cdot b$$

$$
\begin{aligned}
&= \left(a\cdot(c^{-1}\cdot a)^{-1}\right)\cdot b\\
&= \left(a\cdot(c\cdot a^{-1})\right)\cdot b\\
&= \left(a\cdot(c\cdot a)\right)\cdot b
\end{aligned}
$$

on the order hand we have

$$
\begin{aligned}
\left((b*a)*c\right)*a &= \left((a^{-1}\cdot b)*c\right)*a\\
&= \left(c^{-1}\cdot(a^{-1}\cdot b)\right)*a\\
&= a^{-1}\cdot\left(c^{-1}\cdot(a^{-1}\cdot b)\right)\\
&= a\cdot\left(c\cdot(a\cdot b)\right)
\end{aligned}
$$

hence

$$b*\left(\left((a*c)\right)*a\right) = \left(a\cdot(c\cdot a)\right)\cdot b = a\cdot\left(c\cdot(a\cdot b)\right) = \left((b*a)*c\right)*a$$

thus $(G,*)$ satisfy the Bol loop identity.■

Lemma:5

If $(G,\cdot)$ is a Bol loop of exponent two with automorphic inverse property then the it parastroph $(G,\backslash)$ is a Bol loop.

Proof:

$$
\begin{aligned}
\left((b\backslash a)\backslash c\right)\backslash a &= \left((b^{-1}\cdot a)\backslash c\right)\backslash a\\
&= \left((b^{-1}\cdot a)^{-1}\cdot c\right)\backslash a\\
&= \left((b^{-1}\cdot a)^{-1}\cdot c\right)^{-1}\cdot a\\
&= \left((b^{-1}\cdot a)\cdot c^{-1}\right)\cdot a\\
&= \left((b\cdot a)\cdot c\right)\cdot a
\end{aligned}
$$

on the other hand we have

$$
\begin{aligned}
b\backslash\left((a\backslash c)\backslash a\right) &= b\backslash\left((a^{-1}\cdot c)\backslash a\right)\\
&= b\backslash\left((a^{-1}\cdot c)^{-1}\cdot a\right)\\
\text{(Automorphic inverse property)}\quad &= b\backslash\left((a\cdot c^{-1})\cdot a\right)\\
&= b^{-1}\cdot\left((a\cdot c^{-1})\cdot a\right)\\
&= b\cdot\left((a\cdot c)\cdot a\right)
\end{aligned}
$$

(loop of exponent two)

Using the fact that $(G,\cdot)$ is a Bol loop we have: $\left((b\backslash a)\backslash c\right)\backslash a = b\backslash\left((a\backslash c)\backslash a\right)$

■

Lemma:6

If $(G,\cdot)$ is a Bol loop of exponent two with automorphic inverse property then the it parastroph $(G,\circ)$ is a Bol loop.

Proof:

$$
\begin{aligned}
\left((b\circ a)\circ c\right)\circ a &= \left((a\cdot b^{-1})\circ c\right)\circ a\\
&= \left(c^{-1}\cdot(a\cdot b^{-1})\right)\circ a
\end{aligned}
$$

$$= a^{-1} \cdot \left(c^{-1} \cdot (a \cdot b^{-1})\right)$$
$$= a \cdot \left(c \cdot (a \cdot b)\right)$$

and we also have

$$b \circ \left((a \circ c) \circ a\right) = b \circ \left((c \cdot a^{-1}) \circ a\right)$$

(By definition $= b \circ \left(a \cdot (c \cdot a^{-1})^{-1}\right)$

$$= b \circ \left(a \cdot (c^{-1} \cdot a)\right) \qquad \text{(AIPL)}$$
$$= \left(a \cdot (c^{-1} \cdot a)\right) \cdot b^{-1}$$
$$= \left(a \cdot (c \cdot a)\right) \cdot b \qquad \text{(Exponent two)}$$

Hence by our hypothesis the loop $(G,\circ)$ satisfy the Bol loop identity so it is a Bol loop. ■

Theorem:B

If $(G,\cdot)$ is a Bol loop of exponent with the automorphic inverse property then its parastrophs are Bol loops.

Proof: See proofs of lemma:2;3;4;5 and 6.■

Example:2

Let $G = \{1,2,3,4,5,6,7,8\}$ and let $(G,\cdot)$ be the loop with the following operation table

·	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	8	7	5	6
4	4	3	2	1	7	8	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	4	3	1	2
8	8	7	6	5	3	4	2	1

In $(G,\cdot)$, $x^2 = 1$ for every $x \in G$. It is a Bol loops of exponent two and hence must have the automorphic inverse property. This particular example is doing the work.

References

[1] G.Bol, *Gewebe and Gruppen*, Math Ann. 114 (1937), 414-431.

[2] R.P.Burn, *Finite Bol loops*, Math. Proc. Camb. Phil. Soc. 84 (1978), 377-385. MR.58 (1979) #11194

[3] R.P.Burn, *Finite Bol loops II*, Math. Proc. Camb. Phil. Soc. 89 (1981), 445-455. MR.82g (1982) ,#20109

[4] O. Chein, H. O. Pflugfelder, J. D. H. Smith, *Quasigroups and loops: Theory and Applications*, Heldermann Verlap, Berlin 1990.

[5] J. Duplak, *A parastrophic Equivalence In Quasgroups,* Quasigroups and Related System, (2000), 7-14.

[6] L.Fook, *Moufang loops of order p,* Nanta Math.7 (1974). No.2, 33-34.MR.51 (1976)#5826

[7] V.V. Guhan and K. K. Sokhatski, *A representation of Parastrophs of loops and quasigroups,* Journal of discrete Math. And Applications (2004), 535-542

[8] T.G. Jaiyeola, *Parastrophic invariance of Smarandache Quasigroups,* Scientia Magna Journal,2, 1, 1-8.

[9] Kenneth Kunen, Quasigroups, Loops, and associative law, J. Algebra, 185(1) (1996), 194-204.

[10] D.J. Mengue Mengue, *Parastrophs of Quasigroups,* Jour. Of Mathematical Sciences Vol. 21, No.3 (2010). 339-344.

[11] H. Niederreiter, K. H. Robinson, *Bol loops of order pq,* Math. Proc. Camb. Phil. Soc. 89 (1981), 241-256. MR.82c:20121

[12] H. O. Pflugfelder, *Quasigroups and loops Intrduction,* Berlin Sigma Series In Pure Mathematics, (1990).

[13] D. A. Robinson, *Bol loops,* Trans. Amer. Math. Soc. 123 (1966), 341-354. MR. 33 (1967) #2755.

[14] A. Sade, *Quasigroupes Parastrophiques,* Math. Nachr. 20 (2009), 73-106

[15] F. N. Sokhatski, *On Isotopes of A Group,* Journal Ukrainian Math. Journal (1995), 1585-1590.