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The Neighbor Degree Sum - Distance Indices on Cactus Families of Graph

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Abstract

A graph can be recognized by a numeric number, a polynomial or a matrix that represents the whole graph and these representations are aimed to be uniquely defined for the graph. The whole structure of the graph is characterized by numeric quantity called topological index. In this article, we initiate the study on topological index based on neighbor degree sum - distance of a graph G are named as the Neighbor Schultz index $NDD(G)$, the Neighbor Gutman index $NZZ(G)$ and the Neighbor Wiener-Albertson index $NWA(G)$. Here, we computed these indices on certain cactus families of graphs and did graphical interpetation, which are useful to theoretical chemistry to measure the topological properties of molecules.

Key words and phrases : Neighbor schultz index , Neighbor Gutman index, Neighbor Wiener Albertson index and Cactus families of graph

2020 Mathematics Subject Classification: 05C07, 05C12, 05C31

1. INTRODUCTION

Throughout this paper, we considered simple, finite, connected, undirected, without loops or multiple edge graphs. Assume that G is a connected graph with $|V(G)| = p$ vertices and $|E(G)| = q$ edges. The maximum and minimum degree of the graphs represented by $\Delta(G)$ and $\delta(G)$ respectively. For a vertex $v \in V(G)$ and $d(v)$ denotes the degree of v. For any $u, v \in V(G)$, the distance $d(u, v)$ is defined as the length of shortest path between u and v. The eccentricity of the vertex v is the distance to a vertex farthest from v. The minimum eccentricity is called radius, $r(G)$ and maximum eccentricity is called diameter, $diam(G)$. For basic graph-theoretical terminology and notation we follow[6].

The Wiener number $W(G)$, also known as the Wiener index in chemical or mathematical chemistry literature, is one of the oldest and most well-studied distance-based graph invariants associated with a connected graph G. It is defined [15] as the sum of distances over all unordered vertex pairs in G, namely,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v)$$

Dobrynin and Kochetova [5] and Gutman [8] in independently proposed a vertex - degree weighted version of Wiener index called degree distance, which is defined for a connected graph G as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]\, d(u, v)$$

and

$$ZZ(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u)d(v)]\, d(u, v).$$

where $d(u)$ is the degree of the vertex u in G and $d(u, v)$ is the shortest distance between the vertices u and v.
In order to improve the structure discriminating ability and maintain certain irregularity measuring ability, Albertson [2] and Abdo et al.,[1] defined the following irregular indices as follows

$$Alb(G) = irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)|$$

$$irr_t(G) = \sum_{(u,v) \subseteq V(G)} |d(u) - d(v)|.$$

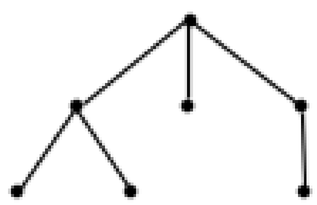

Figure 1: The graph with different irregularities.

From the above figure 1, we get $Irr(G) = 8$ and $Irr_t(G) = 21$. We can say

$Irr(G) \neq Irr_t(G)$.
The Neighbourhood Zagreb index of a graph G was introduced by Mondal et al., [11] and is defined as

$$M_N = \sum_{u \in V(G))} s(u)^2.$$

T V Asha and B Chaluvaraju [3] introduced new topological indices based on neighbor degree sum distance based indices of a graph G are denoted as Neighbor Schultz index $NDD(G)$, Neighbor Gutman index $NZZ(G)$ and Neighbor Wiener-Albertson index $NWA(G)$. These are defined as

$$NDD(G) = \sum_{(u,v) \subseteq V(G)} [s(u) + s(v)]\, d(u,v)$$

$$NZZ(G) = \sum_{(u,v) \subseteq V(G)} [s(u)\, s(v)]\, d(u,v)$$

$$NWA(G) = \sum_{(u,v) \subseteq V(G)} |s(u) - s(v)|\, d(u,v).$$

Where $s(u) = \sum_{v \in N(u)} d(v)$ is the degree sum of neighbor vertex set $N(G)$ and $d(u, v)$ is the shortest distance between the vertices u and v. $NWA(G) = 0$ iff G is regular and $NWA(G) \neq 0$ if G is not a regular graph.

We computed these indices on certain families of graphs along with some bounds and charactersization [3]. Further we exatend this concept to certain cactus families of graphs.

CACTUS FAMILIES OF A GRAPH

Cactus chains are a type of basic linear polymer that we look at in this paper. Cactus graphs were first known as Husimi trees, and they first appeared in the scientific literature some sixty years ago, in publications by Husimi and Riddell dealing with cluster integrals in the theory of condensation in statistical mechanics [7, 9, 13]. A cactus is called triangular if all of its blocks are triangles. If each C_3 of a triangular cactus has at most two cut - vertices and each cut - vertex is shared by precisely two triangles, the cactus is said to be chain triangular. When we replace triangles in G with C_4, we get square cactus, which has a block of C_4. When we replace the triangles in G with C_6, we get a hexagonal cactus whose block is C_6. It's important to note that the internal squares' connections to their neighbours may vary. We call such a square an ortho - square if its cut - vertices are adjacent, and a para - square if the cut - vertices are not adjacent. The way the internal hexagonal connects to its neighbours may be different. We call a hexagonal Meta hexagonal if its cut - vertices are adjacent, and a para-hexagonal if the cut - vertices are not contiguous. The length z of the chain triangular, para chain square, ortho chain square, para chain hexagonal and Meta chain hexagonal cactus graph are denoted by T_n, P_n, O_n, L_n and M_n, respectively. where n is the number of cycles present in the graph. For more details, we refer to [12, 14].

Theorem 1.1. *Let T_n be the chain triangular cactus graph with $n \geq 3$. Then*

$$NDD(T_n) = \frac{40n^3 + 60n^2 - 4n - 24}{3},$$
$$NZZ(T_n) = \frac{200n^3 - 186n^2 + 706n - 240}{3}, \text{ and}$$
$$NWA(T_n) = \frac{4n^3 + 36n^2 - 76n}{3}.$$

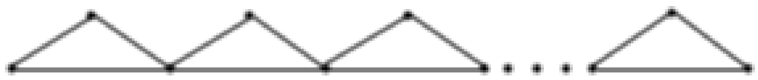

Figure 2: The chain triangular cactus graphs T_n.

Proof. Let u and v be any two vertices of T_n and $d(u, v) = d$. We have

(i) If $d(u, v) = 1$, then there exist 2 pair of verices with $s(u) = s(v) = 6$, 4 pair of vertices with $s(u) = 6, s(v) = 10$, 2 pair of vertices with $s(u) = 8, s(v) = 10$, $2(n - 3)$ pair of vertices with $s(u) = 8, s(v) = 12$, 2 pair of vertices with $s(u) = 10, s(v) = 12$ and $n - 4$ pair of vertices with $s(u) = 12, s(v) = 12$.

(ii) If $d(u, v) = k$, such that $2 \leq k \leq n - 1$, if $n > k$ then there exist 4 pair of verices with $s(u) = 6, s(v) = 8$, if $n = k + 1$ then there exist 4 pair of vertices with $s(u) = 6, s(v) = 10$, if $n > k + 1$ there exist 4 pair of vertices with $s(u) = 6, s(v) = 12$, $n - k - 1$ pair of vertices with $s(u) = 8, s(v) = 8$, if $n > k+1$ there exists 2 pair of vertices with $s(u) = 8, s(v) = 10$, $2(n-k-2)$ pair of vertices with $s(u) = 8, s(v) = 12$, if $n = k + 2$ there exist 1 pair of vertices with $s(u) = 10, s(v) = 10$, if $n > k + 2$ there exists 2 pair of vertices with $s(u) = 10, s(v) = 12$ and $(n - k - 3)$ pair of vertice with $s(u) = 12, s(v) = 12$.

(iii) If $d(u, v) = n$, then there exist 4 pair of verices with $s(u) = s(v) = 6$.

From the definition of $NDD(G)$, we have

$$\begin{aligned} NDD(T_n) &= \sum_{u,v \in V(G)} [s(u) + s(v)]d(u, v) \\ &= \sum_{d(u,v)=1} [s(u) + s(v)]d(u, v) + \sum_{d(u,v)=k,k=2}^{n-1} [s(u) + s(v)]d(u, v) \\ &+ \sum_{d(u,v)=n} [s(u) + s(v)]d(u, v) \end{aligned}$$

$$NDD(T_n) = \frac{40n^3 + 60n^2 - 4n - 24}{3}.$$

From the definition of $NZZ(G)$, we have

$$\begin{aligned} NZZ(T_n) &= \sum_{u,v \in V(G)} [s(u) \cdot s(v)]d(u,v) \\ &= \sum_{d(u,v)=1} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=k,k=2}^{n-1} [s(u) \cdot s(v)]d(u,v) \\ &+ \sum_{d(u,v)=n} [s(u) \cdot s(v)]d(u,v) \\ NZZ(T_n) &= \frac{200n^3 - 186n^2 + 706n - 240}{3}. \end{aligned}$$

From the definition of $NWA(G)$, we have

$$\begin{aligned} NWA(T_n) &= \sum_{u,v \in V(G)} |s(u) - s(v)|d(u,v) \\ &= \sum_{d(u,v)=1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=k,k=2}^{n-1} |s(u) - s(v|]d(u,v) \\ &+ \sum_{d(u,v)=n} |s(u) - s(v)|d(u,v) \\ NWA(T_n) &= \frac{4n^3 + 36n^2 - 76n}{3}. \end{aligned}$$

□

Observation 1.1. *In view of the above theorem, we present the graphical representation of the Neighbor degree distance based indices of chain triangular cactus graph T_n; $n \geq 3$ with respect to $NWA(T_n) \leq NDD(T_n) \leq NZZ(T_n)$ as follows.*

Theorem 1.2. *Let P_n be the para chain square cactus graph with $n \geq 3$. Then*

$$\begin{aligned} NDD(P_n) &= \frac{112n^3 + 96n^2 + 440n - 984}{3}, \\ NZZ(P_n) &= \frac{448n^3 - 192n^2 + 2144n - 3984}{3}, \text{ and} \\ NWA(P_n) &= 48n^2 - 56n - 8. \end{aligned}$$

Proof. Let u and v be any two vertices of P_n and $d(u,v) = d$. Then

(i) If $d(u,v) = 1$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$, 4 pair of vertices with $s(u) = 6, s(v) = 8$ and $4(n-2)$ pair of vertices with $s(u) = s(v) = 8$.

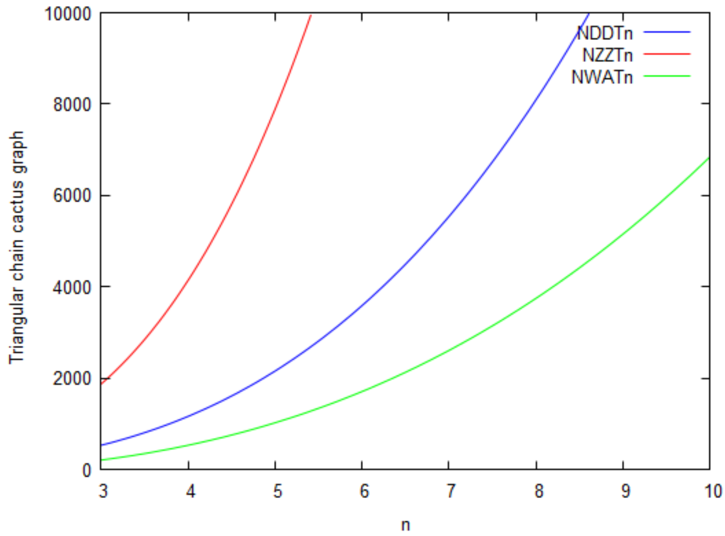

Figure 3: Graphical representation of T_n.

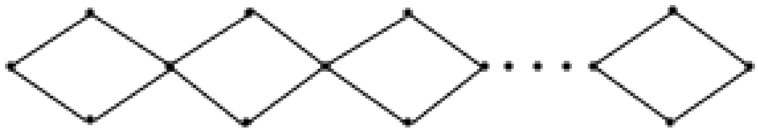

Figure 4: The para chain square cactus graphs P_n

(ii) If $d(u,v) = 2$, then there exist 2 pair of verices with $s(u) = s(v) = 6$, 2 pair of vertices with $s(u) = 4, s(v) = 8$ and 4 pair of vertices with $s(u) = 6, s(v) = 8$ and $2(3m-8)$ pair of vertices with $s(u) = 8, s(v) = 8$.

(iii) If $d(u,v) = k$ and $k = 2s(2 \leq s \leq n-1)$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 8$, 4 pair of verices with $s(u) = 6, s(v) = 8$ and $3n - 2s - 7$ pair of verices with $s(u) = s(v) = 8$.

(iv) If $d(u,v) = k$ and $k = 2s+1(1 \leq s \leq n-2)$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 8$, 4 pair of verices with $s(u) = 6, s(v) = 8$ and $4(n-s-2)$ pair of verices with $s(u) = s(v) = 8$.

(v) If $d(u,v) = 2n-1$, then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$.

(vi) If $d(u,v) = 2n$, then there exist one pair of verices with $s(u) = s(v) = 4$.

By the definition of $NDD(G)$, we have

$$NDD(P_n) = \sum_{u,v \in V(G)} [s(u) + s(v)]d(u,v)$$

$$= \sum_{d(u,v)=1} [s(u) + s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u) + s(v)]d(u,v)$$
$$+ \sum_{d(u,v)=k=2s,s=2}^{n-1} [s(u) + s(v)]d(u,v) + \sum_{d(u,v)=k=2s+1,s=1}^{n-2} [s(u) + s(v)]d(u,v)$$
$$+ \sum_{d(u,v)=2n-1} [s(u) + s(v)]d(u,v) + + \sum_{d(u,v)=2n} [s(u) + s(v)]d(u,v)$$
$$NDD(P_n) = \frac{112n^3 + 96n^2 + 440n - 984}{3}.$$

By the definition of $NZZ(G)$, we have

$$NZZ(P_n = \sum_{u,v \in V(G)} [s(u)s(v)]d(u,v)$$
$$= \sum_{d(u,v)=1} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u)s(v)]d(u,v)$$
$$+ \sum_{d(u,v)=k=2s,s=2}^{n-1} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=k=2s+1,s=1}^{n-2} [s(u)s(v)]d(u,v)$$
$$+ \sum_{d(u,v)=2n-1} [s(u) \cdot s(v)]d(u,v) + + \sum_{d(u,v)=2n} [s(u) \cdot s(v)]d(u,v)$$
$$NZZ(P_n) = \frac{448n^3 - 192n^2 + 2144n - 3984}{3}.$$

By the definition of $NWA(G)$, we have

$$NWA(P_n) = \sum_{u,v \in V(G)} |s(u) - s(v)|d(u,v)$$
$$= \sum_{d(u,v)=1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=2} |s(u) - s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=k=2s,s=2}^{n-1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=k=2s+1,s=1}^{n-2} |s(u) - s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=2n-1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=2n} |s(u) - s(v)|d(u,v)$$
$$NWA(P_n) = 48n^2 - 56n - 8$$

□

Observation 1.2. *In view of the above theorem, we present the graphical representation of Neighbor degree distance based indices of para chain square cactus graphs P_n ; $n \geq 3$ with respect to $NWA(P_n) \leq NDD(P_n) \leq NZZ(P_n)$ as follows.*

Theorem 1.3. *Let O_n be the ortho-chain square cactus graph with $n > 4$. Then*

$$NDD(O_n) = 24n^3 + 82n^2 - 170n + 524,$$

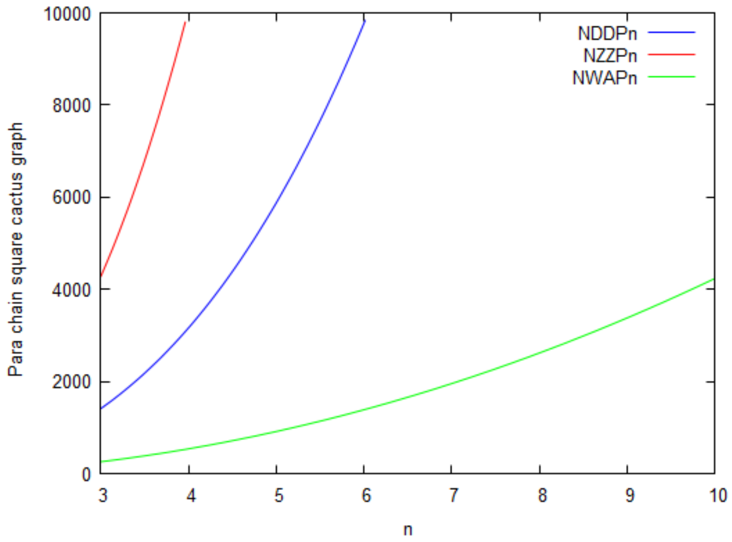

Figure 5: Graphical representation of P_n.

$$NZZ(O_n) = 84n^3 + 232n^2 - 512n + 1328, \textit{and}$$
$$NWA(O_n) = 4n^3 + 10n^2 - 34n + 140.$$

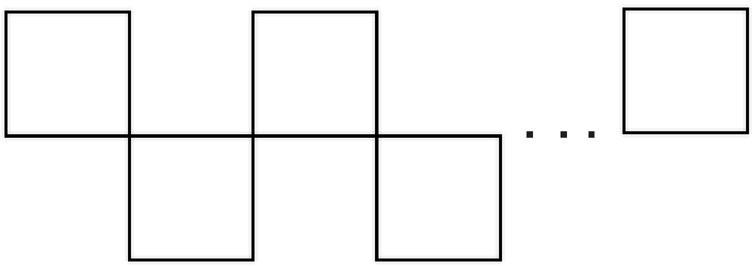

Figure 6: The ortho-chain square cactus graphs O_n

Proof. Let u and v be any two vertices of O_n and $d(u, v) = d$. We have the following cases:

(i) If $d(u, v) = 1$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$, $n-2$ pair of vertices with $s(u) = 6, s(v) = 6$, $2n$ pair of vertices with $s(u) = 6, s(v) = 10$ and $n - 2$ pair of vertices with $s(u) = s(v) = 10$.

(ii) If $d(u, v) = 2$, then there exist 2 pair of verices with $s(u) = 4, s(v) = 10$, $n + 3$ pair of vertices with $s(u) = s(v) = 6$, $2(2n - 3)$ pair of vertices with $s(u) = 6, s(v) = 10$ and $n - 3$ pair of vertices with $s(u) = s(v) = 10$.

(iii) If $d(u, v) = 3$ then there exist 2 pair of verices with $s(u) = 4, s(v) = 10$, $n + 6$ pair of verices with $s(u) = s(v) = 6$, $2(2n - 5)$ pair of verices with $s(u) = 6, s(v) = 10$ and $n - 4$ pair of vertices with $s(u) = s(v) = 10$.

(iv) If $d(u, v) = k$ $(4 \leq k \leq n - 1)$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$, 2 pair of verices with $s(u) = 4, s(v) = 10$, $4(n - k) + 8$ pair of verices with $s(u) = s(v) = 6$ and $6(n - k)$ pair of vertices with $s(u) = 6, s(v) = 10$.

(v) If $d(u, v) = n$, then there exist 2 pair of verices with $s(u) = 4, s(v) = 10$ and 9 pair of vertices with $s(u) = s(v) = 6$.

(vi) If $d(u, v) = n + 1$, then there exist 6 pair of verices with $s(u) = 4, s(v) = 6$.

(vi) If $d(u, v) = n + 2$, then there exist one pair of verices with $s(u) = s(v) = 4$.

By the definition of $NDD(G)$, we have

$$\begin{aligned}
NDD(GO_n) &= \sum_{u,v \in V(G)} [s(u) + s(v)]d(u, v) \\
&= \sum_{d(u,v)=1} [s(u) + s(v)]d(u, v) + \sum_{d(u,v)=2} [s(u) + s(v)]d(u, v) \\
&+ \sum_{d(u,v)=3} [s(u) + s(v)]d(u, v) + \sum_{d(u,v)=k,k=4}^{n-1} [s(u) + s(v)]d(u, v) \\
&+ \sum_{d(u,v)=n} [s(u) + s(v)]d(u, v) + \sum_{d(u,v)=n+1} [s(u) + s(v)]d(u, v) \\
&+ \sum_{d(u,v)=n+2} [s(u) + s(v)]d(u, v) \\
NDD(O_n) &= 24n^3 + 82n^2 - 170n + 524.
\end{aligned}$$

By the definition of $NZZ(G)$, we have

$$\begin{aligned}
NZZ(O_n) &= \sum_{u,v \in V(G)} [s(u) \cdot s(v)]d(u, v) \\
&= \sum_{d(u,v)=1} [s(u) \cdot s(v)]d(u, v) + \sum_{d(u,v)=2} [s(u) \cdot s(v)]d(u, v) \\
&+ \sum_{d(u,v)=3} [s(u) \cdot s(v)]d(u, v) + \sum_{d(u,v)=k,k=4}^{n-1} [s(u) \cdot s(v)]d(u, v) \\
&+ \sum_{d(u,v)=n} [s(u) \cdot s(v)]d(u, v) + \sum_{d(u,v)=n+1} [s(u) \cdot s(v)]d(u, v) \\
&+ \sum_{d(u,v)=n+2} [s(u) \cdot s(v)]d(u, v)
\end{aligned}$$

$$NZZ(O_n) = 84n^3 + 232n^2 - 512n + 1328.$$

By the definition of $NWA(G)$, we have

$$\begin{aligned} NWA(O_n) &= \sum_{u,v \in V(G)} |s(u) - s(v)| d(u,v) \\ &= \sum_{d(u,v)=1} |s(u) - s(v)| d(u,v) + \sum_{d(u,v)=2} |s(u) - s(v)| d(u,v) \\ &+ \sum_{d(u,v)=3} |s(u) - s(v)| d(u,v) + \sum_{d(u,v)=k,k=4}^{n-1} |s(u) - s(v)| d(u,v) \\ &+ \sum_{d(u,v)=n} |s(u) - s(v)| d(u,v) + \sum_{d(u,v)=n+1} |s(u) - s(v)| d(u,v) \\ &+ \sum_{d(u,v)=n+2} |s(u) - s(v)| d(u,v) \end{aligned}$$

$$NWA(O_n) = 4n^3 + 10n^2 - 34n + 140.$$ □

Observation 1.3. *In view of the above theorem, we present the graphical representation of the neighbor degree distance based indices of ortho-chain square cactus graph* O_n *;* $n > 4$ *with respect to* $NWA(O_n) \leq NDD(O_n) \leq NZZ(O_n)$ *as follows.*

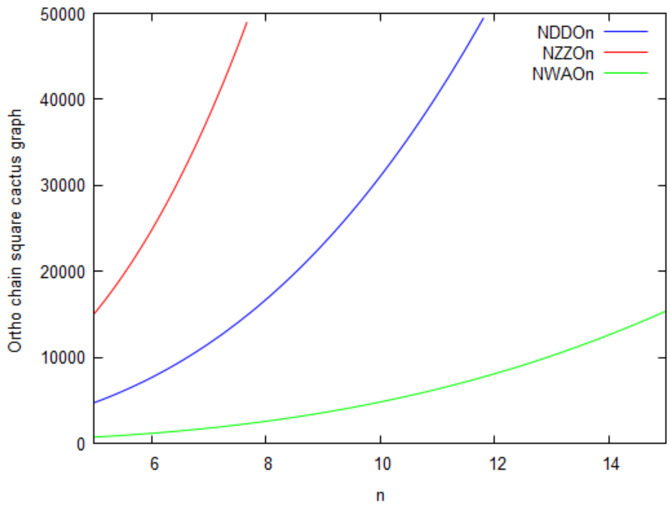

Figure 7: Graphical representation of O_n.

Theorem 1.4. *Let* L_n *be the para-chain hexagonal cactus with* $n > 4$*. Then*

$$\begin{aligned} NDD(L_n) &= 76n^3 + 408n^2 - 500n - 80, \\ NZZ(L_n) &= 224n^3 + 1056n^2 - 1528n - 224, \text{ and} \\ NWA(L_n) &= 4n^3 - 144n^2 + 212n - 16. \end{aligned}$$

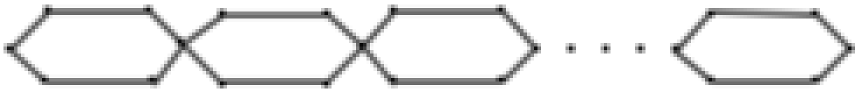

Figure 8: The para-chain hexagonal cactus graphs L_n

Proof. Let u and v be any two vertices of L_n and $d(u, v) = d$. We have the following cases:

(i) If $d(u, v) = 1$ then there exist 4 pair of verices with $s(u) = s(v) = 4$, 4 pair of vertices with $s(u) = 4, s(v) = 6$ and $2(n-2)$ pair of vertices with $s(u) = s(v) = 6$.

(ii) If $d(u, v) = 2$, then there exist 2 pair of verices with $s(u) = 4, s(v) = 4$, 4 pair of vertices with $s(u) = 4, s(v) = 6$, 4 pair of vertices with $s(u) = 4, s(v) = 8$, $6(n-1)$ pair of vertices with $s(u) = s(v) = 6$ and $4(n-2)$ pair of vertices with $s(u) = 6, s(v) = 8$.

(iii) If $d(u, v) = 3$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$, 2 pair of verices with $s(u) = 4, s(v) = 8$, $8(n-2)$ pair of verices with $s(u) = s(v) = 6$ and $n-2$ pair of vertices with $s(u) = s(v) = 8$.

(iv) If $d(u, v) = k$ and $k = 3s$ $(2 \leq s \leq n-1)$ then there exist 8 pair of verices with $s(u) = 4, s(v) = 6$, 2 pair of vertices with $s(u) = 4, s(v) = 8$, $8(n-s-1)$ pair of verices with $s(u) = s(v) = 6$, $(n-s-1)$ pair of verices with $s(u) = s(v) = 8$.

(v) If $d(u, v) = k$ and $k = 3s+1$ $(1 \leq s \leq n-2)$ then there exist 12 pair of verices with $s(u) = 4, s(v) = 6$, $4(n-s-2)$ pair of vertices with $s(u) = s(v) = 6$ and $4(n-2s)$ pair of vertices with $s(u) = 6, s(v) = 8$.

(vi) If $d(u, v) = k$, and $k = 3s+2$ $(1 \leq s \leq n-2)$ then there exist 4 pair of verices with $s(u) = 4, s(v) = 6$, 4 pair of verices with $s(u) = 4, s(v) = 8$ and $4(n-s-1)$ pair of verices with $s(u) = s(v) = 6$.

(vii) If $d(u, v) = 3n-2$, then there exist 2 pair of verices with $s(u) = s(v) = 4$ and 4 pair of verices with $s(u) = 4, s(v) = 6$.

(viii) If $d(u, v) = 3n-1$, then there exist 4 pair of verices with $s(u) = s(v) = 4$.

(ix) If $d(u, v) = 3n$, then there exist one pair of verices with $s(u) = s(v) = 4$.

By the definition of $NDD(G)$, we have

$$NDD(L_n) = \sum_{u,v \in V(G)} [s(u) + s(v)]d(u, v)$$

$$
\begin{aligned}
&= \sum_{d(u,v)=1} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=k=3s,s=2}^{n-1} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=k=3s+1,s=1}^{n-2} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=k=3s+2,s=1}^{n-2} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3n-2} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=3n-1} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3n} [s(u)+s(v)]d(u,v)
\end{aligned}
$$

$$NDD(L_n) = 76n^3 + 408n^2 - 500n - 80.$$

By the definition of $NZZ(G)$, we have

$$
\begin{aligned}
NZZ(L_n) &= \sum_{u,v \in V(G)} [s(u) \cdot s(v)]d(u,v) \\
&= \sum_{d(u,v)=1} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u) \cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=k=3s,s=2}^{n-1} [s(u) \cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=k=3s+1,s=1}^{n-2} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=k=3s+2,s=1}^{n-2} [s(u) \cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3n-2} [s(u) \cdot s(v)]d(u,v) + \sum_{d(u,v)=3n-1} [s(u)s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3n} [s(u) \cdot s(v)]d(u,v)
\end{aligned}
$$

$$NZZ(L_n) = 224n^3 + 1056n^2 - 1528n - 224.$$

By the definition of $NWA(G)$, we have

$$
\begin{aligned}
NWA(L_n) &= \sum_{u,v \in V(G)} |s(u) - s(v)|d(u,v) \\
&= \sum_{d(u,v)=1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=2} |s(u) - s(v)|d(u,v) \\
&+ \sum_{d(u,v)=3} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=k=3s,s=2}^{n-1} |s(u) - s(v)|d(u,v)
\end{aligned}
$$

$$+ \sum_{d(u,v)=k=3s+1,s=1}^{n-2} |s(u)s(v)|d(u,v) + \sum_{d(u,v)=k=3s+2,s=1}^{n-2} |s(u)-s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=3n-2} |s(u)-s(v)|d(u,v) + \sum_{d(u,v)=3n-1} |s(u)-s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=3n} |s(u)-s(v)|d(u,v)$$

$NWA(L_n) = 4n^3 - 144n^2 + 212n - 16.$ □

Observation 1.4. *In view of the above theorem, we present the graphical representation of the neighbor degree distance based indices ofpara chain hexagonal cactus graphs L_n; $n > 4$ with respect to $NWA(L_n) \leq NDD(L_n) \leq NZZ(L_n)$ as follows.*

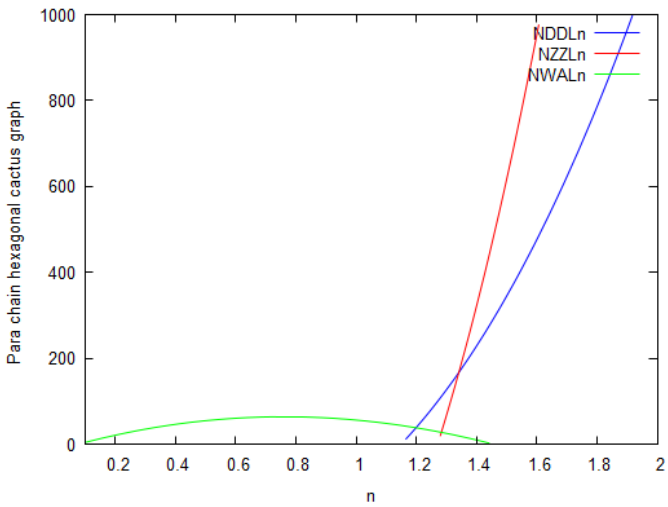

Figure 9: Graphical reprasentation of L_n.

Theorem 1.5. *For $n > 4$, let M_n be the meta-chain hexagonal cactus graph.Then*

$$NDD(M_n) = \frac{152n^3 + 540n^2 + 1420n - 1680}{3}$$
$$NZZ(M_n) = 160n^3 + 416n^2 + 1392n - 1992$$
$$NWA(M_n) = \frac{16n^3 + 60n^2 + 356n - 288}{3}$$

Proof. Let u and v be any two vertices of M_n and $d(u,v) = d$. We have the following cases:

(i) If $d(u,v) = 1$ then there exist 4 pair of verices with $s(u) = s(v) = 4$, $2n$ pair of vertices with $s(u) = 4, s(v) = 6$, $2n$ pair of vertices with $s(u) = 6, s(v) = 8$ and $2(n-2)$ pair of vertices with $s(u) = s(v) = 8$.

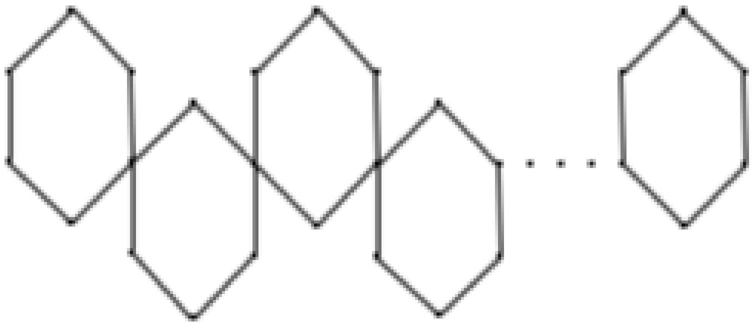

Figure 10: The meta-chain hexagonal cactus graphs M_n

(ii) If $d(u,v) = 2$, then there exist 2 pair of verices with $s(u) = 4, s(v) = 4$, 4 pair of vertices with $s(u) = 4, s(v) = 6$, $2n$ pair of vertices with $s(u) = 4, s(v) = 8$, $2n + 1$ pair of vertices with $s(u) = s(v) = 6$, $4n - 6$ pair of vertices with $s(u) = 6, s(v) = 8$ and $2n - 5$ pair of vertices with $s(u) = s(v) = 8$, .

(iii) If $d(u,v) = 3$ then there exist $2n + 6$ pair of verices with $s(u) = 4, s(v) = 6$, $2n$ pair of verices with $s(u) = 6, s(v) = 8$, $2(n - 3)$ pair of verices with $s(u) = s(v) = 8$ and $3n - 2$ pair of vertices with $s(u) = 4, s(v) = 8$.

(iv) If $d(u,v) = 4$ then there exist $n + 1$ pair of verices with $s(u) = s(v) = 4$, $2n$ pair of verices with $s(u) = 4, s(v) = 8$, $4n - 10$ pair of verices with $s(u) = 6, s(v) = 8$ and $2(n + 1)$ pair of vertices with $s(u) = s(v) = 6$.

(v) If $d(u,v) = k$ and $k = 2s$ $(3 \leq s \leq n - 1)$ then there exist $n - s + 3$ pair of verices with $s(u) = s(v) = 4$, 4 pair of vertices with $s(u) = 4, s(v) = 6$, $4(n - s + 1)$ pair of verices with $s(u) = s(v) = 6$, $(n - s - 1)$ pair of verices with $s(u) = s(v) = 8$.

(vi) If $d(u,v) = k$ and $k = 2s + 1$ $(2 \leq s \leq n - 2)$ then there exist 2 pair of verices with $s(u) = s(v) = 4$, 8 pair of vertices with $s(u) = 4, s(v) = 6$, $2(n - s - 1)$ pair of vertices with $s(u) = 4, s(v) = 8$ and $4(n - s) - 2$ pair of vertices with $s(u) = 6, s(v) = 8$.

(vii) If $d(u,v) = 2n - 1$, then there exist 8 pair of verices with $s(u) = 4, s(v) = 6$ and 2 pair of verices with $s(u) = 4, s(v) = 8$.

(viii) If $d(u,v) = 2n$, then there exist 4 pair of verices with $s(u) = s(v) = 4$ and 4 pair of verices with $s(u) = 4, s(v) = 6$.

(ix) If $d(u,v) = 2n + 1$, then there exist 4 pair of verices with $s(u) = s(v) = 4$.

(x) If $d(u,v) = 2n + 2$, then there exist one pair of verices with $s(u) = s(v) = 4$.

By the definition of $NDD(G)$, we have

$$
\begin{aligned}
NDD(M_n) &= \sum_{u,v\in V(G)} [s(u)+s(v)]d(u,v) \\
&= \sum_{d(u,v)=1} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=4} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=k=2s,s=3}^{n-1} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=k=2s+1,s=2}^{n-2} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=2n-1} [s(u)+s(v)]d(u,v) + \sum_{d(u,v)=2n} [s(u)+s(v)]d(u,v) \\
&+ \sum_{d(u,v)=2n+1} [s(u)+s(v)]d(u,v) ++ \sum_{d(u,v)=2n+2} [s(u)+s(v)]d(u,v)
\end{aligned}
$$

$$NDD(M_n) = \frac{152n^3+540n^2+1420n-1680}{3}.$$

By the definition of $NZZ(G)$, we have

$$
\begin{aligned}
NZZ(M_n) &= \sum_{u,v\in V(G)} [s(u)\cdot s(v)]d(u,v) \\
&= \sum_{d(u,v)=1} [s(u)\cdot s(v)]d(u,v) + \sum_{d(u,v)=2} [s(u)\cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=3} [s(u)\cdot s(v)]d(u,v) + \sum_{d(u,v)=4} [s(u)\cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=k=2s,s=3}^{n-1} [s(u)s(v)]d(u,v) + \sum_{d(u,v)=k=2s+1,s=2}^{n-2} [s(u)s(v)]d(u,v) \\
&+ \sum_{d(u,v)=2n-1} [s(u)\cdot s(v)]d(u,v) + \sum_{d(u,v)=2n} [s(u)\cdot s(v)]d(u,v) \\
&+ \sum_{d(u,v)=2n+1} [s(u)\cdot s(v)]d(u,v) + \sum_{d(u,v)=2n+2} [s(u)\cdot s(v)]d(u,v)
\end{aligned}
$$

$$NZZ(M_n) = 160n^3+416n^2+1392n-1992.$$

By the definition of $NWA(G)$, we have

$$
\begin{aligned}
NWA(M_n) &= \sum_{u,v\in V(G)} |s(u)-s(v)|d(u,v) \\
&= \sum_{d(u,v)=1} |s(u)-s(v)|d(u,v) + \sum_{d(u,v)=2} |s(u)-s(v)|d(u,v)
\end{aligned}
$$

$$+ \sum_{d(u,v)=3} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=4} |s(u) - s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=k=2s,s=3}^{n-1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=k=2s+1,s=2}^{n-2} |s(u) - s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=2n-1} |s(u) - s(v)|d(u,v) + \sum_{d(u,v)=2n} |s(u) - s(v)|d(u,v)$$
$$+ \sum_{d(u,v)=2n+1} |s(u) - s(v)|d(u,v) + + \sum_{d(u,v)=2n+2} |s(u) - s(v)|d(u,v)$$
$$NWA(M_n) = \frac{16n^3 + 60n^2 + 356n - 288}{3}. \quad \square$$

Observation 1.5. *In view of the above theorem, we present the graphical representation of the neighbor degree distance based indices of meta-chain hexagonal cactus graphs* M_n*;* $n > 4$ *with respect to* $NWA(M_n) \leq NDD(M_n) \leq NZZ(M_n)$ *as follows.*

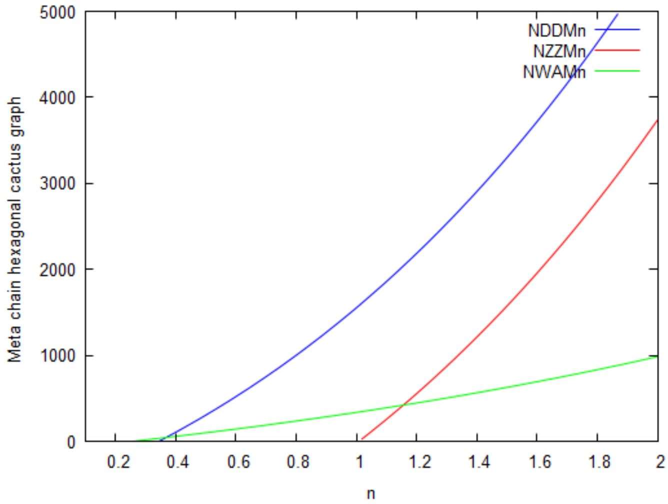

Figure 11: Graphical representation of M_n.

CONCLUSION AND OPEN PROBLEMS

We initiated the study of neighborhood degree - distance based indices. We obtain some computed values of some stanadard graphs, bounds and characterisation. Also, we obtained these indices for some cactus families of graph and its graphical interpretation. Numerous problems, including the following, are raised by this research regarding comparative advantages, applications, and mathematical perspectives.

1. Find the extremal values and extremal graphs of the neighbor degree-distance based indices.

2. Find the exact value of $NDD(G)$, $NZZ(G)$ and $NWA(G)$ of some chemical graphs/derived graphs/product graphs.

3. Find the relationship between among the $NDD(G)$, $NZZ(G)$ and $NWA(G)$ of a graph G and other degree based topological indices.

4. Explore some results towards QSPR/QSAR/QSTR Model.

5. Characterize the $NDD(G)$, $NZZ(G)$ and $NWA(G)$ in terms of other global and local degree based topological indices.

REFERENCES

[1] H. Abdo, S. Brandt, D. Dimitrov, The total irregularity of a graph, Discrete Math. Theoret. Comput. Sci. **16**, (2014), 201–206.

[2] M. O. Albertson. The irregularity of a graph. Ars Comb., **46**, (1997), 219–225.

[3] T V Asha, B Chaluvaraju , The Neighbor Degree Sum - Distance Indices of a Graph, Communicated

[4] M. Chellali, *Bounds on the 2-domination number in cactus graphs*, Opuscula Math. **2**, (2006), 5–12.

[5] A. A. Dobrynin and A. A. Kochetova, *Degree distance of a graph: A degree analog of the Wiener index*, Journal of Chemical Information and Computer Sciences, **34(5)**, (1994), 1082–1086.

[6] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass, (1969).

[7] F. Harary and B. Uhlenbeck, *On the number of Husimi trees*, I, Proc. Nat. Acad. Sci. **39**, (1953), 315–322.

[8] S. Klavžar, I. Gutman, *Wiener number of vertex-weighted graphs and a chemical applicat*, Discr. Appl. Math. **80**, (1997), 73–81.

[9] K. Husimi, *Note on Mayer's theory of cluster integrals*, J. Chem. Phys. **18** , (1950), 682–684.

[10] S. Majstorovic, T. Doslic and A. Klobucar, *k-domination on hexagonal cactus chains*, Kragu jevac J. Math. **36(2)**, (2012), 335-347.

[11] S. Mondal, N. De and A. Pal, *The neighbourhood Zagreb index of product graphs and its chemical interest*, arXiv, (2018), 1805.05273.

[12] Pattabiraman Kandan, and Manzoor Ahmed Bhat, *Reciprocal version of product degree distance of cactus graphs*, TWMS J. App. and Eng. Math. V.11, Special Issue, (2020), 228-239.

[13] R.J. Riddell, *Contributions to the theory of condensation*, Ph.D. Thesis, Univ. of Michigan, Ann Arbor, (1951).

[14] Sadeghieh, A., Ghanbari, N., and Alikhani, S, *Computation of Gutman index of some cactus chains*, Electronic Journal of Graph Theory and Applications, **6(1)**, (2018), 237-872.

[15] H. Wiener, *Structural determination of paraffin boiling point*, J. Amer. Chem. Soc., **69**, (1947), 17–20.

[16] Zhu, Zhongxun, Ting Tao, Jing Yu, and Liansheng Tan. ”On the Harary index of cacti.” Filomat 28, no.**3** , (2014), 493-507.