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Some aspects of finite dimensional complex valued normed linear spaces

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Abstract

In this paper finite dimensional complex valued normed linear space is studied. One fundamental lemma is established and completeness, compactness etc. are studied in finite dimensional complex valued normed linear spaces.
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1. INTRODUCTION

The history of functional analysis is not very old. The idea grew up in early twentieth century. Researchers felt the necessity of this subject during the studies of integration theory and integral equations. However, Banach was the pioneer of formal functional analysis. In 1922, he defined normed linear spaces as a set of axioms. Since then, mathematicians keep on trying to find a proper generalization of this concept. Different authors came up with different types of metrics and norms over the years (for reference please see [1],[2],[12],[13],[14],[15],[16],[17],[18],[19]).
Complex-valued metric space, introduced by Akbar Azam et al.[1] in 2011, has played an important role in the development of functional analysis. Alternatively complex-valued normed linear space, presented by G. A. Okeke[2] has played a great role in the development of functional analysis. Different authors studied various results in complex-valued metric spaces and complex-valued normed linear spaces with different approaches (for reference please see[7],[8],[9],[10],[11]).
The study of finite dimensional normed linear space[3],[4],[5],[6] related to completeness, compactness etc. plays a pivotal role to establish different famous results in normed linear space.

In this work, we have studied different results in complex-valued normed linear space. Finally, we have established some theorems about completeness, compactness in finite dimensional complex-valued normed linear space. Also a lemma analogous to Riesz's lemma has been examined in complex-valued normed linear space by us to establish the compactness of unit ball is equivalent to the finite dimensionality of the complex-valued normed linear space.

2. PRELIMINARIES

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $preceq$ on C as follows:

$$z_1 \preceq z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2), \Im(z_1) \leq \Im(z_2).$$

Let X be a nonempty set. Suppose that the mapping $d : X \times X \to C$, satisfies:

1. $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. Let X be a linear space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R} or \mathbb{C}$. A complex valued function $\|\| : X \to \mathbb{C}$ such that

1. $\|x\| = 0$ if and only if $x = 0, x \in X$,
2. $\|kx\| = |k|\|x\|, \forall k \in \mathbb{K}, x \in X$,
3. $\|x + y\| \preceq \|x\| + \|y\|, \forall x, y \in X$

then $\|.\|$ is called a complex valued norm on X and $(X, \|.\|)$ is called a complex valued normed linear space. Let $E = \mathbb{C}$ be the set of complex numbers. Define $\| \cdot \| : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $\|z_1 - z_2\| = \|x_1 - x_2\| + \iota \|y_1 - y_2\| \ \forall z_1, z_2 \in \mathbb{C}$ where $z_1 = x_1 + \iota y_1, z_2 = x_2 + \iota y_2$. Clearly, $(\mathbb{C}, \| \cdot \|)$ is a complex valued normed linear space. A point $x \in E$ is called an interior point of a set $A \subset E$ if there exist $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in E : \|x - y\| < r\} \subset A.$$

A point $x \in E$ is called a limit point of the set A whenever for each $0 \prec r \in \mathbb{C}$, we have

$$B(x, r) \cap (A - E) \neq \phi.$$

Let $(E, \|\cdot\|)$ be a complex valued normed linear space. The set $A \subset E$ is said to be open if each element of A is an interior point of A. In a complex valued normed linear space $(E, \|\cdot\|)$, a subset B of E is said to be closed if it contains each of its limit point. Suppose x_n is a sequence in E. If for all $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\|x_n - x_{n+m}\| \prec c$, then $\{x_n\}$ is called a Cauchy sequence in $(E, \|\cdot\|)$. If every Cauchy sequence is convergent in $(E, \|\cdot\|)$, then $(E, \|\cdot\|)$ is called a complex valued Banach space.

3. MAIN RESULT

Let $(X, \|\cdot\|)$ be a complex valued normed linear space. We define $\|\cdot\|_{sqrt} : X \to \mathbb{R}$ by

$$\|x\|_{sqrt} = \frac{1}{\sqrt{2}}|\|x\||.$$

Then $\|\cdot\|_{sqrt}$ is a norm on X.

Proof. 1. For every $x \in X$,

$$\begin{aligned} \|x\|_{sqrt} = 0 \\ &\iff \frac{1}{\sqrt{2}}|\|x\|| = 0 \\ &\iff |\|x\|| = 0 \\ &\iff \|x\| = 0 \\ &\iff x = 0_X. \end{aligned}$$

2. Let us take $c \in X$. Then

$$\begin{aligned} \|cx\|_{sqrt} &= \frac{1}{\sqrt{2}}|\|cx\|| \\ &= \frac{1}{\sqrt{2}}||c|\|x\|| \\ &= \frac{1}{\sqrt{2}}|c||\|x\|| \\ &= |c|\frac{1}{\sqrt{2}}|\|x\|| \\ &= |c|\|x\|_{sqrt}. \end{aligned}$$

3. Let us take $x, y \in X$. Then $\|x + y\| \preceq \|x\| + \|y\|$. So $|\|x + y\|| \leq |\|x\| + \|y\||$.

Thus we have

$$\begin{aligned}\|x+y\|_{sqrt} =&\frac{1}{\sqrt{2}}|\|x+y\||\\ \leq&\frac{1}{\sqrt{2}}|\|x\|+\|y\||\\ \leq&\frac{1}{\sqrt{2}}(|\|x\||+|\|y\||)\\ =&\frac{1}{\sqrt{2}}|\|x\||+\frac{1}{\sqrt{2}}|\|y\||\\ =&\|x\|_{sqrt}+\|y\|_{sqrt}.\end{aligned}$$

Hence from it fillows that $\|\cdot\|_{sqrt}$ is a norm on X. □

Let $(X, \|\cdot\|)$ be a complex valued normed linear space. A subset A of X is said to be bounded if $\exists M \succ 0$ such that $\|x\| \preceq M \; \forall x \in A$. Let $(X, \|\cdot\|)$ be a complex valued normed linear space and $A \subset X$. Then the closure $\overline{A}$ is defined by the following:

$$\overline{A} = \{x \in X : \text{there is a sequence } \{x_n\} \text{ in } A \text{ converging to} x\}.$$

Let $(X, \|\cdot\|)$ be a complex valued normed linear space. Let A, B be two bounded sets in X. Then

[(i)]

1. $A + B$ is bounded.
2. cA is bounded for each $c \in \mathbb{K}$.

Proof. 1. Since A, B are two bounded sets in X, so $\exists M_1, M_2 \succ 0$ such that $\|x\| \preceq M_1 \; \forall x \in A$ and $\|x\| \preceq M_2 \; \forall x \in B$. Suppose $z \in A + B$. Then $z = x + y$ for some $x \in A$ and $y \in B$.

$$\begin{aligned}\text{Now } \|z\| = \|x+y\| \preceq&\|x\|+\|y\|\\ \preceq&M_1+M_2 \; \forall z \in A+B.\end{aligned}$$

Thus $A + B$ is bounded.

2. Since A is bounded, so there is $M \succ 0$ such that $\|x\| \preceq M \; \forall x \in A$. If $c = 0$ then $cx = \theta$ for every $x \in A$. Hence the result holds.
Suppose $c \neq 0$. Then $M|c| \succ 0$.

$$\text{Now } \|cx\| = |c|\|x\| \preceq M|c| \; \forall x \in A.$$

Hence cA is bounded.

□

Let $(X, \|\cdot\|)$ be a complex valued normed linear space and $A \subset X$. A is said to be convex if $sx + ty \in A$ for every $x, y \in A$ and $0 \leq s, t \leq 1$ and $s + t = 1$. Let $(X, \|\cdot\|)$ be a complex valued normed linear space and $A \subset X$. Then the convex hull of A, denoted by $Co(A)$ is the set of all convex combinations of the elements of A. Let $(X, \|\cdot\|)$ be a complex valued normed linear space. Let A, B be two convex subsets of X. Then

1. $A + B$ is convex.
2. cA is convex.

Proof. 1. Let $x, y \in A + B$. Then $x = a_1 + b_1, y = a_2 + b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.
   Let us take $s, t \in [0, 1]$ with $s + t = 1$. Then we have $sa_1 + ta_2 \in A$ and $sb_1 + tb_2 \in B$. This implies that $sx + ty = sa_1 + ta_2 + sb_1 + tb_2 \in A + B$. Hence $A + B$ is convex.

2. Suppose $c = 0$. Then the result is obvious.
   Let $c \neq 0$. Let us take $x, y \in cA$. Then $x = cx_1, y = cy_1$ for some $x_1, y_1 \in A$. Since A is convex, so $sx_1 + ty_1 \in A$ for $s, t \in [0, 1]$ and $s + t = 1$. Now $sx + ty = scx_1 + tcy_1 = c(sx_1 + ty_1)$. So it is obvious that $sx + ty \in cA$. Hence cA is convex.

□

Let $(X, \|\cdot\|)$ be a complex valued normed linear space and A be a bounded subset of X. Then

1. $\overline{A}$ is bounded.
2. $Co(A)$ is bounded.
3. $\overline{Co(A)}$ is bounded.

Proof. Since A is bounded, so $\exists M \succ 0$ such that $\|x\| \preceq M \; \forall x \in A$.

1. Suppose $y \in \overline{A}$. Then there is a sequence $\{y_n\}$ in A such that $\{y_n\}$ convrerges to y. By assumption we have $\|y_n\| \preceq M\ \forall n \in \mathbb{N}$. Now $\|y\| \preceq \|y - y_n\| + \|y_n\|$ i.e. $\|y\| \preceq \|y_n - y\| + \|y_n\|$. Since $\{y_n\}$ converges to y so we have $\|y_n - y\| \to 0$ as $n \to \infty$. Hence it follows that $\|y\| \preceq M$. Here $y \in \overline{A}$ is arbitrary, so $\overline{A}$ is bounded.

2. Let $y \in Co(A)$. Then $y = sy_1 + ty_2$ for $y_1, y_2 \in A$ where $0 \leq s, t \leq 1$ and $s + t = 1$. Now $\|y\| = \|sy_1 + ty_2\| \preceq |sy_1| + |ty_2| = |s|\|y_1\| + |t|\|y_2\| \preceq sM + tM = (s+t)M = M$. So $\|y\| \preceq M$. Since $y \in Co(A)$ is arbitrary thus it follows that $Co(A)$ is bounded.

3. The proof follows from that of 1 and 2.

□

Let $(X, \|\cdot\|)$ be a complex valued normed linear space and $A \subset X$. A is said to be compact if every sequence in A has a convergent subsequence which converges to some point in A. Let $(X, \|\cdot\|)$ be a complex valued normed linear space. Let $\{e_1, e_2, \cdots, e_n\}$ be a linearly independent set of vectors in X. Then $\exists a > 0$ such that for every set of scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$\Re(\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|) \geq a \sum_{i=1}^{n} |\alpha_i|. \tag{1}$$

Proof. If $\alpha_i = 0$ for every $i = 1, 2, \cdots, n$ then it is obvious.
Now we assume that at least one α_i is non-zero. Set $s = \sum_{i=1}^{n} |\alpha_i|$. Then $s > 0$. Then (1) is equivalent to

$$\Re(\|\sum_{i=1}^{n} \beta_i e_i\|) \geq a \tag{2}$$

where $\{\beta_1, \beta_2, \cdots, \beta_n\}$ is a set of scalars with $\sum_{i=1}^{n} |\beta_i| = 1$. If possible suppose (2) does not hold. Then for every $a > 0$ there is a set of scalars $\{\beta_1, \beta_2, \cdots, \beta_n\}$ whwere $\sum_{i=1}^{n} |\beta_i| = 1$ such that the following holds:

$$\Re(\|\sum_{i=1}^{n} \|\beta_i e_i\|) < a.$$

Thus for $a_m = \frac{1}{m}$, there are sequences $\{\beta_i^{(m)}\}$ of scalars with $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ such that

$$\Re(\|\sum_{i=1}^{n} \|\beta_i^{(m)} e_i\|) < \frac{1}{m}.$$

$$\text{So } \Re(\|x_{n,m}\|) < \frac{1}{m}. \tag{3}$$

Let $\epsilon > 0$ be given. Let us choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Then $\frac{1}{m} < \epsilon \,\forall m \geq k$. Then from it follows that $\Re(\|x_{n,m}\| < \epsilon \,\forall m \geq k$. As a consequence we can conclude that

$$\lim_{m\to\infty} \Re(\|x_{n,m}\|) = 0. \tag{4}$$

Since $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ so $|\beta_i^{(m)}| \leq 1$ for each $i = 1, 2, \cdots, n$. Hence $\{\beta_i^{(m)}\}$ is a bounded sequence of scalars for each $i = 1, 2, \cdots, n$. Then by Bolzano-Weierstrass theorem, $\{\beta_i^{(m)}\}$ has a convergent subsequence of scalars converging to some β_i for every $i = 1, 2, \cdots, n$. So $\{\beta_1^{(m)}\}$ has a convergent subsequence with limit β_1 and suppose $\{x_{1,m}\}$ be the corresponding subsequence of $\{x_m\}$. By the similar argument we obtain a subsequence $\{x_{2,m}\}$ of $\{x_{1,m}\}$ for which the corresding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2. Proceeding in similar way at the n-th step a subsequence $\{x_{n,m}\}$ of $\{x_m\}$ can be obtained which is of the form $x_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} e_i$ where $\gamma_i^{(m)} \to \beta_i$ as $m \to \infty$ and $\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1$.
Let us take $x = \sum_{i=1}^{n} \beta_i e_i$. Then $x \in X$ and $x_{n,m} - x = \sum_{i=1}^{n} (\gamma_i^{(m)} - \beta_i) e_i$. Now $\|x_{n,m} - x\| = \|\sum_{i=1}^{n} (\gamma_i^{(m)} - \beta_i) e_i\| \preceq \sum_{i=1}^{n} |\gamma_i^{(m)} - \beta_i| \|e_i\|$. Since $\lim\limits_{m\to\infty} \gamma_i^{(m)} = \beta_i$ for each $i = 1, 2, \cdots, n$ so $\lim\limits_{m\to\infty} \|x_{n,m} - x\| = 0$. Thus we have,

$$\lim_{m\to\infty} \Re(\|x_{n,m} - x\|) = 0. \tag{5}$$

Again

$$\begin{aligned} &\|x\| \preceq \|x - x_{n,m}\| + \|x_{n,m}\| \\ \Longrightarrow\ &\Re(\|x\|) \leq \Re(\|x - x_{n,m}\|) + \Re(\|x_{n,m}\|). \end{aligned}$$

Thereafter from (4) and (5) we have,

$$\Re(\|x\|) = 0. \tag{6}$$

Here $\gamma_i^{(m)} \to \beta_i$ as $m \to \infty$ for each $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1$ so we get $\sum_{i=1}^{n} |\beta_i| = 1$. This ensures that at least one β_i is nen-zero. Consequently $\sum_{i=1}^{(n)} \beta_i e_i$ i.e. x is a non-zero vector in X. Thus it follows that $\|x\| \succ 0$. Hence $\Re(\|x\|) > 0$. Clearly this contradicts (6).
Therefore our assumption is wrong and hence (2) holds. □

Let $(X, \|\cdot\|)$ be a complex valued normed linear space. Let $\{e_1, e_2, \cdots, e_n\}$ be a linearly independent set of vectors in X. Then $\exists a > 0$ such that for every set of scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$\Im(\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|) \geq a \sum_{i=1}^{n} |\alpha_i|. \tag{7}$$

Proof. The proof is similar to that of Lemma 3.10. □

Let $(X, \|\cdot\|)$ be a complex valued normed linear space. Let $\{e_1, e_2, \cdots, e_n\}$ be a linearly independent set of vectors in X. Then $\exists c \succ 0$ such that for every set of scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\| \succeq c \sum_{i=1}^{n} |\alpha_i|. \tag{8}$$

Proof. From Lemma 3.10 we have, $\exists a > 0$ such that for every set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that

$$\Re(\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|) \geq a \sum_{i=1}^{n} |\alpha_i|. \tag{9}$$

Similarly from Lemma 3.11, there is $b > 0$ such that for every set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that

$$\Im(\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|) \geq b \sum_{i=1}^{n} |\alpha_i|. \tag{10}$$

Let us take $c = (a, b)$. Then $c \succ 0$. Thus from (9)and (10) it follows that

$$\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\| \succeq c \sum_{i=1}^{n} |\alpha_i|.$$

□

Every finite dimensional complex valued normed linear space is complete.

Proof. Let $(x, \|\cdot\|)$ be a finite dimensional complex valued normed linear space. Suppose $\dim(X) = n$ and $\{e_1, e_2, \cdots, e_n\}$ is a basis for X.
Let us a consider a Cauchy sequence $\{x_m\}$ in $(X, N_C, *)$. Suppose $x_m = \sum_{i=1}^{n} \alpha_i^{(m)} e_i$ where $\{\alpha_i^{(m)}\}$ is a suitable set of scalars for each $i = 1, 2, \cdots, n$.Then $x_m - x_p = \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(p)}) e_i$. Then by Lemma 3.12, $\exists r \succ \theta$ such that

$$\|x_m - x_p\| \succeq r \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}|. \tag{11}$$

Since $\{x_m\}$ is Cauchy so for $s \succ \theta, \exists m_0 = m_0(s) \in \mathbb{N}$ such that

$$\|x_m - x_p\| \prec s \ \forall m, p \geq m_0. \tag{12}$$

Thus from (11) and (12) we have

$$\begin{aligned}
&r\sum_{i=1}^{n}|\alpha_i^{(m)}-\alpha_i^{(p)}| \prec s\ \forall m,p\geq m_0\\
\Longrightarrow\ &|r|\sum_{i=1}^{n}|\alpha_i^{(m)}-\alpha_i^{(p)}| < |s|\ \forall m,p\geq m_0\\
\Longrightarrow\ &\sum_{i=1}^{n}|\alpha_i^{(m)}-\alpha_i^{(p)}| < \frac{|s|}{|r|}\ \forall m,p\geq m_0\\
\Longrightarrow\ &|\alpha_i^{(m)}-\alpha_i^{(p)}| < \frac{|s|}{|r|}\ \forall m,p\geq m_0, \text{for every } i=1,2,\cdots,n.
\end{aligned}$$

Hence $\{\alpha_i^{(m)}\}$ is a Cauchy sequenc of scalars for each $i = 1, 2, \cdots, n$. So $\alpha_i^{(m)}$ is convergent for every $i = 1, 2, \cdots, n$. Let us suppose that $\lim_{m\to\infty} \alpha_i^{(m)} = \alpha$ for each $i = 1, 2, \cdots, n$. We consider $x = \sum_{i=1}^{n} \alpha_i e_i$. Then $x \in X$.
Here $x_m - x = \sum_{i=1}^{n}(\alpha_i^{(m)} - \alpha_i)e_i$. So $\|x_m - x\| = \|\sum_{i=1}^{n}(\alpha_i^{(m)} - \alpha_i)e_i\| \preceq \sum_{i=1}^{n} \|(\alpha_i^{(m)} - \alpha_i)e_i\| = \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i|\|e_i\|$. Since $\{\alpha_i^{(m)}\}$ converges to α_i for each $i = 1, 2, \cdots, n$ so we have $\lim_{m\to\infty} x_m = x$. Thus $\{x_m\}$ is convergent. Therefore X is complete. □

Let $(X, \|\cdot\|)$ be a finite dimensional complex-valued normed linear space and $A \subset X$. Then A is compact iff A is closed and bounded.

Proof. First suppose that A is compact. Let us consider a sequence $\{x_m\}$ in A such that $\{x_m\}$ converges to some $x \in X$. Since A is compact, so $\{x_m\}$ has a convergent subsequence $\{x_{m_k}\}$ which converges to some point in A. It is very clear that $\{x_{m_k}\}$ converges to x. Therefore $x \in A$. Hence A is closed.
If possible suppose that A is not bounded. Then for every $M \succ \theta, \exists x \in A$ such that $\|x\| \succ M$. Thus we obtain a sequence $\{x_m\}$ in A such that $x_m \succ m\ell\ \forall m \in \mathbb{N}$.
Here A is compact which suggests that $\{x_m\}$ has a convergent subsequence $\{x_{m_k}\}$ converging to some $x \in X$. Let us take $c \succ \theta$. Then $\exists k_0 \in \mathbb{N}$ such that $\|x_{m_k} - x\| \prec c\ \forall k \geq k_0$. Now $\|x_{m_k}\| = \|x_{m_k} - x + x\| \preceq \|x_{m_k} - x\| + \|x\| \prec c + \|x\|$. So we have $m_k\ell \prec c + \|x\|$. Since $\|x\|$ is a fixed number, so this is impossible.
Hence our assumption is wrong. Therefore A is bounded.
Conversely suppose that A is closed and bounded. Suppose $\dim(X) = n$ and $\{e_1, e_2, \cdots, e_n\}$ be a basis for X. Let us consider a sequence $\{x_m\}$ in A. Suppose $x_m = \sum_{i=1}^{n} \alpha_i^{(m)} e_i$ where $\{\alpha_1^{(m)}, \alpha_2^{(m)}, \cdots, \alpha_n^{(m)}\}$ is a suitable set of scalars. Since A is

bounded so $\exists M \succ \theta$ such that $\|x\| \preceq M\ \forall x \in A$. So $\|x_m\| \preceq M\ \forall m \in \mathbb{N}$. Then

$$\|\sum_{i=1}^{n} \alpha_i^{(m)} e_i\| \preceq M. \tag{13}$$

On the other hand by Lemma 3.12, $\exists r \succ \theta$ such that

$$\|\sum_{i=1}^{n} \alpha_i^{(m)} e_i\| \succeq r \sum_{i=1}^{n} |\alpha_i^{(m)}|. \tag{14}$$

So from (13) and (14) it follows that

$$r \sum_{i=1}^{n} |\alpha_i^{(m)}| \preceq M$$
$$\Longrightarrow \||\alpha_i^{(m)}| \preceq \frac{M}{r} \text{ for every } i = 1, 2, \cdots, n.$$

Thus we have $\{\alpha_i^{(m)}\}$ is a bounded sequence of scalars for each $i = 1, 2, \cdots, n$. Then by Bolzano-Weierstrass theorem we can obtain convergent subsequence $\{\alpha_i^{(m_k)}\}$ that converges to some α_i(say) for each $i = 1, 2, \cdots, n$.
Let us consider $x = \sum_{i=1}^{n} \alpha_i e_i$. Clearly $x \in X$.
Now $\|x_{m_k} - x\| = \| \sum_{i=1}^{n} (\alpha_i^{(m_k)} - \alpha_i) e_i\| \leq \sum_{i=1}^{n} |\alpha_i^{(m_k)} - \alpha_i| \|e_i\|$. Here $\{\alpha_i^{(m_k)}\}$ convereges to α_i so we can conclude that x_{m_k} converges to x. Since A is closed so $x \in A$.
Hence we obtain a subsequence of $\{x_m\}$ which converges to some point in A. Thus A is compact. □

Let Y and Z be subspaces of a normed space X and suppose that Y is a proper and closed subset of Z. Then for every $r = (a, b)$ with $a, b \in (0, 1)$ there is a $z \in Z$ such that $\|z\| = 1$ and $|\|z - y\|| \succeq \max\{a, b\}\ \forall y \in Y$.

Proof. We consider any $v \in Z - Y$ and denote its distance from Y by d. So

$$d = \inf_{y \in Y} \|v - y\|.$$

Y being closed implies that $d \succ \theta$. Let us take $r = (a, b)$ where $a, b \in (0, 1)$. Then denote $t = \max\{a, b\}$. So $t \in (0, 1)$.
Then $\exists y_0 \in Y$ such that

$$d \preceq \|v - y_0\| \prec \frac{d}{t}.$$

Thus we have

$$|d| \leq |\|v - y_0\|| < \frac{|d|}{|t|}. \tag{15}$$

Let $z = c(v - y_0)$ where $c = \frac{1}{\|v-y_0\|}$. Then $\|z\| = 1$. We show that $|\|z - y\|| > t$ for every $y \in Y$.

$$\|z - y\| = \|c(v - y_0) - y\| = |c|\|v - y_0 - \frac{y}{c}\| = |c|\|v - y_1\|$$

where $y_1 = y_0 + \frac{y}{c}$.
It is clear that $y_1 \in Y$.

$$\text{Now } |\|z - y\|| = ||c|\|v - y_1\|| = |c||\|v - y_1\|| \geq |c||a| = \frac{|a|}{|\|v - y_0\||} > |t| = t$$

i.e. $|\|z - y\|| > t$
i.e. $|\|z - y\|| > \max\{a, b\}$.

Since $y \in Y$ is arbitrary, so $|\|z - y\|| > \max\{a, b\}$ $\forall y \in Y$. □

Let $(X, \|\cdot\|)$ be a complex-valued normed linear space. Then the closed unit ball $M = \{x \in X : \|x\| \preceq 1\}$ is compact iff X is finite dimensional.

Proof. First suppose that X is finite dimensional. Then it is obvious that M is compact. Conversely suppose that M is compact. If possible suppose that $\dim(X) = \infty$.
Let us choose any x_1 of norm 1. This x_1 generates a one dimensional subspace X_1 of X which is a closed and proper subspace of X since $\dim(X) = \infty$. Let us take $r = \frac{1}{2}\ell$. By Riesz's lemma, there is an $x_2 \in X$ of norm 1 such that $|\|x_2 - x_1\|| > \max\{\frac{1}{2}, \frac{1}{2}\}$. The elements x_1, x_2 generates a two dimensional proper and closed subspace X_2 of X. Again by Riesz's lemma, there is an x_3 of norm 1 such that for every $x \in X_2$ we have $|\|x_3 - x\|| > \max\{\frac{1}{2}, \frac{1}{2}\}$. In particular, $|\|x_3 - x_1\|| > \frac{1}{2}$ and $|\|x_3 - x_2\|| > \frac{1}{2}$. Proceeding in similar way we obtain a sequence $\{x_n\}$ of elements $x_n \in M$ such that

$$|\|x_m - x_n\|| > \frac{1}{2}(m \neq n).$$

It is obvious that $\{x_n\}$ cannot have a convergent subsequence. This contradicts the compactness of M. Hence our assumption $\dim(X) = \infty$ is wrong. Therefore $\dim(X) < \infty$. □

4. CONCLUSION

This work consists of some results on finite dimensional complex valued normed linear spaces. The study of finite dimensional normmed linear spaces has a great impact for developing different significant results of functional analysis. So we hope that the present work will contribute to study various results in complex valued normed linear spaces.

REFERENCES

[1] A. Azam, B. Fisher and M. Khan. Common fixed point theorems in complex valued metric spaces. *Numerical Functional Analysis and Optimization*, 32(3), 243-253, 2011.

[2] Godwin Amechi Okeke. Iterative approximation of fixed points of contraction mappings in complex valued Banach spaces. *Arab Journal of Mathematical Sciences*, 25(1), 83-105, 2019.

[3] Erwin Kreyszig. Introductory functional analysis with applications. *John Wiley & Sons*, 1991.

[4] Haim Brezis. Functional analysis, Sobolev spaces and partial differential equations. *Springer*, 2(3), 2011.

[5] Alexandre Ern and Jean-Luc Guermond Finite elements II. *Springer*, 2011.

[6] Barbara D Maccluer. Elementary functional analysis. *Springer*, 253, 2009.

[7] Wutiphol Sintunavarat and Poom Kumam. Generalized common fixed point theorems in complex valued metric spaces and applications. *Journal of inequalities and Applications*, 2012(1), 84, 2012.

[8] Pari Amiri and Mohammad Esmael Samei. Existence of Urysohn and Atangana–Baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators. *Chaos, Solitons & Fractals*, 165 ,2022.

[9] Godwin Amechi Okeke, Mujahid Abbas and Manuel De La Sen. Approximation of the Fixed Point of Multivalued Quasi-Nonexpansive Mappings via a Faster Iterative Process with Applications. *Discrete Dynamics in Nature and Society* 2020(1), 2020.

[10] Godwin Amechi Okeke and Cyril Ifeanyichukwu Ugwuogor. Iterative construction of the fixed point of Suzuki's generalized nonexpansive mappings in Banach spaces. *Fixed Point Theory* 23.2 (2022): 633-652.

[11] Godwin Amechi Okeke and Mujahid Abbas. Fejer monotonicity and fixed point theorems with applications to a nonlinear integral equation in complex valued Banach spaces. *Applied General Topology*, 21(1), 135-158, 2020.

[12] Siegfried Gahler. 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten*, 26, 115-148, 1963.

[13] Siegfried Gahler. Lineare 2-normierte Räume. *Mathematische Nachrichten*, 28, : 1-43, 1964.

[14] Ali Abou Bakr and Sahar Mohamed. On various types of cone metric spaces and some applications in fixed point theory. *International Journal of Nonlinear Analysis and Applications*, 14(1), 163-184, 2023.

[15] Thabet Abdeljawad, Duran Turkoglu and Muhib Abuloha. Some theorems and examples of cone Banach spaces. *J. Comput. Anal. Appl*, 12(4), 739-753, 2010.

[16] Bapurao C Dhage. Generalized metric spaces and topological structure I. *Analele Atintifice ale Universitatii Al. I. Cuza din lasi. Serie Noua Mathematica*, 46(3), 2000.

[17] H. Gunawan and M. Mashadi. On n-normed spaces. *International journal of mathematics and mathematical sciences*, 27(10), 631-639, 2001.

[18] Zead Mustafa and Brailey Sims. A new approach to generalized metric spaces. *Journal of Nonlinear and convex Analysis*, 7(2), 289, 2006.

[19] Kamran Alam Khan. Generalized normed spaces and fixed point theorems. *Journal of Mathematics and Computer Science*, 13, 157-167, 2014.

[20] Riya Deyand Tarapada Bag. Some Topological Results on Complex-Valued Fuzzy Normed Linear Spaces. New Mathematics and Natural Computation, 20(02), 365-381, 2024.