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Dom-Chromatic Number of join of path with Ladder related Graphs

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Abstract

The Dom - chromatic number of a graph brings together the two fundamental ideas of graph theory namely domination and coloring. By merging these two concepts, it gives a strong tool for studying and interpreting complex network structures across different applications. A proper vertex coloring of a graph $G(V, E)$ is an assignment of colors to its vertices so that no two adjacent vertices have the same color. A group of vertices with the same color forms a color class. A non-empty subset S of the vertex set V of G is said to be a dominating set if for every vertex v in $V - S$, there is a vertex u in S such that u is adjacent to v. A dominating set is said to be a dom coloring set if it contains at least one vertex from each color class. The minimum cardinality of a dom-coloring set is said to be its dom chromatic number and is denoted by $\gamma_{dc}(G)$. The join of two graphs G_1, G_2 denoted as $G_1 + G_2$, is a graph formed by taking the disjoint union of G_1 and G_2 and then adding an edge between each vertex of G_1 to every vertices of G_2. In this paper, we determine the dom chromatic number of join of path with ladder, open ladder, triangular ladder, open triangular ladder, Mobius ladder and circular ladder graphs.

Keywords: Join of graphs, Dom chromatic number, Dom coloring set, Dominating set.

1. INTRODUCTION

Graph Theory has emerged as a powerful tool for analyzing complex structures in Mathematics and real-world systems. The two important key concepts of graph theory are domination and coloring, both of which have been extensively studied for their theoretical depth and practical relevance. Domination ensures that all vertices in a graph are covered through a strategically chosen subset, while coloring organizes the graph into distinct classes that avoid conflicts between adjacent vertices. Francis Guthrie introduced graph coloring in 1852 through the Four Color Problem, asking if any planar graph can be colored with four colors so that adjacent regions differ. Arthur Cayley published the first academic study on graph coloring in 1879, focusing on map coloring [1]. He credited his student Augustus De Morgan, who had heard the problem from Francis Guthrie in 1852. In 1962, Ore introduced domination conditions in his book Theory of Graphs [14]. In recent years, various graph-theoretic concepts merging domination and coloring have emerged. One such problem is called dom coloring problem where the dom chromatic number is dertermined by selecting minimum dom coloring set, that satisfies certain condition called DC - condition. The dom-coloring problem was introduced in 2010 by T. N. Janakiraman and M. Poobalaranjani by defining the dom-coloring set S, such that $\chi(S) = \chi(G)$ [6] . Work on the dom-chromatic number of cycle-related graphs was reported in 2020 [16], and for splitting graphs of path and comb graphs in 2021 [9] for caterpillar, coconut tree, lobster, tadpole, pan and lollipop graphs, mobius ladder and circular ladder in 2022 [10, 11], for wrapped butterfly & bloom graphs in 2023 [12]. In many practical scenarios, it is not enough to simply dominate or simply color a network. Instead, one needs a subset of controllers (dominating set) that also interacts with every functional group (color class). This combined parameter provides a stronger framework for analyzing structural properties and has potential applications in areas such as communication networks, optimization problems, social or biological systems where both coverage and conflict-free assignment are essential in such applications. In this paper the dom-chromatic number for the join of a path with ladder, open ladder, triangular ladder, open triangular ladder, Mobius ladder and circular ladder graphs are determined. These applications motivated us to carry our research in this field.

2. PRELIMINARIES

This section focuses on some basic definitions and results on dom chromatic number.

Definition 2.1. [1] *A proper vertex coloring of G assigns different colors to adjacent vertices. A set of vertices sharing a color is a color class.*

Definition 2.2. [1] *The chromatic number $\chi(G)$) is the minimum number of colors needed* *to color G so that adjacent vertices differ.*

Definition 2.3. [14] *A non-empty subset $D' \subset V(G)$ is a dominating set if every $v \in V(G) - D'$ is adjacent to some $u \in D'$. If D' has the minimum vertices, then it is the minimum dominating set, whose size is the domination number denoted by $\gamma(G)$.*

Definition 2.4. [6] ***DC - condition*** *In a k -coloring of G, a dominating set D' is a dom-coloring set if $D' = \{v \in V(G)/v$ belongs to atleast one color class $C_i, 1 \leq i \leq k$, and $D' \cap C_i \neq \phi, \forall i\}$. This is the dom-coloring condition (DC-condition). The size of the smallest such set is the dom-chromatic number, denoted by $\gamma_{dc}(G)$.*

Definition 2.5. [5] *The join of two disjoint graphs G_1 & G_2 denoted as $G_1 + G_2$, is a graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to every vertices of G_2.*

Definition 2.6. [2] *A walk is a finite sequence $W = v_o e_1 v_1 e_2 v_2 \ldots, e_k v_k$, whose terms are alternately vertices $v_i, 0 \leq i \leq k$ and edges $e_i, 1 \leq i \leq k$ such that, the ends of e_i are v_{i-1} and v_i. If the vertices $v_0, v_1, v_2, \ldots, v_k$ are distinct, W is a path.*

Definition 2.7. [13] *The ladder graph L_n is defined as $L_n = P_n \times K_2$ where P_n is a path with n vertices, K_2 is a complete graph with two-vertices and $'\times'$ denotes the operation cartesian product.*

Definition 2.8. [15] *Open ladder $O(L_n), n \geq 2$ is obtained from two paths of length $n-1$ with $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup u_i v_i : 2 \leq i \leq n-1$.*

Definition 2.9. [7] *A triangular ladder $TL_n, n \geq 2$ is a graph obtained from L_n by adding the edges $u_i v_{i+1}, 1 \leq i \leq n–1$. The vertices of L_n are u_i and v_i. u_i and v_i are the two paths in the graph L_n where $i = \{1, 2, 3, \ldots, n\}$.*

Definition 2.10. [15] *An open Triangular ladder $O(TL_n), n \geq 2$ is obtained from an open ladder $O(L_n)$ by adding the edges $u_i v_{i+1}, 1 \leq i \leq n-1$.*

Definition 2.11. [8] *A Mobius ladder graph M_n is a cubic graph with $2n$ vertices and $3n$ edges obtained from the ladder $P_n \times P_2$ by joining together the opposite end points of the two copies of P_n.*

Definition 2.12. [4] *A circular ladder graph $CL(n)$ is defined as the cartesian product $C_n \times K_2$ where K_2 is the complete graph on two vertices and C_n is the cycle graph on n vertices.*

Theorem 2.1. [3] *For any graph G,*

$$\max\{\gamma(G), \chi(G)\} \leq \gamma_{dc}(G) \leq \gamma(G) + \chi(G) - 1.$$

3. MAIN RESULTS

This section deals with ladder related graphs. More specifically, it deals with the join of a path with ladder, open ladder, triangular ladder, open triangular ladder, mobius ladder, circular ladder graphs. Here, we determine the dom chromatic number for the resulting graphs. Also the bound is proved to be sharp.

Theorem 3.1. *For $m, n \geq 2$, the dom chromatic number of $P_m + L_n$ is $\gamma_{dc}(P_m + L_n) = 4$.*

Proof. Let P_m be a path graph on m vertices and L_n be a ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + L_n$ of two disjoint graphs P_m & L_n, is a graph derived from $P_m \cup L_n$ and then joining edges from each vertex of P_m to every vertices of L_n. This graph $P_m + L_n$ has $m + 2n$ vertices and $2mn + m + 3n - 3$ edges.

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + L_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly untill all the vertices are colored. Color the vertices of L_n in $P_m + L_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq m\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq m\}$. It ensures that no two adjacent vertices in $P_m + L_n$ receives the same color. Refer Figure 1. Hence, the graph $P_m + L_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + L_n)$. For every vertex in $V(P_m + L_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set, This set D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2\}$ which contains minimum of one vertex from every color class of $P_m + L_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + L_n) = 4$. Refer Figure 1.

□

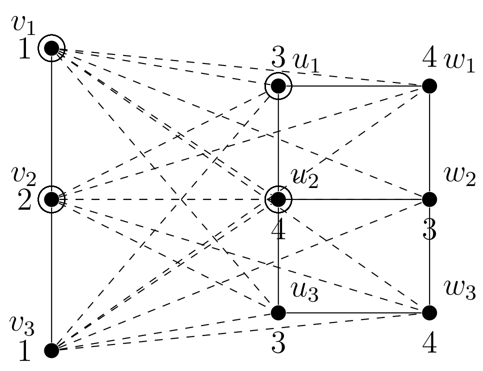

Figure 1: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2\}$, $\gamma_{dc}(P_3 + L_3) = 4$.

Theorem 3.2. *For $m, n \geq 2$, the dom chromatic number of $P_m + ML_n$ is*

$$\gamma_{dc}(P_m + ML_n) = \begin{cases} 4, & n \text{ is odd} \\ 6, & n \text{ is even} \end{cases}$$

Proof. Let P_m be a path graph on m vertices and ML_n be a Mobius ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + ML_n$ of two disjoint graphs P_m & ML_n, is a graph derived from $P_m \cup ML_n$ and then joining edges from each vertex of P_m to every vertices of ML_n. This graph $P_m + ML_n$ has $m + 2n$ vertices and $2mn + m + 3n - 1$ edges.

Case (i): $m \geq 2, n$ **is odd**

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + ML_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly until all the vertices are colored. Color the vertices of ML_n in $P_m + ML_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq n\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq n\}$. It ensures that no two adjacent vertices in $P_m + ML_n$ receives the same color. Refer Figure 3. Hence, the graph $P_m + ML_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + ML_n)$. For every vertex in $V(P_m + ML_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2\}$ which contains minimum of one vertex from every color class of $P_m + ML_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + ML_n) = 4$. Refer Figure 3.

Case (ii): $m \geq 2, n$ **is even**

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + ML_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly until all the vertices are colored. Color the vertices of ML_n in $P_m + ML_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq n-1\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq n-1\}$ and use the color 5 & 6 for the vertices u_n & w_n respectively. It ensures that no two adjacent vertices in $P_m + ML_n$ receives the same color. Refer Figure 2. Hence, the graph $P_m + ML_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + ML_n)$. For every vertex in $V(P_m + ML_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2, w_n\}$ which contains minimum of one vertex from every color class of $P_m + ML_n$.

That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + ML_n) = 5$. Refer Figure 2.

□

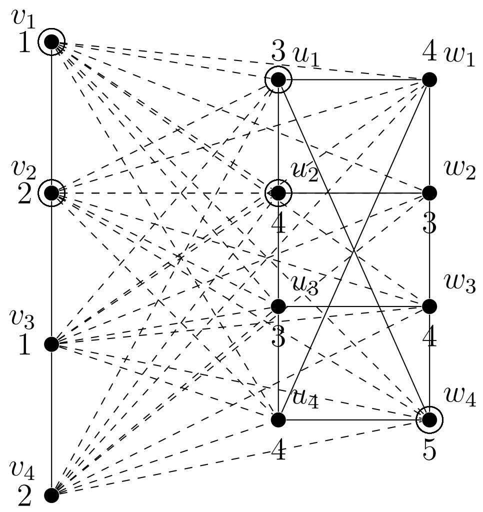

Figure 2: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2, w_5\}$, $\gamma_{dc}(P_4 + ML_4) = 5$.

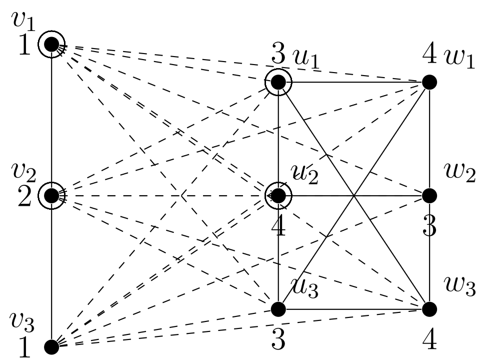

Figure 3: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2\}$, $\gamma_{dc}(P_3 + ML_3) = 4$.

Theorem 3.3. *For $m, n \geq 2$, the dom chromatic number of $P_m + CL_n$ is*

$$\gamma_{dc}(P_m + ML_n) = \begin{cases} 4, & n \; is \; even \\ 6, & n \; is \; odd \end{cases}$$

Proof. Let P_m be a path graph on m vertices and CL_n be a circular ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + CL_n$ of two disjoint graphs P_m & CL_n, is a graph derived from $P_m \cup CL_n$ and then joining edges from each vertex of P_m to every vertices of CL_n. This graph $P_m + CL_n$ has $m + 2n$ vertices and $2mn + m + 3n - 1$ edges.

Case (i): $m \geq 2, n$ **is even**

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + CL_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly until all the vertices are colored. Color the vertices of CL_n in $P_m + CL_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq n\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq n\}$. It ensures that no two adjacent vertices in $P_m + CL_n$ receives the same color. Refer Figure 4. Hence, the graph $P_m + CL_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + CL_n)$. For every vertex in $V(P_m + CL_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2\}$ which contains minimum of one vertex from every color class of $P_m + CL_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + CL_n) = 4$. Refer Figure 4.

Case (ii): $m \geq 2, n$ **is odd**

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + CL_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly until all the vertices are colored. Color the vertices of CL_n in $P_m + CL_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq n-1\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq n-1\}$ and use color 5 & 6 for the vertices u_n & w_n respectively. It ensures that no two adjacent vertices in $P_m + CL_n$ receives the same color. Refer Figure 5. Hence, the graph $P_m + CL_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + CL_n)$. For every vertex in $V(P_m + CL_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2, u_n, w_n\}$which contains minimum of one vertex from every color class of $P_m + CL_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + CL_n) = 6$. Refer Figure 5.

□

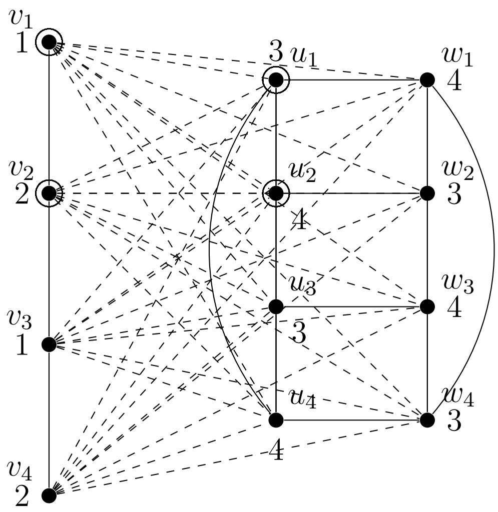

Figure 4: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2\}$, $\gamma_{dc}(P_4{+}CL_4) = 4$.

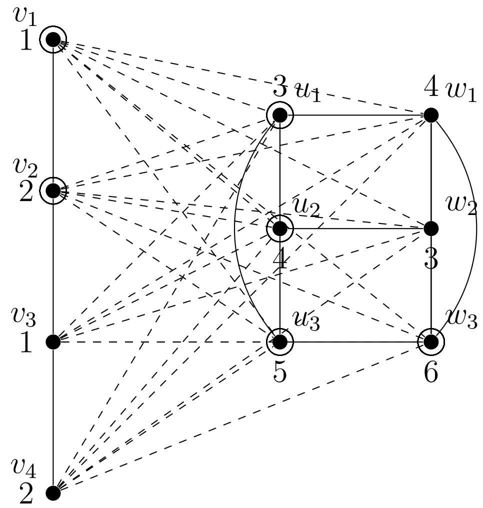

Figure 5: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2, u_3, w_3\}$, $\gamma_{dc}(P_4 + CL_3) = 6$.

Theorem 3.4. *For* $m, n \geq 2$*, the dom chromatic number of* $P_m + OL_n$ *is* $\gamma_{dc}(P_m + OL_n) = 4$.

Proof. Let P_m be a path graph on m vertices and OL_n be a open ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + OL_n$ of two disjoint graphs P_m & OL_n, is a graph derived from $P_m \cup OL_n$ and then joining edges from each vertex of P_m to every vertices of OL_n. This graph $P_m + OL_n$ has $m + 2n$ vertices and $2mn + m + 3n - 5$ edges.

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + OL_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly untill all the vertices are colored. Color the vertices of OL_n in $P_m + OL_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq m\}$ and using the colors 4, 3 in the order $\{w_j, 1 \leq j \leq m\}$. It ensures that no two adjacent vertices in $P_m + OL_n$ receives the same color. Refer Figure 6. Hence, the graph $P_m + OL_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + OL_n)$. For every vertex in $V(P_m + OL_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2\}$ which contains minimum of one vertex from every color class of $P_m + OL_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + OL_n) = 4$. Refer Figure 6.

□

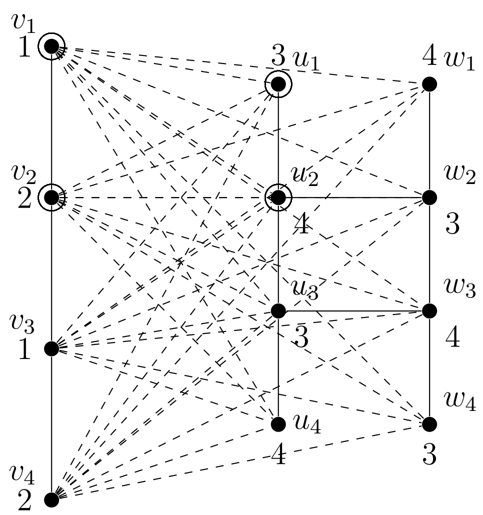

Figure 6: Encircled vertices form dom coloring set $\{v_1, v_2, u_1, u_2\}$, $\gamma_{dc}(P_4 + OL_4) = 4$.

Theorem 3.5. *For $m, n \geq 2$, the dom chromatic number of $P_m + TL_n$ is $\gamma_{dc}(P_m + TL_n) = 5$.*

Proof. Let P_m be a path graph on m vertices and TL_n be a triangular ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + TL_n$ of two disjoint graphs P_m & TL_n, is a graph derived from $P_m \cup TL_n$ and then joining edges from each vertex of P_m to every vertices of TL_n. This graph $P_m + TL_n$ has $m + 2n$ vertices and $2mn + m + 4n - 4$ edges.

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + TL_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly untill all the vertices are colored. Color the vertices of TL_n in $P_m + TL_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq m\}$ and using the colors 4, 5 in the order $\{w_j, 1 \leq j \leq m\}$. It ensures that no two adjacent vertices in $P_m + TL_n$ receives the same color. Refer Figure 7. Hence, the graph $P_m + TL_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + TL_n)$. For every vertex in $V(P_m + TL_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2, w_2\}$ which contains minimum of one vertex from every color class of $P_m + TL_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + TL_n) = 5$. Refer Figure 7.

□

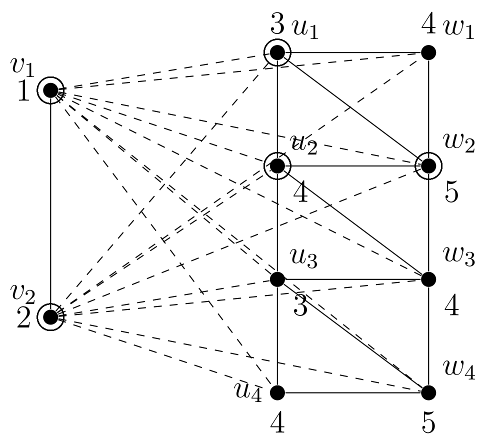

Figure 7: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2, w_2\}$, $\gamma_{dc}(P_2 + TL_4) = 5$.

Theorem 3.6. *For any m & $n = 2$, the dom chromatic number of $P_m + OTL_3$ is $\gamma_{dc}(P_m + OTL_2) = 4$.*

Proof. The proof is obvious. □

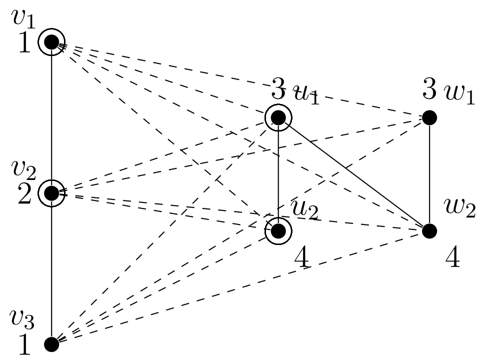

Figure 8: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2\}$, $\gamma_{dc}(P_3+OTL_2) = 4$.

Theorem 3.7. *For any* m & $n = 3$*, the dom chromatic number of* $P_m + OTL_3$ *is* $\gamma_{dc}(P_m + OTL_3) = 5$.

Proof. The proof is obvious. □

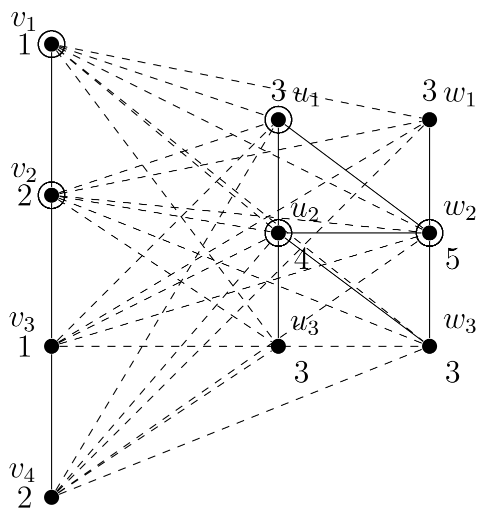

Figure 9: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2, w_2\}$, $\gamma_{dc}(P_4 + OTL_3) = 5$.

Theorem 3.8. *For* $m \geq 2$ & $n \geq 4$*, the dom chromatic number of* $P_m + OTL_n$ *is* $\gamma_{dc}(P_m + OTL_n) = 6$.

Proof. Let P_m be a path graph on m vertices and OTL_n be a open triangular ladder graph on $2n$ vertices, labelled as $\{v_i, 1 \leq i \leq m\}$ & $\{u_j, w_j, 1 \leq j \leq n\}$ respectively.

The join $P_m + OTL_n$ of two disjoint graphs P_m & OTL_n, is a graph derived from $P_m \cup OTL_n$ and then joining edges from each vertex of P_m to every vertices of OTL_n. This graph $P_m + OTL_n$ has $m + 2n$ vertices and $2mn + m + 3n - 4$ edges.

Clearly, P_m is 2 colorable. Color the vertices of P_m in $P_m + OTL_n$ using two colors namely 1, 2 in the order $\{v_i, 1 \leq i \leq m\}$ repeatedly until all the vertices are colored. Color the vertices of OTL_n in $P_m + OTL_n$ using two colors namely 3, 4 in the order $\{u_j, 1 \leq j \leq n\}$ and using the colors 5, 6 in the order $\{w_j, 1 \leq j \leq n\}$. It ensures that no two adjacent vertices in $P_m + OTL_n$ receives the same color. Refer Figure 10. Hence, the graph $P_m + OTL_n$ is properly colored.

Let $D = \{v_1, u_1\}$ be a subset of $V(P_m + OTL_n)$. For every vertex in $V(P_m + OTL_n) - D$ there exist a vertex in D, which are adjacent to each other. Hence, D is a minimum dominating set. This D doesn't satisfy DC-condition. Hence, choose $D' = D \cup \{v_2, u_2, w_1, w_2\}$ which contains minimum of one vertex from every color class of $P_m + OTL_n$. That is, D' satisfies DC- condition. Therefore, D' is a dom – coloring set with cardinality $\gamma_{dc}(P_m + OTL_n) = 6$. Refer Figure 10.

□

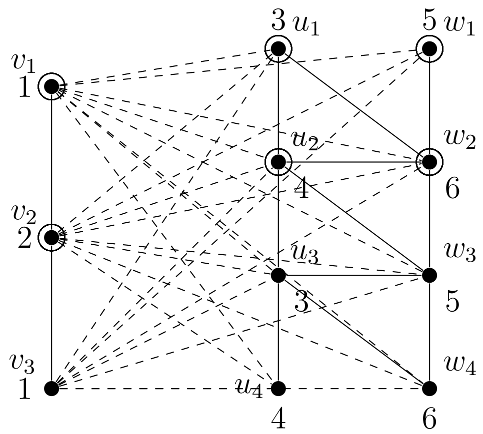

Figure 10: Encircled vertices form a dom coloring set $\{v_1, v_2, u_1, u_2, w_1, w_2\}$, $\gamma_{dc}(P_3 + OTL_4) = 6$.

4. APPLICATION

The dom-chromatic number of the join of a path with ladder related graph can be illustrated using a transportation setting. Let the path graph P_m with vertex set $\{v_i, 1 \leq i \leq m\}$ represent a series of railway stations $\{v_i\}$ and the ladder graph L_n $\{u_i, w_i, 1 \leq i \leq n\}$, represent a pedestrian bridge with two parallel walkways $\{u_i, w_i\}$ connected by steps $\{u_i w_i\}$. In the join $P_m + L_n$, every station $\{v_i\}$ is adjacent to all bridge points $\{u_i, w_i\}$, giving rise to complete interconnection. The dom-chromatic number in this case refers to the minimum cardinality of a dom coloring set, which is a dominating set that includes at least one vertex from each color class. In the transportation analogy, it corresponds to identifying the smallest collection of properly colored stations and bridge points such that every location is under surveillance and each color class contributes at least one dominating vertex, thereby ensuring complete monitoring of the system under proper coloring constraints.

5. CONCLUSIONS AND FUTURE WORK

The two major areas of Graph Theory, domination and coloring are integrated to study the dom-coloring problem of the join of a path with ladder, open ladder, triangular ladder, open triangular ladder, Mobius ladder and circular ladder graphs. We have determined the dom chromatic number of these resulting graphs and proved that the bound is sharp. Exploring the dom-chromatic number in this study can also be extended to other graph operations, such as the lexicographic and tensor products, which remains open for future work.

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