

On The Probability That An Element of Metacyclic 2-Group of Positive Type Fixes A Set and Its Generalized Conjugacy Class Graph

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Abstract

The probability that an element of a group fixes a set is considered as one of the extensions of commutativity degree that can be obtained by some group actions on a set. We denote G as a metacyclic 2-group of positive type of nilpotency of class at least three and Ω as the set of all subsets of all commuting elements of G of size two in the form of (a, b) , where a and b commute and each of order two. In this paper, we compute the probability that an element of G fixes a set in which G acts regularly on Ω . Then the results are applied to graph theory, more precisely to generalized conjugacy class graph.

Introduction

Throughout this paper, Γ denotes a simple undirected graph and G denotes a finite non-abelian group. In the following, the definition of the commutativity degree is stated.

Definition 1.1

[1] Let G be a finite non-abelian group. Suppose that x and y are two random elements of G . The probability that two random elements commute (also known as the commutativity degree) is given as follows.

$$P(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

The first investigation of the commutativity degree of symmetric groups was done in 1968 by Erdos and Turan [2]. Few years later, Gustafson [3] and MacHale [4] found an upper bound for the commutativity degree of all finite non-abelian groups in which $P(G) \leq \frac{5}{8}$. The concept of the commutativity degree has been generalized and extended by several authors. In 1975, a new concept was introduced by Sherman [5], namely the probability of an automorphism of a finite group fixes an arbitrary element in the group. The definition of this probability is given as follows:

Definition 1.2

[5] Let G be a group. Let X be a non-empty set of G (G is a group of permutations of X). Then the probability of an automorphism of a group fixes a random element from X is defined as follows:

$$P_G(X) = \frac{|\{(g, x) | gx = x, \forall g \in G, x \in X\}|}{|G||X|}.$$

In 2011, Moghaddam [6] explored Sherman's definition and introduced a new probability, which is called the probability of an automorphism fixes a subgroup element of a finite group, the probability is stated as follows:

$$P_{A_G}(H, G) = \frac{|\{(\alpha, h) | h^\alpha : h \in H, \alpha \in A_G\}|}{|H||G|},$$

where A_G is the group of automorphisms of a group G . It is clear that if $H = G$, then $P_{A_G}(G, G) = P_{A_G}$.

Recently, Omer *et al* [7] extended the work in [5] by defining the probability that a group element fixes a set. In this paper, the probability that a group element fixes a set is found for metacyclic 2-groups of positive type of nilpotency class at least three.

Next, we state some basic concepts in group theory that are needed in this paper.

Definition 1.3

[8] Suppose G is a finite group that acts on a set S . A group G acts regularly on S if the action is transitive and $\text{Stab}_G(s) = 1$ for all $s \in S$. Also, the action is regular if for s_1 and s_2 belong to S , there exists an element $g \in G$ such that $gs_1 = s_2$.

Definition 1.4

[9] Let G acts on a set S and $x \in S$. The orbit of x denoted by $\text{cl}(x)$ is the sub set $\text{cl}(x) = \{gx : g \in G\} \subseteq S$.

Next, we provide some concepts related to metacyclic p -groups. Throughout this paper, p denotes a prime number.

Definition 1.5

[10] A group G is called a metacyclic if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic.

In 1973, King [11] gave the presentation of metacyclic p -groups, as given in the follows:

$$G \cong \langle a, b : a^{p^{\alpha_1}} = 1, b^{p^{\alpha_2}} = a^n, [b, a] = a^m \rangle, \text{ where } \alpha_1, \alpha_2 \geq 0, m > 0, n \leq p^{\alpha_1}, p^{\alpha_1} \mid n(m-1).$$

In 2005, Beuerle [12] separated the classification into two parts, namely for the non-abelian metacyclic p -groups of class two and class at least three. Based on [12], the metacyclic p - groups of nilpotency class two are then partitioned into two families of non-isomorphic p -groups stated as follows:

$$G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-\gamma}} \rangle, \text{ where } \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma \text{ and } \beta \geq \gamma \geq 1.$$

$$G \cong Q_8.$$

Meanwhile, the metacyclic p -groups of nilpotency class of at least three (p is an odd prime) are partitioned into the following groups:

$$G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [b, a] = a^{p^{\alpha-\gamma}} \rangle, \text{ where } \alpha, \beta, \gamma \in \mathbb{N}, \alpha - 1\gamma < 2 \text{ and } \alpha \leq \beta.$$

$$G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b, a] = a^{p^{\alpha-\gamma}} \rangle, \text{ where } \alpha, \beta, \gamma, \varepsilon \in \mathbb{N}, \alpha - 1\gamma < 2\gamma, \alpha \leq \beta \text{ and } \alpha \leq \beta + \varepsilon.$$

Moreover, metacyclic p -groups are also classified into two types, namely negative and positive type [12]. The following notations for these two types which are used in this paper are represented as follows:

$$G(\alpha, \beta, \omega, \gamma, \pm) \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b, a] = a^t \rangle, \text{ where } \alpha, \beta, \gamma, \varepsilon \in \mathbb{N}, t = p^{\alpha-\gamma} \pm 1.$$

If $t = p^{\alpha-\gamma} - 1$, then the group is called a metacyclic of negative type and it is of positive type if $t = p^{\alpha-\gamma} + 1$. Thus, $G(\alpha, \beta, \varepsilon, \gamma, -)$ denotes the metacyclic group of negative type, while $G(\alpha, \beta, \varepsilon, \gamma, +)$ denotes the positive type. These two notations are shortened to $G(p, +)$ and $G(p, -)$, respectively ([11], [12]).

Since this paper focuses on metacyclic p -groups of positive type, thus the following theorem gives the presentations of the two groups of positive type.

Theorem 1.1

[12] Let G be a metacyclic 2-group of positive type of nilpotency class of at least three. Then G is isomorphic to one of the following types:

$$(1.1.1) G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle, \alpha, \beta, \gamma \in \mathbb{N}, \beta \geq \gamma, 1 + \gamma < \alpha < 2\gamma.$$

$$(1.1.2) G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta, \alpha \leq \beta + \varepsilon.$$

The followings are some basic concepts of graph theory that are needed in this paper. These concepts can be found in one of the references ([13], [14]).

A graph Γ is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, Γ is called undirected. Two vertices are adjacent if they are linked by an edge. A complete graph is a graph where each ordered pair of distinct n vertices are adjacent, denoted by K_n , while it is null if there is no vertices in Γ . The graph is called empty if there are no edges linked between its vertices. In this paper, K_e denotes empty graph, while K_0 denotes null graph.

In 1990, a new graph called graph related to conjugacy class was introduced by Bertram *et al.* [15]. The vertices of this graph are non central conjugacy classes i.e $|V(\Gamma)| = K(G) - |Z(G)|$, where $K(G)$ is the number of conjugacy class of a group and $Z(G)$ is the center of a group G . A pair of vertices of this graph are connected by an edge if their cardinalities are not coprime.

In 2013, Omer *et al.* [16] extended the work on the conjugacy class graph by introducing a generalized conjugacy class graph whose vertices are non-central orbits under group action on a set.

This paper is structured as follows: Section 1 provides some fundamental concepts of group theory and graph theory which are used in this paper. In Section 2, we state some of previous works, which are related to the commutativity degree, in particular related to the probability that a group element fixes a set and conjugacy class graph. The main results are presented in Section 3.

Preliminaries

In this section, we provide some previous works related to the commutativity degree, more precisely to the probability that an element of a group fixes a set and graph theory, in particular to graph related to conjugacy class.

Recently, Omer *et al.* [7] extended the commutativity degree by defining the probability that an element of a group fixes a set of size two, given in the following.

Definition 2.1

[7] Let G be a group. Let S be a set of all subsets of commuting elements of G of size two. If G acts on S , then the probability of an element of a group fixes a set is defined as follows:

$$P_G(S) = \frac{|\{(g, s) \mid gs = s : g \in G, s \in S\}|}{|S||G|}.$$

Theorem 2.1

[7] Let G be a finite group and let X be a set of elements of G of size two in the form of (a,b) where a and b commute. Let S be the set of all subsets of commuting elements of G of size two and G acts on S . Then the probability that an element of a group fixes a set is given by

$$P_G(S) = \frac{K(S)}{|S|}, \text{ where } K(S) \text{ is the number of orbits of } S \text{ in } G.$$

In 2014, Mustafa *et al.* [17] has extended the work in [7] by restricting the order of Ω . The following theorem illustrates their results.

Theorem 2.2

[17] Let G be a finite group and let S be a set of elements of G of size two in the form of (a,b) , where a,b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω . Then the probability that an element of a group fixes a set is given by $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the

number of conjugacy classes of Ω in G .

Recently, Mustafa *et al.* [18] obtained the probability that an element of a group fixes a set for metacyclic 2-groups of negative type of nilpotency class two and class at least three. The probability is also computed under conjugate action for metacyclic 2-groups of positive type of nilpotency class at least three [19]. In addition, Mustafa *et al.* [20] found the probability that a group element fixes a set for semi-dihedral groups and quasi-dihedral groups.

In this paper, the probability that a positive type metacyclic 2-group element fixes a set is computed. Furthermore, the obtained results are then applied to graph theory by using the orbits that are obtained under group action on a set to graph conjugacy class.

Some related works on conjugacy class graph include Bianchi *et al.* [21] who studied the regularity of the graph related to conjugacy classes. In addition, Moreto *et al.* [22] in 2005 classified the groups in which conjugacy classes sizes are not coprime for any five distinct classes. Furthermore, You *et al.* [23] also classified the groups in which conjugacy classes are not set-wise relatively prime for any four distinct classes. Moreover, Moradipour *et al.* [24] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups.

Recently, Omer *et al.* [16] introduced generalized conjugacy class graph, given in the following definition.

Definition 2.2

[16] Let G be a finite group and Ω a set of G . Let A be the set of commuting element in Ω , i.e $\{\omega \in \Omega: \omega g = g\omega, g \in G\}$. Then the generalized conjugacy class graph Γ_G^Ω is defined as a graph whose vertices are non-central orbits under group action on a set,

that is $|V(\Gamma_G^{\Omega_c})| = K(\Omega) - |A|$. Two vertices ω_1 and ω_2 in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime, i.e. $\gcd(\omega_1, \omega_2) \neq 1$.

Moreover, the generalized conjugacy class graph is determined for the symmetric groups and alternating groups [25].

Main Results

This section provides our main results. First, the probability that an element of metacyclic 2-groups of positive type of nilpotency class at least three fixes a set is computed. Then the second part relates the obtained results to generalized conjugacy class graph.

Throughout this section, let S be a set of elements of G of size two in the form of (a, b) , where a and b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by regular action.

The Probability That a Metacyclic 2-Group Element Fixes a Set

Theorem 3.1

Let G be a finite group of type (1.1.1), $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, $\beta \geq \gamma$. If G acts regularly on Ω , then $P_G(\Omega) = 1$.

Proof

If G acts regularly on Ω , then there exists an element $g \in G$ such that $g\omega_1 = \omega_2$ for $\omega_1, \omega_2 \in \Omega$. The elements of order two in G are $a^{2^{\alpha-1}}$, $b^{2^{\beta-1}}$ and $a^{2^{\alpha-1}}b^{2^{\beta-1}}$. Therefore, the elements of Ω are stated as follows: Two elements in the form of $(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}})$, $0 \leq i < 2^\alpha$, and one element in the form $(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}})$. Then $|\Omega| = 3$.

. By regular action, there exists an element $g \in G$ for $\omega_1, \omega_2 \in \Omega$. such that $g\omega_1 = \omega_2$.

The orbits of Ω can be described as follows: One orbit in the form of $(a^{2^{\alpha-1}}, b^{2^{\beta-1}})$, one orbit in the form of $(a^{2^{\alpha-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}})$ and one orbit in the form of $(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}})$.

Thus there are three orbits. Using Theorem 2.2, then $P_G(\Omega) = 1$.

Theorem 3.2

Let G be group of type (1.1.2), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\epsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, $1 + \gamma < \alpha < 2\gamma$, $\gamma \leq \beta$, $\alpha \leq \beta + \epsilon$. If G acts regularly on Ω , then $P_G(\Omega) = 1$.

Proof

If G acts regularly on Ω , then there exists an element $g \in G$ such that $g\omega_1 = \omega_2$ for $\omega_1, \omega_2 \in \Omega$. $g \in G$. Thus, the elements of order two in G are $a^{7 \times 2^{\alpha-4} + 2^{\alpha-1}} b^{2^{\beta-1}}$, $a^{7 \times 2^{\alpha-4}} b^{2^{\beta-1}}$ and $a^{2^{\alpha-1}}$. Hence, the elements of Ω are stated as follows: Two elements are in the form of $(a^{2^{\alpha-1}}, a^{7 \times 2^{\alpha-4} + 2^{\alpha-1} i} b^{2^{\beta-1}})$, $0 \leq i \leq 1$, and there is one element in Ω which are in the form of $(a^{7 \times 2^{\alpha-4}} b^{2^{\beta-1}}, a^{7 \times 2^{\alpha-4} + 2^{\alpha-1}} b^{2^{\beta-1}})$, from which it follows that $|\Omega| = 3$. By regular action, there exists an element $g \in G$ for $\omega_1, \omega_2 \in \Omega$ such that $g\omega_1 = \omega_2$. The orbits of Ω can be described as follows: One orbit in the form of $(a^{2^{\alpha-1}}, a^{7 \times 2^{\alpha-4}} b^{2^{\beta-1}})$, one orbit in the form of $(a^{2^{\alpha-1}}, a^{7 \times 2^{\alpha-4} + 2^{\alpha-1}} b^{2^{\beta-1}})$ and one orbit in the form of $(a^{7 \times 2^{\alpha-4}} b^{2^{\beta-1}}, a^{7 \times 2^{\alpha-4} + 2^{\alpha-1}} b^{2^{\beta-1}})$. Thus there are three orbits. Using Theorem 2.2, then $P_G(\Omega) = 1$.

The Generalized Conjugacy Class Graph

In this section, the results are related to generalized conjugacy class graph. First, the generalized conjugacy class graph of metacyclic 2-groups of positive type of nilpotency class of at least three is found, starting with the group of type (1.1.1).

Theorem 3.3

Let G be a finite group of type (1.1.1), $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, $\beta \geq \gamma$. If G acts regularly on Ω , then $\Gamma_G^{\Omega_c} = K_e$.

Proof

Based on Theorem 3.1, there are three orbits all of size one. Using Definition 2.2, $\Gamma_G^{\Omega_c}$ consists of three isolated vertices, which are in the form of $(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}})$, $(a^{2^{\alpha-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}})$ and $(a^{2^{\alpha-1}}, b^{2^{\beta-1}})$. The proof then follows. ■

Theorem 3.4

Let G be group of type (1.1.2), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\epsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, $1 + \gamma < \alpha < 2\gamma$, $\gamma \leq \beta$, $\alpha \leq \beta + \epsilon$. If G acts regularly on Ω , then

$$\Gamma_G^{\Omega_c} = \begin{cases} K_e, & \text{if } \alpha > \beta + 1, \\ K_0, & \text{if } \alpha \leq \beta + 1. \end{cases}$$

Proof

In the first case, the proof is similar to the proof of Theorem 3.3. When $\alpha \leq \beta + 1$ and based on Theorem 3.3 in [19] $P_G(\Omega) = 1$ under conjugate action and by using

Theorem 3.1 in [19] all commuting element of G of order two are in the center of G . Thus, $K(\Omega) = |\Omega| = A$. Using the vertices adjacency of the generalized conjugacy class graph, Γ_G^{Ω} is null. ■

Conclusion

In this paper, the probability that a group element fixes a set under regular action is found for metacyclic 2-groups of positive type of nilpotency class of at least three. The results obtained are then applied to generalized conjugacy class graph.

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