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2-Dominator Coloring in Graphs

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Abstract
The concept of dominator chromatic number was introduced by Gera et al. [4]. A dominator coloring of *G* is a proper coloring in which every vertex of *G* dominates every vertex of at least one color class. The $\chi_d(G)$ dominator chromatic number, is the minimum number of colors, which a dominator coloring of G requires. A *k*-dominator coloring of *G* is a proper coloring in which every vertex of *G* dominates at least *k*color classes. The $\chi_{d,k}(G)$ *k*-dominator chromatic number, minimum number of colors, is required for a *k*-dominator coloring of *G*. In this paper, we significantly concentrate on 2-dominator coloring on circulant graphs. Further, different results were obtained on this coloring parameter.

Key words: Dominator coloring, dominator chromatic number, 2-dominatorchromatic number, circulant graphs.

Introduction

By *G*(*V, E*), we mean a finite and undirected graph *G* with neither loops nor multiple edges, *V* is called as the vertex set of *G* and *E* is called as the edge set of *G*. Coloring of graphs and finding the domination number of graphs are major research areas in graph theory that have been well studied [5], [6]. A proper coloring of a graph G is an assignment of colors to the vertices of *G* in such a way that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the minimum number of colors required for a proper coloring of *G*.

A set $S \subset V$, of vertices in a graph *G* = (*V, E*) is called a dominating set if every vertex $v \in V$ is either an element of *S* or is adjacent to an element of *S*[3]. The domination number $\gamma(G)$ of a graph *G* equals the minimum cardinality of a dominating set in *G*[3] and the corresponding dominating set is called a γ-set. Gera et al. [4] introduced the concept of dominator chromatic number, whichcombines the concept of domination and coloring. A dominator coloringof*G* is a proper coloring in which every vertex of *G* dominates every vertex of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of *G*.

In this paper, we introduce *k*-dominator coloring in graphs. A *k*-dominator coloring of *G* is a proper coloring in which every vertex of *G* dominate at least *k* color classes. The *k*-dominator chromatic number $\chi_{d,k}(G)$ is the minimum number of colors required for a *k*-dominator coloring of *G*. In this paper, we particularly concentrate on *2*-dominator coloring. Note that a graph *G* admit *k*-dominator coloring, then the minimum degree $\delta(G) \geq k-1$ (Since a vertex may dominate itself when it forms a color class). In the case of2-dominator coloring, *G* could not have isolated vertices.

Let Γ be a finite group with e as the identity and *X* be a symmetric generating set(if $a \in X$ then $a^{-1} \in X$) with $e \notin X$. The Cayley graph *G* = Cay(Γ , *X*), where *V (G)* = Γ and *E(G)* = $\{(x, xa)/ x \in V(G), a \in X\}$. Since *X* is a generating set for, *Cay*(Γ , *X*) is a connected and regular graph of degree |*X*|. Cayley graph constructed out of a finite cyclic group of order *n* is calledacirculant graph and it is denoted by *Cir*(*n, X*) where *X* is a generating set of Z_n [2].

Even though Cayley graphs are extensively dealt in various literatures, only few authors have worked on domination in Cayley graphs. To understand the concept of domination for Cayley graphs one can refer to [7], [8], [9], [10]. Throughout this paper, *n* is a natural number, $Z_n = \{0,1,2,.........,n-1\}$ and *G* =*Cir*(*n, X*), where *X* is a generating set given by $X = \{1,2,..........,x,n-x,.....,n-2,n-1\}$ where $1 \leq x \leq \frac{n-1}{2}$ and the binary operation $\oplus_n$ is the addition modulo *n* in Z_n. Any finite cyclic group is isomorphic to Z_n with respect to the operation $\oplus_n$, addition modulo *n*.

We also need the following results to analyse the paper further.

Theorem 1. 1

Let *G* = *Cay*(Z_n , *A*)where $A = \{1, n-1, n-2, ..., k, n-k\}$ and *n, k* are positive integers with $1 \leq k \leq \frac{n-1}{2}$. Then $\gamma(G) = \left\lceil \frac{n}{|A|+1} \right\rceil$.

Further $D = \{0,(2k+1),2(2k+1),3(2k+1),.........,(l-1)(2k+1)\}$ where $l = \left\lceil \frac{n}{|A|+1} \right\rceil$, is a γ-set of G[2].

Theorem1. 2

Let G be a connected graph of order n. Then $\chi_d(G) = n$ if and only if G is the complete graph K_n[1].

Theorem 1. 3

Let Gbe a connected graph. Then [1]

$$max\{\chi(G),\gamma(G)\} \le \chi_d(G) \le \chi(G)+\gamma(G)$$

Dominator Coloring Number In A Class of Circulant Graphs

The below result shows that a collection of circulant graphs reaches the upper bound.

Theorem 2. 1

Let $G=Cir(n, X)$ where $X = \{1,2,.........,x,n-x,.....,n-2,n-1\}$. Then

$$\chi_d(G) \le \left\lceil \frac{n}{|A|+1} \right\rceil + 2x.$$

Proof.

Note that $|X| = 2x$ and let $M = \left\lceil \frac{n}{|X|+1} \right\rceil = \left\lceil \frac{n}{2x+1} \right\rceil$. First we color all these vertices of $C = \{0,(2x+1),2(2x+1),3(2x+1),.........,(M-1)(2x+1)\}$ by M different colors. The remaining vertices are colored with $2x$ colors namely, 1, 2, . . . , $2x$ as follows:

Let $v \in V(G) - C$. By using the division algorithm, we can write $c = (2x + 1)a + b$ for some a and b satisfying $0 \le a \le M - 1$ and $1 \le b \le 2x$. We have the following cases:

Case 1: When $x \le b \le 2x$.

In this case c(v) = v mod(2x + 1).

Case 2: When $1 \le b < x$.

Subcase 2. 1: If $0 \le a \le M - 2$, then

$c(v) = v\mathrm{mod}(2x + 1)$.

Subcase 2. 2: If $a = M - 1$, then

$c(v) = v\mathrm{mod}(2x + 1) + x$.

Note that, for any u,

$N[u] = \{u, u+1, u+2, . , u+x, u+1, u+2, . . . , u+(n-x)\}$. Thus the above coloring is a proper coloring with $M + 2x = \left\lceil \frac{n}{|X|+1} \right\rceil + 2x$ colors. It remains to prove thatthe abovecoloring is a dominator coloring.

In Theorem1. 1, it is proved that $C = \{0,(2x+1),2(2x+1),3(2x+1),.........,(M-1)(2x+1)\}$ is a dominating set. We use different colors for all these vertices and those colors are not used to color any other vertices not in C. Thus all the vertices in $V(G) - C$ adjacent to at least one vertex of C; that is adjacent to a color class. Thus the above coloring is a dominator coloring.

Theorem2. 2

Let $G=Cir(n, X)$where $X = \{1,2,.........,x,n-x,.....,n-2,n-1\}$.Suppose n divides x+1 and 2x+1 then $\chi_d(G) = \chi(G) + \gamma(G)$.

Proof.

By Theorem 1. 3, it is enough to show that $\chi_d(G) \ge \chi(G) + \gamma(G)$. Since any consecutive $x + 1$ vertices of$G$ are mutually adjacent, we have $\chi(G) \ge x + 1$. Also, by coloring the vertices of G by $c(v) = v \pmod{(x + 1)}$, we can get a proper coloring and hence $\chi(G) \le x + 1$. Thus $\chi(G) = x + 1$. ByTheorem 1. 1, we have $\gamma(G) = \left\lceil \frac{n}{2x+1} \right\rceil$. Consider the 2$x$ + 1 vertices C = {1, 2, . . . , 2x + 1}. Since 1, 2, …. , x + 1 are mutually adjacent we should color all these vertices by different colors. Now to make the vertex x + 1 having adjacency with a color class, there are two ways. One way is notusing the color of the vertex x + 1 anywhere else in the graph.

Otherwise we should not use atleast one of the colors 1, 2, . . . , x + 1 to color the vertices outside C. This will happen for every 2x+1 vertices. Since n divides 2x+1. Thus $\gamma(G) \ge (x+1) + \left\lceil \frac{n}{2x+1} \right\rceil = \chi(G) + \gamma(G)$.

2-Dominator Coloring

In this section, we study the 2-Dominator Coloring number. Note that for any graph of order n ($\ge$ 2), $2 \le \chi_{d,2}(G) \le n$. From the definition of complete graph and 2-Dominator Coloring number, the next result follows.

Theorem 3. 1

Let G be a connected graph of order n($\ge$ 2), then $\chi_{d,2}(G) = n$ if and only if G is the complete graph K_n.

Proof.

Since $\chi_{d,2}(G) = n$, in any 2-dominator coloring all the vertices receive different colors. Hence G is complete. The converse is trivial.

Theorem 3. 2

Let G be a connected graph of order n($\ge$ 2). Then $\chi_{d,2}(G) = 2$ if and only if G is isomorphic to K_2.

Proof.

Suppose $\chi_{d,2}(G) = 2$. LetC_1, C_2 be the two color classes obtained from a 2-dominator coloring of G. Let $v \in C_1$. The v is adjacent with two color classes. Suppose the number of elements in C_1 exceeds one then v cannot dominate the color class C_1 and hence the coloring is not a 2-Dominator Coloring. Thus C_1 and C_2 have exactly one element each and hence G is isomorphic to K_2. The converse follows from the previous theorem.

Theorem 3. 3

Let G be a connected graph such that $\chi_{d,2}(G)=3$ and there are atleast two vertices in each of the corresponding three color classes. Then G is a complete tripartite graph.

Proof.

Let G be a connected graph with $\chi_{d,2}(G)=3$ and C_1, C_2, C_3 be three color classes obtained from a 2-Dominator coloring of G. Let $u \in C_1$. Since there exists another vertex $v(\neq u)$ in C_1, u cannot dominate the color class C_1. Hence u must be adjacent with all the vertices of C_2 and C_3. Similarly, every vertex is adjacent to all the vertices of all the other color classes. This means that G is a complete tripartite graph.

Remark 3. 4

a) The converse of the above theorem need not be true. For example, consider the following graph G.

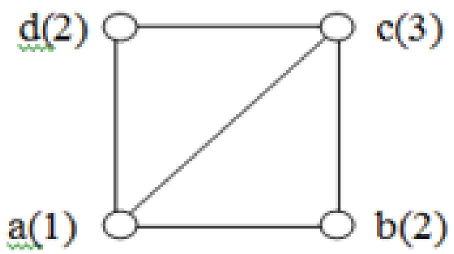

Fig. 1. G

In the above figure, if we take $C_1 = \{a\}$, $C_2 = \{b, d\}$ and $C_3 = \{c\}$. Then G is a complete tripartite graph. Since there are three mutually adjacent vertices, $\chi_{d,2}(G)=3$ (a 3-Dominator coloringhas been given in the Fg. 1. by using 3 colors, namely 1, 2 and 3). But each of C_1 and C_3 have only one vertex.

b) Let G be a connected graph such that $\chi_{d,2}(G)=3$ and each of the color classes having exactly one vertex. Then G is $K_3$and hence G is complete tripartite.

Theorem 3. 5

Let G be a connected graph such that $\chi_{d,2}(G)=3$ such that exactly two color classes contain only one vertex. Then $G = K_{1,1,C}$ or $G = K_{1,1,C}-\{e\}$, where e is only edge connecting the two partitions having one vertex.

Proof.

Let G be a connected graph such that $\chi_{d,2}(G)=3$. Let C_1, C_2, C_3 be the corresponding color classes of G such that $|C_1|=1, |C_2|=1$ and $|C_3|=c$. By hypothesis G is a tripartite graph with partitions C_1, C_2 and C_3. Also each of C_1 and C_2 contain one element. Let $v \in C_3$. Since C_3has atleast two vertices, v will not dominate the color class C_3. Thus v must be adjacent with all the vertices of C_1 and C_2. Thus every vertex of C_3will dominate C_1 and C_2. Also this will make that every vertex of C_1 to dominate C_1 and C_3 as well as every vertex of C_2 to dominate C_2 and C_3. This means that $G = K_{1,1,c}$or $G = K_{1,1,c} - \{e\}$, where e is only edge connecting the two partitions C_1 and C_2.

Theorem 3. 6

Let G be a connected graph such that $\chi_{d,2}(G)=3$ and exactly one color class contain only one vertex. Then $G = K_{1,b,c}$ for some integers $b, c \geq 2$.

Proof.

Let G be a connected graph such that $\chi_{d,2}(G)=3$. Let C_1, C_2, C_3 be the corresponding color classes of G such that $|C_1|=1, |C_2|=b$ and $|C_3|=c$.

Let $v \in C_2$. Since C_2 has at least two vertices, v will not dominate the color class C_2. Thus v must be adjacent with all the vertices of C_3 and C_1. Similarly every vertex of C_3 must be adjacent with all the vertices of C_2 and C_1. Thus $G = K_{1,b,c}$.

Theorem 3. 7

Let T be a tree of $n(\geq 2)$. Then $\chi_{d,2}(T)=\chi(T)$ if and only if $T = K_2$.

Proof.

Note that $\chi(T)=2$ where T is a tree of order $n(\geq 2)$. Thus from Theorem 3. 2, $\chi_{d,2}(T)=\chi(T)$.

The next result shows a 2-Dominator chromatic number of a class of Circulant graphs.

Theorem3. 8

Let $G=Cir(n, X)$ where $X=\{1,2,\ldots\ldots\ldots,x,n-x,\ldots\ldots\ldots,n-2,n-1\}$ and n is a multiple of $2x$. Then $\chi_{d,2}(G) \leq \left\lceil \frac{n}{x} \right\rceil + 2(x-1)$.

Proof.

Let $M = \left\lceil \frac{n}{x} \right\rceil$. First we color all these vertices of $C=\{0,x,2x,3x,\ldots\ldots\ldots\ldots\ldots\ldots,(M-1)x\}$ by M different colors. The remaining vertices are colored with $2(x-1)$ colors namely, $1, 2, \ldots, x-1, x+1, \ldots, 2x-1$as follows:

Let $v \in V(G)-C$. Then by division algorithm, it is easy to write $c = (2x)a+b$ for some a and b satisfying $0\leq a \leq l-1$ and $1 \leq b \leq 2x-1$, where $l = \left\lceil \frac{n}{2x} \right\rceil$. We define $c(v) = v \bmod (2x)$.

Since for any u, $N[u] = \{u, u+1, u+2, \ldots, u+x, u+1, u+2, \ldots, u+(n-x)\}$, the coloring defined above is a proper coloring with $M + 2(x-1) = \left\lceil \frac{n}{x} \right\rceil + 2(x-1)$ colors. It remains to show that the above coloring is a 2-Dominator coloring.

Let $v \in V(G)$

Case 1: If $v \in C$, then $v=i(x)$ for some $1 \leq i \leq M - 1$. In this case v dominates the color classes $\{i(x)\}$, $\{(i+1)(x)\}$ when $0 \leq i \leq M-2$ and v dominates the colorclasses $\{i(x)\}$, $\{0\}$ when$i = M-1$.

*Case 2:*Let $v \in V(G) - C$. Then $v = i(x) + j$ for some $1 \leq i \leq M - 1$ and $1 \leq j \leq x\text{-}1$. In this case v dominates the color classes $\{i(x)\}$, $\{(i+1)(x)\}$ when $0 \leq i \leq M-2$ and v dominates the color classes $\{i(x)\}$, $\{0\}$ when $i = M-1$. Thus the above coloring is a 2-Dominator coloring and hence the result.

Conclusion

In this paper, we introduced 2-dominator coloring and derived some results in 2-dominator coloring of circulant graphs and also we obtain several results on this coloring parameter. This will be initiative for the study of *k*-dominator coloring.

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