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On the Comparison of Stability Analysis with Phase Portrait for a Discrete Prey-Predator System

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Abstract

Prey-predator interaction can be defined as consumption of predator against prey. It is well known that when the predator absent, the growth of prey population will increase. In the absence of prey, the predator population goes extinct. In this paper, we consider the interaction between prey and predator in a discrete prey-predator system. For this system, we would like to study the dynamic behavior of solutions (or in particular fixed points) for this system. The objective of this project is to investigate the stability of fixed points using stability analysis and phase portrait. After that, we compare the results of stability obtained from both approaches. Our observations indicate that results of stability using both approaches are same. This means that both approaches are suitable to explain the behavior of the solutions, especially when deal with real world problems. This research is important for the sustaining of interacting species of prey and predator populations in an ecosystem.

INTRODUCTION

Dynamical system can be defined as a system that evolve with time and it consists of possible states, together with a rule that determine the present state in terms of past state. Dynamical systems are mathematical models of real systems such as the climate, brain, electronic circuit, laser, population, etc. In mathematics, dynamical system is a function which describes the time dependence of a point in geometrical. There are two types of dynamical system which are flows and maps. Flows are described by continuous time while maps are decribed by discrete time. In this paper, we focus on the discrete-time model where the general form of equation is given as follows:

$$x_{n+1} = f(x_n), \quad (1)$$

where x is the state and f is the evolution or map.

Ecology is the relationship between organisms and their environments. Prey-predator is one of the interaction in ecosystem. Prey is an organism which the predator eats and predator is defined as an organism that eats another organism. Some examples of prey and predator are fish and sharks, rat and snake and rabbit and fox. The first mathematical model describing interacting population was proposed by Alfred Lotka in 1925 and Vito Volterra in 1926. The model of prey-predator system is called as a Lotka-Volterra. The Lotka-Volterra model is expressed by the following equations (Reni Sagaya Raj et al., 2013b):

$$x' = ax - bxy,$$

$$y' = -cy + dxy,$$

where x is the population density of prey, y is the population density of predator and a, b, c, d are all positive parameters.

Interaction between prey and predator is a crucial study in explaining the population dynamics, where this refers to the changes in the size of populations of organisms through time. Lynch (2014) states that there are two types of outcome from the prey-predator interaction. First, there is coexistence where the two species live in harmony. Secondly, there is mutual exclusion in which one of the species becomes extinct.

In this paper, we use the discrete-time model. There are reasons why discrete-time models are better than the continuous-time models. According to Zhao et al. (2016), the discrete-time models are more reasonable to be used for the case where there are non-overlapping generations. Besides that, the discrete time models are more efficient for computation and numerical simulations. The aims of this paper are to find the stability of fixed points of a discrete prey-predator system by means of stability analysis and phase portrait. We will then compare the results from both approaches.

MODEL

In this project, we consider the following system of difference equations which describes the interactions between two species from Reni Sagaya Raj et al. (2013b):

$$x_{n+1} = rx_n(1 - x_n) - ax_ny_n,$$

$$y_{n+1} = -cy_n + bx_ny_n, \quad (2)$$

where $r, a, b, c > 0$ and where x_n is the population of the prey at time n and y_n is the population of the predator at time n. The following are the parameters used in the model:

Parameter	Description
r	The natural growth rate of the prey in the absence of predators
a	The rate of predation on the prey which depends on the likelihood that a predator encounters a prey
b	The efficiency and propagation rate of the of predation in the presence of prey
c	The natural death rate of the predator in the absence of prey

Previously, Elsadany et al. (2012) investigate system (2) where they find local stability of fixed points, bifurcation, chaotic behavior, Lyapunov exponents and fractal dimension of the solution of a discrete prey-predator system. In 2013, Reni Sagaya Raj et al. (2013a) who also considered a discrete model of prey-predator, have plot the phase portrait and bifurcation diagram for a certain range of parameter. Their results show that the prey population has chaotic behaviour over time. Recently, Zhao et al. (2016) considered the same model where necessary and sufficient conditions of the existence and stability of fixed points are proved. Their results show that this model undergoes a flip bifurcation. Moreover, they extend the analysis by stabilizing the chaotic orbits using feedback control method.

STABILITY ANALYSIS

The solutions of discrete-time model is called the fixed point. In this section, we find the fixed points for system (2) and investigate their stability.

Definition 1 (Lynch, 2014). A *fixed point* of system (1) is a point which satisfies $x_{n+1} = f(x_n) = x_n$, for all n.

From the above definition, we are able to find the fixed points or in biological case, the number of populations of prey and predator in future time. For each fixed point, we may find its stability using the following theorem:

Theorem 1 (Lynch, 2014). Suppose that the map $f_\mu(x)$ has a fixed point at x^*. Then the fixed point is *stable* if $\left|\frac{d}{dx} f_\mu(x^*)\right| < 1$ and it is *unstable* if $\left|\frac{d}{dx} f_\mu(x^*)\right| < 1$.

The above theorem is for the one-dimensional system. For two-dimensional system, we refer to the following proposition (Reni Sagaya Raj et al., 2013b):

Proposition 1. The characteristic roots λ_1 and λ_2 are called *eigenvalues* of the fixed points (x^*, y^*). Then,

1. (x^*, y^*) is sink (stable) if $|\lambda_{1,2}| < 1$;
2. (x^*, y^*) is source (unstable) if $|\lambda_{1,2}| > 1$;
3. (x^*, y^*) is saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

Note that saddle implies unstable.

Steps towards finding the stability of fixed points

In this section, we study the behaviour of the system (2) about each fixed points. We compute the stability of the system (2) by using Jacobian matrix corresponding to each fixed points. We will determine whether the fixed points are stable, unstable or saddle from the values of eigenvalues obtained from Jacobian matrix. The process of finding the stability is discussed in the following steps:

Step 1: Finding fixed points

We solve system (2) as follows:

$$rx_n(1 - x_n) - ax_ny_n = x_n,$$
$$-cy_n + bx_ny_n = y_n.$$

Proposition 2. Then we obtain that the system (2) has

i) One extinction fixed point $E_0 = (0,0)$,

ii) One exclusion fixed point $E_1 = (\frac{r-1}{r}, 0)$ and

iii) One coexistence fixed point $E_2 = \left(\frac{c+1}{b}, \frac{r(b-c-1)}{ab} - \frac{1}{a}\right)$.

From the above proposition, E_0 means that there will be no prey and predator in the future in an ecosystem and E_2 means that only one species is survive where in this case the survivor is the prey. Finally E_2 has both solutions for prey and predator which indicates that both species sustain forever. All these outcomes have been expected by Lynch (2014) as we discussed in the introduction. According to Reni Sagaya Raj et al. (2013), we use the parameter $r = 2.41, a = 1.19, b = 3.91$ and $c = 0.45$. Therefore the fixed points are $E_0 = (0,0)$, $E_1 = (0.59,0)$, $E_2 = (0.37,0.43)$.

Step 2: Finding the partial derivative

The general formula for Jacobian matrix for system (2) is

$$J(x_{n+1,}y_{n+1}) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix}.$$

The general form of Jacobian matrix for system (2) is

$$J(x,y) = \begin{pmatrix} r - 2rx - ay & -ax \\ by & bx - c \end{pmatrix}.$$

Step 3: Classify eigenvalues from step 2.

We estimate the eigenvalues of Jacobian matrix J at E_0. The Jacobian matrix at E_0 is of the form

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -c \end{pmatrix}.$$

Hence, the eigenvalues of $J(E_0)$ are $\lambda_1 = r$ and $\lambda_2 = -c$. E_0 is stable if $|\lambda_{1,2}| < 1$ which implies $r < 1$ and $c < 1$. E_0 is unstable if $|\lambda_{1,2}| > 1$which implies $r > 1$ and $c > 1$. The Jacobian matrix J at E_1 is given by

$$J(E_1) = \begin{pmatrix} 2-r & a\left(\frac{1-r}{r}\right) \\ 0 & b\left(\frac{r-1}{r}\right) - c \end{pmatrix}.$$

Hence, the eigenvalues of the matrix $J(E_1)$ are $\lambda_1 = 2 - r$ and $\lambda_2 = b\left(\frac{r-1}{r}\right) - c$. Thus, E_1 is stable if $|\lambda_{1,2}| < 1$ and unstable if $|\lambda_{1,2}| > 1$

The fixed point E_2 has the Jacobian

$$J(E_2) = \begin{pmatrix} 1 - \dfrac{r(1+c)}{b} & -a\left(\dfrac{1+c}{b}\right) \\ \dfrac{r(b-1-c)-b}{a} & 1 \end{pmatrix}.$$

By using the same values of parameters, we obtain the following eigenvalues and their corresponding stability for E_0, E_1 and E_2 for system (2).

Table 1: The results on the stability of fixed points for system (2)

Fixed points	Eigenvalues	Stability
E_0	$\lambda_1 = 2.41$, $\lambda_2 = -0.45$	Saddle
E_1	$\lambda_1 = 1.84$, $\lambda_2 = -0.41$	Saddle
E_2	$\lambda_{1,2} = 0.5531 \pm 0.7409i$	Stable

We discuss the results in Table 1 based on Proposition 2. E_0 is saddle since $|\lambda_1| > 1$ and $|\lambda_2| < 1$. E_1 is also saddle since $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Meanwhile E_2 has a pair of complex conjugate eigenvalues with positive real parts. Since both real parts of eigenvalues are less than 1, therefore E_2 is stable.

PHASE PORTRAIT

The behavior of fixed points in a system can also be shown by plotting the phase portrait. In this section, we plot the phase portrait for system (2) by using Maple.

Definition 2 (Kenneth, 2008). Let (x_n, y_n) be a solution to the discrete system**.** As n varies, the solution (x_n, y_n) describes a curve in the xy-plane called a *trajectory*. The xy-plane is called phase plane.

The phase portrait is a representative sampling of trajectories of the system.

Definition 3. The *phase portrait* is a two-dimensional figure showing how qualitative behaviour of system (2) is determined as x and y vary with n.

Hence, the eigenvalues of the matrix $J(E_2)$are

$$\lambda_1 = \frac{1}{2}\,\frac{2b - r(c+1)\sqrt{4b^2 + 8rbc + 4rb + r^2c^2 + 2r^2c + r^2 - 4rb^2c - 4rb^2 + 4brc^2 + 4b^2c}}{b}$$

and

$$\lambda_2 = \frac{1}{2}\,\frac{-2b + r(c+1)\sqrt{4b^2 + 8rbc + 4rb + r^2c^2 + 2r^2c + r^2 - 4rb^2c - 4rb^2 + 4brc^2 + 4b^2c}}{b}$$

Thus, the fixed point of E_2 is stable if $|\lambda_{1,2}| < 1$ and unstable fixed point if $|\lambda_{1,2}| > 1$.

Some examples of sketch of stability using phase portrait

Here we show some examples of phase portraits along with several graphs of x_1 versus x_2 which are given below. We show that the three fixed points which are stable, unstable and saddle points in Figure 1.

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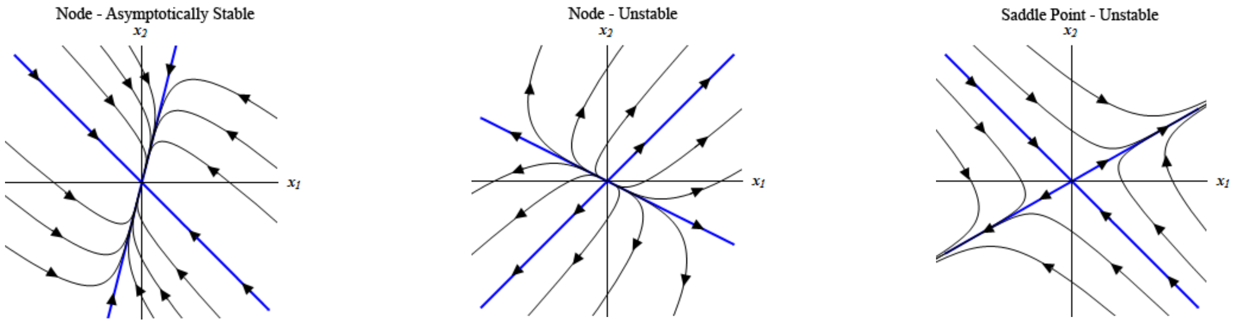
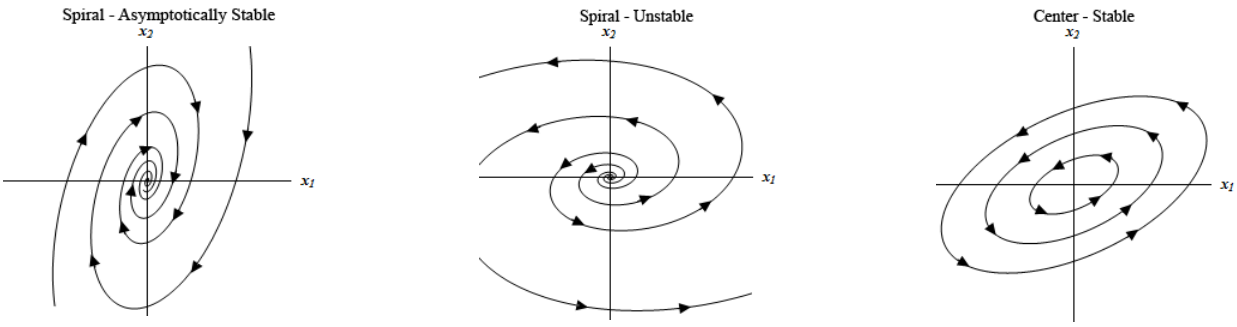


a) stable b) unstable c) saddle point

Figure 1: Schematic diagrams for stable, unstable and saddle points



a) spiral in b) spiral out c) centre

Figure 2: Schematic diagrams for points with complex eigenvalues

For the case of complex eigenvalues, there are some examples of graph stability. Suppose the eigenvalues are $a \pm bi$, where a and b are real with $a \neq 0$ and $b > 0$. We show for this case in Figure 2.

Result for phase portrait for system (2)

In this section, we discuss about the stability of fixed points between prey and predator based on phase portrait diagram obtained using Maple. We mark the three fixed points E_0, E_1 and E_2 in Figure 3 with small empty circle. The small arrows filled throughout the figure represent the trajectories of initial conditions chosen in the system. From this figure, the trajectories approach E_0 along y-axis but repel from E_0 along x-axis. Since there are mixed of directions of in and out, therefore we say that E_0 is a saddle point. For E_1, we can see that the trajectories approach E_1 from both left and right. However, there are also trajectories that move away from E_1. Since E_1 has both in and out directions of the trajectories, E_1 is also a saddle point. Meanwhile E_2 is a stable point since the trajectories move towards E_2 from all directions. This indicates that the number of populations of prey and predator stabilize to a constant value after long period of time.

From the above discussion, we observe that the results of stability for fixed points obtained using phase portrait are the same as by using the stability analysis approach.

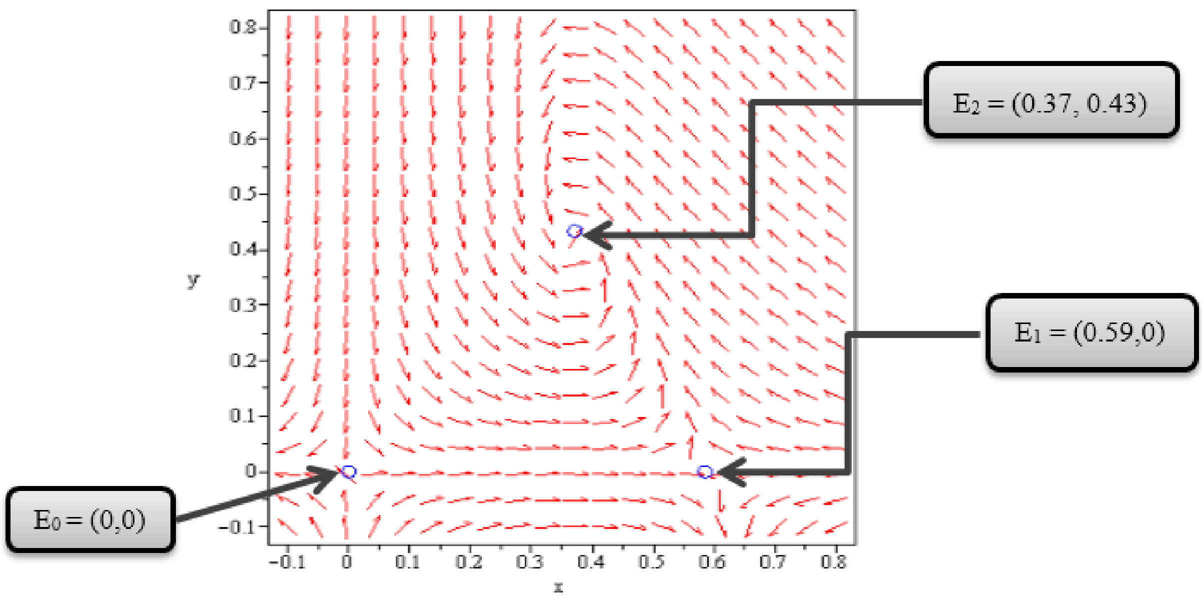


Figure 3: The points of three fixed points for system (2).

CONCLUSION

In the present work, we have considered a 2-dimensional discrete prey-predator model and obtained three fixed points for the system (2). The stability of the fixed points obtained from two approaches has been discussed. The results show that system (2) has two saddle and one stable points. Based on the results of this model, we hope to apply this model in real cases of marine ecosystem where we will be able to provide the values of parameters that affecting the prey and predator populations.

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