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Differential equations associated with twisted q-tangent polynomials

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Abstract

In this paper, we study linear differential equations arising from the generating func tions of twisted q-tangent polynomials. We give explicit identities for the twisted q-tangent polynomials.

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INTRODUCTION

Recently, many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, Hermite polynomials, Laguerre polynomials, and tangent numbers and polynomials. These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, physics, and applied engineering(see [1-11]).

We first give the definitions of the twisted q-tangent numbers and polynomials. It should be mentioned that the definition of twisted q-tangent numbers $T_{n,\zeta,q}$ and polynomials $T_{n,\zeta,q}(x)$ can be found in [5]. Let r be a positive integer, and let ζ be rth root of unity. The twisted q-tangent numbers $T_{n,\zeta,q}$ and polynomials $T_{n,\zeta,q}(x)$ are defined by means of the generating functions:

$$\frac{2}{\zeta q e^{2t}+1}=\sum_{n=0}^{\infty}T_{n,\zeta,q}\frac{t^n}{n!}$$

$$\left(\frac{2}{\zeta q e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_{n,\zeta,q}(x)\frac{t^n}{n!}. \tag{1.1}$$

The twisted q-tangent polynomials of higher order, $T_{n,\zeta,q}^{(k)}(x)$ are defined by means of the following generating function

$$\left(\frac{2}{\zeta q e^{2t}+1}\right)^k e^{xt}=\sum_{n=0}^{\infty}T_{n,\zeta,q}^{(k)}(x)\frac{t^n}{n!}. \tag{1.2}$$

The twisted q-tangent numbers of higher order, $T_{n,\zeta,q}^{(k)}$ are defined by the following generating function

$$\left(\frac{2}{\zeta q e^{2t}+1}\right)^k=\sum_{n=0}^{\infty}T_{n,\zeta,q}^{(k)}\frac{t^n}{n!}. \tag{1.3}$$

When $k=1$, above (1.2) and (1.3) will become the corresponding definitions of the twisted q-tangent polynomials $T_{n,\zeta,q}(x)$ and the twisted q-tangent numbers $T_{n,\zeta,q}$ (see [5]).

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [4, 7]). In this paper, we study linear differential equations arising from the generating functions of twisted q-tangent polynomials. We give explicit identities for the twisted q-tangent polynomials.

DEFERENTIAL EQUATIONS ASSOCIATED WITH TWISTED *Q*-TANGENT POLYNOMIALS

In this section, we study linear differential equations arising from the generating functions of twisted q-tangent polynomials. Let

$$\begin{aligned}\mathrm{H}&=\mathrm{H}(t,\zeta,q)=\frac{2}{\zeta q e^{2t}+1},\\ \mathrm{F}&=\mathrm{F}(t,\zeta,q,x)=\left(\frac{2}{\zeta q e^{2t}+1}\right)e^{xt}.\end{aligned} \tag{2.1}$$

Then, by (2.1), we get

$$H^{(1)}=\frac{d}{dt}H(t,\zeta,q)=\frac{d}{dt}\left(\frac{2}{\zeta q e^{2t}+1}\right)=-\zeta q\left(\frac{2}{\zeta q e^{2t}+1}\right)^2 e^{2t}.$$

Hence we have

$$H^{(1)}=-\zeta q H^2 e^{2t}.$$

By (2.1), we obtain

$$\begin{aligned}F^{(1)}=\frac{d}{dt}F(t,\zeta,q,x)&=\frac{d}{dt}\left(\frac{2}{\zeta q e^{2t}+1}\right)e^{xt}=-\zeta q\left(\frac{2}{\zeta q e^{2t}+1}\right)^2 e^{(x+2)t}+x\left(\frac{2}{\zeta q e^{2t}+1}\right)e^{xt}\\ &=(-\zeta q H e^{2t}+x)F(t,\zeta,q,x)\end{aligned} \tag{2.2}$$

$$\begin{aligned}F^{(2)} &= \left(\frac{d}{dt}\right)^2 F(t,\zeta,q,x)\\
&= \left(-\zeta q H^{(1)} e^{2t} - 2x\zeta q H e^{2t}\right)F + (-\zeta q H e^{2t} + x)F^{(1)}\\
&= (-1)^2 2\zeta^2 q^2 H^2 e^{4t} F + (-1)2\zeta q x H e^{2t} F + (-1)2\zeta q H e^{2t} F + x^2 F\\
&= ((-1)^2 2\zeta^2 q^2 H^2 e^{4t} + (-1)(2\zeta q x + 2\zeta q) H e^{2t} + x^2) F(t,\zeta,q,x)\end{aligned}$$

and

$$\begin{aligned}F^{(3)} &= \left(\frac{d}{dt}\right)^3 F(t,\zeta,q,x)\\
&= (-1)^2 4\zeta^2 q^2 H H^{(1)} e^{4t} F + (-1)^2 8\zeta^2 q^2 H^2 e^{4t} F + (-1)^2 2\zeta^2 q^2 H^2 e^{4t} F^{(1)}\\
&\quad +(-2)(\zeta q x + \zeta q) H^{(1)} e^{2t} F + (-4)(\zeta q x + \zeta q) H e^{2t} F\\
&\quad +(-2)(\zeta q x + \zeta q) H e^{2t} F^{(1)}\\
&= (-1)^3 6\zeta^3 q^3 H^3 e^{6t} F(t,\zeta,q,x) + (-1)^2 (8\zeta^2 q^2 + 2\zeta^2 q^2 x + 2\zeta^2 q^2 x + 2\zeta^2 q^2) H^2 e^{4t} F(t,\zeta,q,x)\\
&\qquad +(-1)(4\zeta q x + 4\zeta q + \zeta q x^2) H e^{2t} F(t,\zeta,q,x) + x^3 F(t,\zeta,q,x).\end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,\zeta,q,x) = \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x), \qquad (N = 0,1,2,\cdots) \tag{2.4}$$

Taking the derivative with respect to *t* in (2.4), we obtain

$$\begin{aligned}F^{(N+1)} &= \frac{dF^{(N)}}{dt}\\
&= \left(\sum_{i=0}^{N} (-1)^i i a_i(N,\zeta,q,x) H^{i-1} e^{2it} + \sum_{i=0}^{N} (-1)^i i a_i(N,\zeta,q,x) H^i e^{2it}\right) F + \left(\sum_{i=0}^{N} (-1)^i i a_i(N,\zeta,q,x) H^{i-1} e^{2it}\right) F^{(1)}(t,\zeta,q,x)\\
&= \left(\sum_{i=0}^{N} (-1)^{i+1} \zeta q (i+1) a_i(N,\zeta,q,x) H^{i+1} e^{2(i+1)t}\right) F(t,\zeta,q,x) + \left(\sum_{i=0}^{N} (-1)^i (2i + x) a_i(N,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x)\\
&= \left(\sum_{i=0}^{N} (-1)^i (2i + x) a_i(N,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x) + \left(\sum_{i=1}^{N+1} (-1)^i \zeta q i a_{i-1}(N,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x)\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by N + 1 in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} (-1)^i a_i(N+1,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}&\left(\sum_{i=0}^{N} (-1)^i (x + 2i) a_i(N,\zeta,q,x) H^i e^{2it} + \sum_{i=1}^{N+1} (-1)^i \zeta q i a_{i-1}(N,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x)\\
&\qquad = \left(\sum_{i=0}^{N+1} (-1)^i a_i(N+1,\zeta,q,x) H^i e^{2it}\right) F(t,\zeta,q,x).\end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned}a_0(N+1,\zeta,q,x) &= x a_0(N,\zeta,q,x)\\
a_{N+1}(N+1,\zeta,q,x) &= \zeta \mathrm{q} (N+1) a_N(N,\zeta,q,x),\end{aligned} \tag{2.8}$$

and

$$a_i(N+1,\zeta,q,x) = (x+2i)a_i(N,\zeta,q,x) + \zeta q i a_{i-1}(N,\zeta,q,x), \ \ (1 \le i \le N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$\mathrm{F} = F^{(0)} = a_0(0,\zeta,q,x)F(t,\zeta,q,x) = F(t,\zeta,q,x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,\zeta,q,x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} -\zeta q H e^{2t} F(t,\zeta,q,x) + xF(t,\zeta,q,x) &= \textstyle\sum_{i=0}^{1} (-1)^i a_i(1,\zeta,q,x) H^i e^{2it} F(t,\zeta,q,x) \\ &= a_0(1,\zeta,q,x)F(t,\zeta,q,x) + (-1)a_1(1,\zeta,q,x)He^{2t}F(t,\zeta,q,x). \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,\zeta,q,x) = x, \ \ a_1(1,\zeta,q,x) = \zeta q, \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,\zeta,q,x) = x a_0(N,\zeta,q,x) = x^2 a_0(N-1,\zeta,q,x) = \cdots = x^{N+1},$$

and

$$\begin{aligned} a_N(N+1,\zeta,q,x) &= \zeta q (N+1) a_N(N,\zeta,q,x) \\ &= \cdots = \zeta^{(N+1)} q^{(N+1)} (N+1)!. \end{aligned} \tag{2.14}$$

For $i = 1,2,3$ in (2.9), we get

$$a_1(N+1,\zeta,q,x) = \zeta q \sum_{k=0}^{N} (x+2)^k a_0(N-k,\zeta,q,x),$$

$$a_2(N+1,\zeta,q,x) = 2\zeta q \sum_{k=0}^{N-1} (x+4)^k a_1(N-k,\zeta,q,x), and$$

$$a_3(N+1,\zeta,q,x) = 3\zeta q \sum_{k=0}^{N-2} (x+6)^k a_2(N-k,\zeta,q,x).$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,\zeta,q,x) = i\zeta q \sum_{k=0}^{N-i+1} (x+2i)^k a_{i-1}(N-k,\zeta,q,x). \tag{2.15}$$

Now, we give explicit expressions for $a_i(N+1,\zeta,q,x)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_1(N+1,\zeta,q,x) &= \zeta q \sum_{k_1=0}^{N} (x+2)^{k_1} a_0(N-k_1,\zeta,q,x). \\ &= \zeta q \sum_{k_1=0}^{N} (x+2)^{k_1} x^{N-k_1}, \end{aligned}$$

$$\begin{aligned} a_2(N+1,\zeta,q,x) &= 2\zeta q \sum_{k_2=0}^{N-1} (x+4)^{k_2} a_1(N-k_2,\zeta,q,x). \\ &= 2!\,\zeta^2 q^2 \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-k_2-1} (x+4)^{k_2} (x+2)^{k_1} x^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} &a_3(N+1,\zeta,q,x) \\ &\quad = 3\zeta q \sum_{k_3=0}^{N-2} (x+6)^{k_3} a_2(N-k_3,\zeta,q,x). \end{aligned}$$

$$= 3!\,\zeta^3 q^3 \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-k_3-2} \sum_{k_1=0}^{N-k_3-k_2-2} (x+6)^{k_3}(x+4)^{k_2}(x+2)^{k_1} x^{N-k_3-k_2-k_1-2}.$$

Continuing this process, we have

$$a_i(N+1,\zeta,q,x) = i!\,\zeta^i q^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-k_i-i+1} \cdots \sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1} (x+2i)^{k_i} \cdots (x+2)^{k_1} x^{N-k_i-\cdots-k_1-i+1}. \tag{2.16}$$

Note that, here the matrix $a_i(j,\zeta,q,x)$ $0 \le i,j \le N+1$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{N+1} \\ 0 & \zeta q & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 2!\,\zeta^2 q^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 3!\,\zeta^3 q^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N+1)!\,\zeta^{N+1} q^{N+1} \end{pmatrix}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N = 0,1,2,\cdots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) \left(\frac{2}{\zeta q e^{2t}+1} \right)^i e^{2it} \right) F$$

has a solution

$$F = F(t,\zeta,q,x) = \left(\frac{2}{\zeta q e^{2t}+1} \right) e^{xt},$$

where

$$a_0(N,\zeta,q,x) = x^N,$$

$$a_N(N,\zeta,q,x) = N!\,\zeta^N q^N,$$

$$a_i(N,\zeta,q,x) = i!\,\zeta^i q^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-k_i-i} \cdots \sum_{k_1=0}^{N-k_i-\cdots-k_2-i} (x+2i)^{k_i} \cdots (x+2)^{k_1} x^{N-k_i-\cdots-k_1-i}, (1 \le i \le N).$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t,\zeta,q,x) = \sum_{k=0}^{\infty} T_{k+N,\zeta,q}(x) \frac{t^k}{k!}. \tag{2.17}$$

From Theorem 1, (1.3) and (2.17), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} T_{k+N,\zeta,q}(x) \frac{t^k}{k!} = F^{(N)} &= \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) \left(\frac{2}{\zeta q e^{2t}+1} \right)^i e^{2it} \right) F \\ &= \sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) e^{(x+2i)t} \left(\frac{2}{\zeta q e^{2t}+1} \right)^{i+1} \\ &= \sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) \left(\sum_{k=0}^{\infty} T_{k,\zeta,q}^{(i+1)}(x+2i) \frac{t^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,x) T_{k,\zeta,q}^{(i+1)}(x+2i) \right) \frac{t^k}{k!}. \end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0,1,2,\cdots$ and $N = 0,1,2,\cdots$, we have

$$T_{k+N,\zeta,q}(x) = \sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,x) T_{k,\zeta,q}^{(i+1)}(x+2i)$$
$$= \sum_{i=0}^{N}\sum_{l=0}^{k}\binom{k}{l}(-1)^i(2i)^{k-l}a_i(N,\zeta,q,x)T_{l,\zeta,q}^{(i+1)}(x), \tag{2.19}$$

where

$$a_0(N,\zeta,q,x) = x^N,$$
$$a_N(N,\zeta,q,x) = N!\,\zeta^N q^N,$$
$$a_i(N,\zeta,q,x) = i!\,\zeta^i q^i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(x+2i)^{k_i}\cdots(x+2)^{k_1}x^{N-k_i-\cdots-k_1-i},$$
$$(1 \le i \le N).$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0,1,2,\cdots$, we have

$$T_{N,\zeta,q}(x) = \sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,x) T_{0,\zeta,q}^{(i+1)}(x+2i).$$

For $N = 0,1,2,\cdots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,x)\left(\frac{2}{\zeta q e^{2t}+1}\right)^i e^{2it}\right)F$$

has a solution

$$F = F(t,\zeta,q,x) = \left(\frac{2}{\zeta q e^{2t}+1}\right)e^{xt}.$$

Here is a plot of the surface for this solution.

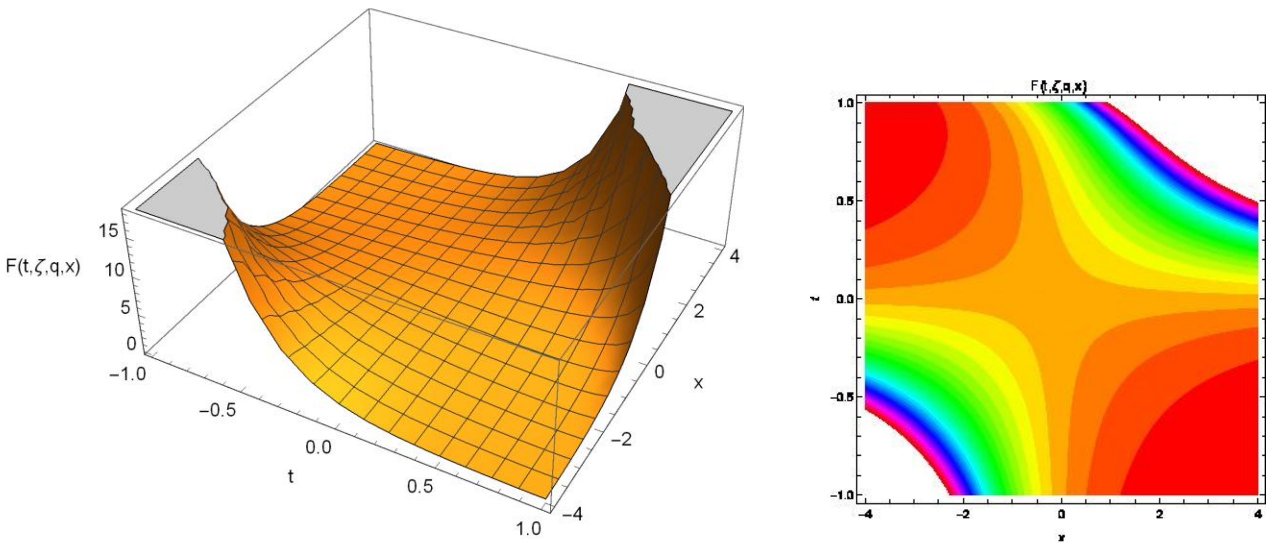

Figure 1: The surface for the solution $F(t,\zeta,q,x)$

In Figure 1, we choose $\zeta = e^{\frac{2\pi i}{2}}, q = \frac{1}{10}$. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

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