International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 24 (2017) pp. 15791-15795
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Solving The Modified Ostrovsky Equation using Pseudo-Spectral Method

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Abstract

In this paper, a nonlinear evolution equation is studied by adding an extra cubic nonlinearity to Ostrovsky equation, known as modified Ostrovsky equation. Ostrovsky equation originally is obtained from Korteweg-de Vries equation with an additional linear non-local term representing the effects of background Earth's rotation. The addition of rotation term has the drastic effect of destroying the solitary wave solution as an initial wave and eventually form a nonlinear wave packet. We solved this equation using Pseudo-Spectral Method by implementing the dealiasing and sponge-layer techniques. With an additional cubic nonlinear term, an unsteady wave packet is emerged as expected as the linear part is unchanged, however the wave packet is clearly separated from the trailing radiation at the end of the calculation and we observed that the radiated wave is emerged together with the wave packet. This means that the modified Ostrovsky equation leads to the formation of wave packet is faster compared to Ostrovsky equation.

Keywords: modified Ostrovsky Equation, cubic nonlinear term, Pseudo-Spectral Method, Wave Packet

INTRODUCTION

In many physical situations, it is essential to take account the fact that nonlinear waves especially Korteweg-de Vries (KdV) equation, with various extension is a canonical model for the description of nonlinear internal waves [1, 2]. When the background rotation is included, the well-known KdV equation is replaced by the Ostrovsky equation [3]. In this paper, an extra cubic nonlinearity is added to the Ostrovsky equation, written

$$(u_t + \alpha\, uu_x + \beta\, u^2 u_x + \lambda\, u_{xxx})_x = \gamma\, u, \tag{1}$$

where $u(x,t)$ is the radial displacement, t is temporal variable, x is a spatial variable, α, β, λ and γ are variable-coefficient of quadratic nonlinearity, cubic nonlinearity, dispersion and rotation effects respectively. If the coefficients $\beta, \gamma = 0$, equation (1) becomes KdV equation:

$$u_t + \alpha\, uu_x + \lambda\, u_{xxx} = 0. \tag{2}$$

which has an exact soliton solution [4]. It is widely known that when $\beta = 0$, equation (1) reduces to the Ostrovsky equation and known that there is no steady solitary wave solution will propagate from the Ostrovksy equation. Instead, a solitary like initial condition collapses due to radiation of inertia gravity waves, with the long time outcome being a nonlinear wave packet [3, 5]. In addition, the coupled Ostrovsky equations arise when two speeds are nearly close and eventually the two wave packets exist [6, 7].

LINEARIZATION AND DISPERSION RELATION

The dispersion relation is obtained by linearising the equation (1) and then seeking the solution in the form of:

$$u = u_0 e^{ik(x - c_p t)} + c.c, \tag{3}$$

where k is the wavenumber, u_0 is the constant amplitude, c_p is the phase velocity and $c.c$ denotes the complex conjugate. Then, the phase velocity and group velocity will be obtained by applying the linearization technique. Here, the linear dispersion relation of the modified Ostrovsky equation is similar with Ostrovsky equation for sinusoidal waves frequency ω and wavenumber k because the cubic nonlinear and quadratic nonlinear terms are neglected.

By linearised equation (1), will yield

$$u_t + \lambda\, u_{xxx} = \gamma \int u \; dx. \tag{4}$$

Next, by substituting and simplifying equation (3) into equation (4), the phase velocity $c_p = \omega/k$ and the group velocity c_g are given by:

$$c_p = \frac{\gamma}{k^2} - \lambda k^2,$$

$$c_g = \frac{d\omega}{dk} = -\frac{\gamma}{k^2} - 3\lambda k^2.$$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 24 (2017) pp. 15791-15795
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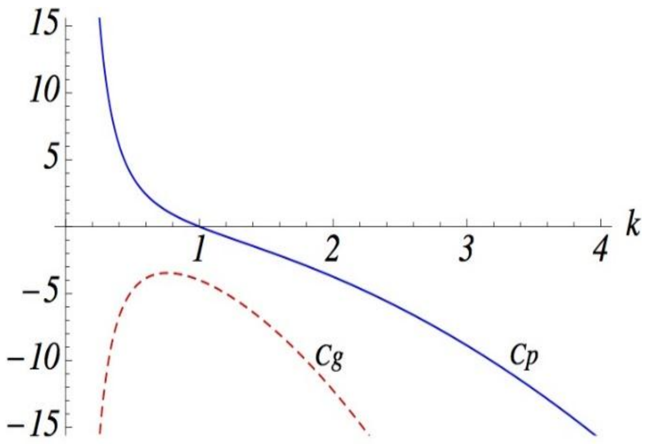


Figure 1: Plot of phase velocity, c_p and group velocity, c_p when $\lambda = \gamma = 1$.

A typical dispersion curve is shown in Figure 1, which based on the linear coefficients equal to unity. It is important to note that the group velocity c_g is always negative for all wavenumbers k when $\lambda, \gamma > 0$. The additional rotation term to the KdV equation is a linear long wave perturbation, and it has the ability to remove the spectral gap in which solitary waves exist for the KdV equation. Hence, no solitary waves are expected to exist. Several studies [3, 8, 9 and 10] have revealed that long time effect of background rotation is the decay of the initial internal solitary wave into the radiation of small amplitude inertia-gravity waves, and the emergence of a propagating nonlinear wave packet. According to Figure 1, there still no spectral gap in dispersion relation and the group velocity, c_g is negative for all wavenumbers.

For example, according to Figure 1, when the value of each coefficients in the modified Ostrovsky equation is equal unity, the maximum value of group velocity, $c_g = -3.4641$ with the wavenumber at $k = 0.7598$ are obtained. This value of group velocity with the negative sign represent the propagation of the wave packet to the left with the speed, approximately at $-3.4641\ m/s$.

NUMERICAL SIMULATIONS: PSEUDO-SPECTRAL METHOD

This modified Ostrovsky equation is being solved using a Pseudospectral method. Initially, the forward scheme obtained from this method is then implemented with dealiasing technique to remove the high frequency (aliasing error) during the simulations by putting roughly $\frac{1}{3}$ of the grid points equal to 0 [12]. Importantly, in Pseudo-Spectral method, the nonlinear terms in the equation should be treated in the real space before we proceed to the Fourier space. According to modified Ostrovsky equation, there exist two nonlinear terms, namely as quadratic nonlinearity, uu_x and cubic nonlinearity, u^2u_x terms.

Pseudospectal method transforms the spatial derivatives of the PDEs by Fourier transform and substitute the temporal derivative by finite- difference approximation. Once again, before we proceed with the numerical discretization, we add a linear damping region ("sponge layer") at each end of the domain to prevent the possibility of radiated waves re-entering the region of interest and interfering with the main wave structure [6].

Indeed, let us consider again the modified Ostrovsky equation (1), with the Fourier transform,

$$\widehat{u}_t + \frac{1}{2} is\alpha\, \hat{u}^2 + \frac{1}{3} is\beta\, \hat{u}^3 - is^3\lambda\, \hat{u} = \frac{\gamma}{is}\hat{u}, \tag{5}$$

where $s = \pi/L$ and L is the real interval.

Then, we can program it in Fourier space as,

$$\hat{u}_t + \frac{1}{2} i\alpha s\, (F(F^{-1}\,\hat{u})^2) + \frac{1}{3} i\beta s(F(F^{-1}\hat{u})^3) - is^3\lambda\, \hat{u} = \frac{\gamma}{is}\hat{u}, \tag{6}$$

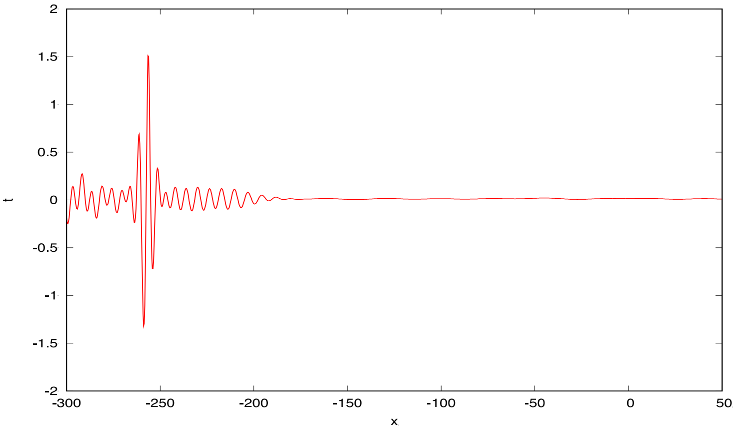
where F is the Fourier Transform and F^{-1} is the inverse Fourier Transform.

NUMERICAL RESULTS AND DISCUSSION

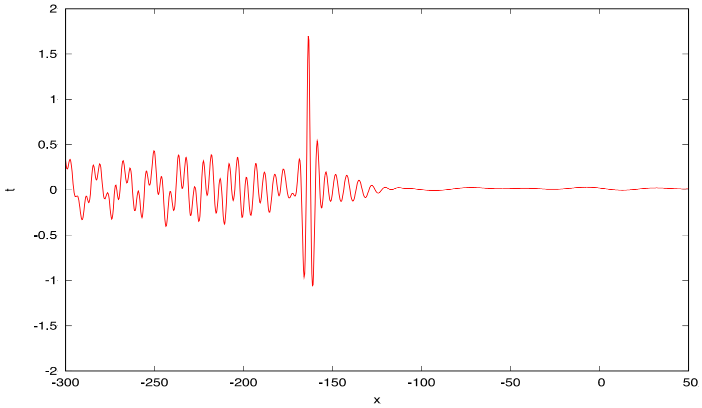
In the numerical simulations, the initial condition is considered as KdV *sech*^{2}-profile when $\beta, \gamma = 1$ in equation (1) given by:

$$u(x,0) = A_0 \operatorname{sech}^2\left(\frac{x}{D}\right), \quad \alpha A_0 D^2 = 12\,\lambda \tag{7}$$

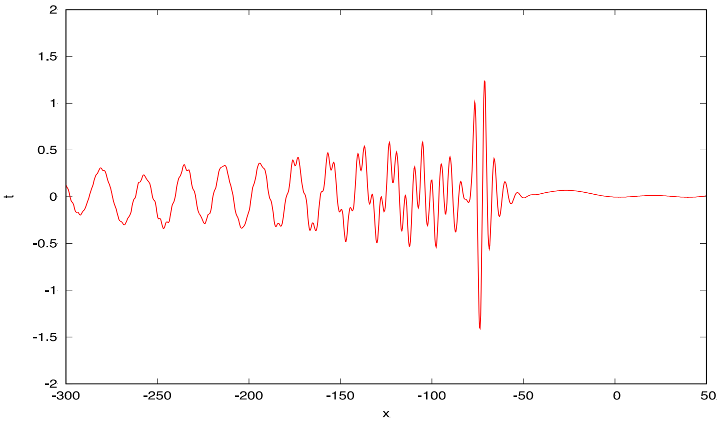
where A_0 is the wave amplitude. Figure 2 shows the numerical solution of modified Ostrovsky equation for the initial KdV solitary wave with the amplitude $A_0 = 4$ and all the coefficients in (1) are equal to unity. It can be seen that an initial solitary wave is rapidly decays into the radiation of inertia-gravity waves and eventually forms a nonlinear envelope wave packet. Numerical experiments have been done by Grimshaw and Helfrich [3], who run the simulation with amplitude between 2 and 32. According to their research, the small amplitude especially $A_0 = 2,$ and 4, the wave packet does not completely separated from the trailing radiation. However, with the modified Ostrovsky equation, when $A_0 = 4$ is taken into account, the separation of the wave packet from the radiated wave is very clear as illustrate in Figure 2. It can be seen that a single unsteady wave packet exists because the coefficient $\lambda\gamma > 0$, however the steady wave packet will happen when the coefficient $\lambda\gamma < 0$ [5].



(a) t=60



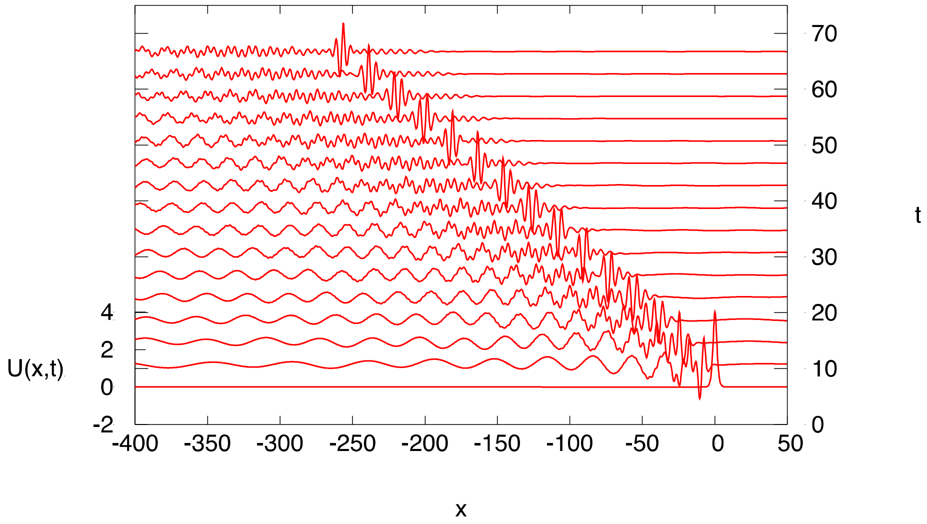
(b) t=40



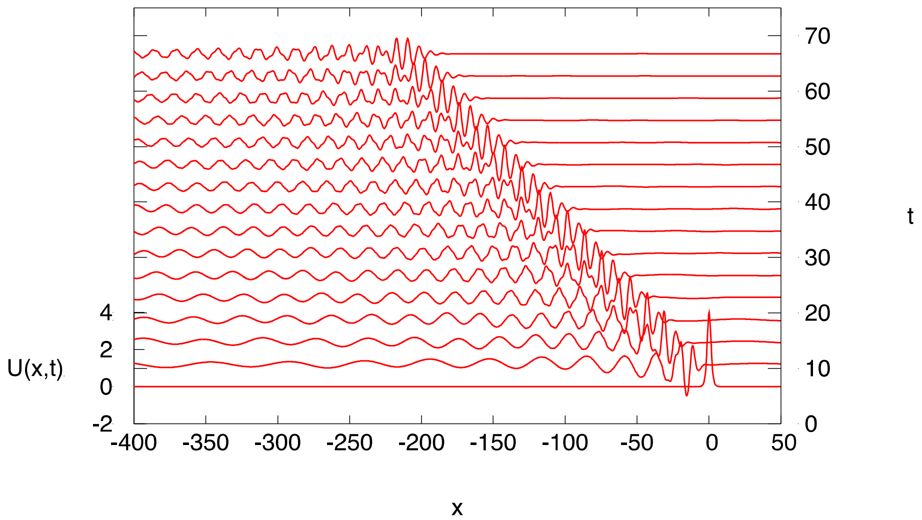
(c) t=20

Figure 2: A cross-sectional simulation results of the modified Ostrovsky equation (1) when $\alpha,\ \beta,\ \lambda, \gamma = 1$ for the initial condition (14) with $A_0 = 4$. (a) $t = 20,$ (b) $t = 40$ and (c) $t = 60$, respectively.

As seen in Figure 2, a single unsteady wave packet propagated to the left with the numerically determined group velocity, approximately as$-4.2537\ m/s$, is quite far to the theoretical prediction group velocity from the dispersion relation$-3.4641\ m/s$. However, the numerical group velocity for Ostrovsky equation when $\beta = 0$ in equation (1) is close to the theoretical prediction as $-3.5049\ m/s$. This is because of the effect of the cubic nonlinearity that can make the wave packet propagates faster.



(a)



(b)

FIGURE 3. (a) A simulation of the modified Ostrovsky equation when $\alpha,\ \beta,\ \lambda,\gamma = 1$ at all time with $A_0 = 4$ and (b) A simulation of the modified Ostrovsky equation when $\alpha,\ \lambda,\gamma = 1$ and $\beta = 0$ at all time with $A_0 = 4$.

Figure 3 illustrates the numerical simulation of both modified Ostrovsky equation with cubic nonlinearity and without cubic nonlinearity terms. In both numerical simulation, we have put all the coefficient as unity and with the amplitude of solitary wave in initial condition as $A_0 = 4$. However, according to the results in Figure 3 (a), we observed that there exist a dispersive wave before the main wave packet, which is in contrast with the numerical simulation without cubic nonlinearity. Besides, the formulation of wave packet in Figure 3 (b) does not clearly separated from the radiating wave until the end of numerical calculation.

CONCLUSION

In the presence work, we have considered the modified Ostrovsky equation with an addition of cubic nonlinear term. This nonlinear evolution equation has been solved numerically by pseudospectal method. The numerical solution shows that the initial solitary wave eventually forms a wave packet because of the existence of the rotation background effect, with an extra formation of dispersive wave before the main figure of wave packet. The result shows that the formation of wave packet was very clear even though we use the small amplitude of the initial solitary wave as $A_0 = 4$ which is do not completely separated from the trailing radiation in Grimshaw and Helfrich [3]. Furthermore, the result also shows that with the existence of the cubic nonlinearity the wave packet can propagates faster.

ACKNOWLEDGMENT

The author would like to thank Ministry of Higher Education Malaysia and Universiti Malaysia Terengganu for the financial support for this work under Reseach grant (RAGS) Vot 57129.

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