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# Estimation of Generalized Long-Memory Stochastic Volatility for High-Frequency Data

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**Abstract**
We consider the generalized long-memory stochastic volatility (GLMSV) model, a relatively general model of stochastic volatility that accounts for persistent (or long-memory) and seasonal (or cyclic) behavior at several frequencies. We employ the decorrelating properties of discrete wavelet packet transform (DWPT) to provide a wavelet-based approximate maximum likelihood estimator that allows for analysis of high-frequency data by simplifying the variance-covariance matrix into a diagonalized matrix, whose diagonal elements have the least distinct variances to compute using a computationally efficient quadrature. We apply the proposed method to the estimation of high-frequency simulated data.

**Keywords:** Stochastic volatility, Long-memory, Wavelets, High-frequency data

## INTRODUCTION

Time-varying volatility is a well-documented feature of signals with applications to high-frequency financial time series (see e.g. [1],[2] and [3]). It has also been observed in the dynamics of some biosignals such as the noise of brain electrical responses [4]. Moreover, time-dependence of traffic flow variation or its volatility, has been recognized in literature (e.g. [5]). In particular, [6] considered modeling and forecasting traffic volatility dynamics in transportation networks using stochastic volatility models.

Long-memory or long-range dependence has been recognized in the time-varying volatility of many signals. This characteristic may not be modeled parsimoniously using the autoregressive conditional heteroscedasticity (ARCH) or stochastic volatility (SV) models. Stochastic volatility (SV) models are both time-varying and stochastic, and serve as the closest equivalent to the autoregressive (AR) process or autoregressive moving average (ARMA) process in the second moment. In this context, the long-memory stochastic volatility (LMSV) model was proposed in [7] by replacing the autoregressive moving average (ARMA) in the SV model by the fractionally integrated (FI) process. However, it does not account for periodic or seasonal components, which are often seen as prominent peaks in the periodogram of the signal. In this case, [2] considered an extension of the LMSV model called the generalized long-memory stochastic volatility (GLMSV) by representing the log-volatility by a $k$-factor Gegenbauer autoregressive moving average ($k$-GARMA) process, which is known to account for $k$ persistent periodicities in the volatility series. It is a relatively general model of stochastic volatility that accounts for persistence (or long-memory) and seasonality (or periodicity) at several frequencies.

In general, it is not possible to obtain an exact expression for the autocovariance of a $k$-GARMA process with two or more poles [8]. Hence, the computation of exact maximum likelihood estimate presents a problem in convergence owing to the size of the variance-covariance matrix, whose dimensions equal the square of the signal length. In the presence of high-frequency data, this computational problem is greatly amplified.

In this paper, we provide an alternative to maximizing the time-domain likelihood function of a generalized long-memory stochastic volatility (GLMSV) model by maximizing the likelihood function in the wavelet domain. We consider an approximate maximum likelihood estimator by diagonalizing the variance-covariance matrix employing the decorrelation property of the wavelet packet coefficients, which is carried out by appropriately selecting wavelet packet coefficients that best decorrelate the series using only the minimum number of levels in the wavelet packet table thereby minimizing the number of variances to compute. We select wavelet basis by testing for white noise starting with smallest level of $j$. We apply Portmanteau test to the wavelet packet coefficients of the log-squared process to obtain the orthonomal basis with the minimum number of elements. This allows for analysis of high-frequency data, which now are ubiquitous in various fields owing to the massive use of computers in data generation and gathering.

This paper is organized as follows. In Section II, we define and discuss the properties of the generalized long-memory stochastic volatility (GLMSV) process. In Section III, we present the covariance structure of the discrete wavelet packet transform (DWPT) of GLMSV. In Section IV, we present a wavelet-based approximate maximum likelihood estimator of the GLMSV parameters. In Section V, an application of the proposed method to the estimation of a high-frequency simulated data is provided. Finally, some concluding remarks are given in Section VI.

## GENERALIZED LONG-MEMORY STOCHASTIC VOLATILITY

The generalized long-memory stochastic volatility (GLMSV) model [2] provides a general framework in modeling volatility dynamics incorporating persistence or long-memory and multiple periodicities or seasonalities. We consider the stochastic volatility model given by

$$r_t = \sigma \exp\{X_t/2\} e_t, \tag{1}$$

with $\sigma > 0$, $\{e_t\}$ are iid shocks with zero mean and unit variance, and $\{X_t\}$ is a *k*-GARMA process, which is independent of $\{e_t\}$. The *k*-factor Gegenbauer autoregressive moving-average (k-GARMA) model is a fairly general model that accounts for long-memory and seasonality at *k* frequencies. From [9], a *k*-GARMA(p,d,u,q) process $\{X_t\}$ is given by

$$\Phi(B)\prod_{i=1}^{k}(1-2u_iB+B^2)^{d_i}X_t = \Theta(B)\varepsilon_t, \tag{2}$$

where B denotes the backshift operator, $\{\varepsilon_t\} \sim iid\ \mathrm{N}(0,\sigma_\varepsilon^2)$, $\Phi(B) = 1-\phi_1B^1-\ldots-\phi_pB^p$ and $\Theta(B)=1+\theta_1B^1+\ldots+\theta_qB^q$ are polynomials of order *p* and *q*, respectively, with all roots inside the unit circle; d and u are vectors of length k, with $d_i \neq 0$ and distinct ui, with $|u_i| \leq 1$, i=1,…,*k*. The components of vector **d** are called the long-memory parameters, and the frequencies corresponding to the seasonal or periodic components, $v_i = \frac{\cos^{-1}(u_i)}{2\pi} \in$ [0,0.5], i=1,2,…,*k,* are called the Gegenbauer frequencies. From [9], a *k*-GARMA(p,d,u,q) process is causal and invertible if for *i*=1,…,*k*

$$|d_i| < \begin{cases} ½, & 0 < v_i < ½ \\ ¼, & v_i = 0 \text{ or } ½ \end{cases}, \tag{3}$$

and the spectral density is given by

$$S_X(f) = \sigma_\varepsilon^2 \left| \frac{\Theta(e^{-i2\pi f})}{\Phi(e^{-i2\pi f})} \right|^2 \prod_{i=1}^{k} \left| 2(\cos(2\pi f) - u_i) \right|^{-2d_i},$$

$$f \in (-0.5, 0.5] \tag{4}$$

Let $Y_t = \log r_t^2$ be the log squared process. Hence, we have

$$Y_t = \mu + X_t + \eta_t, \tag{5}$$

where $\mu = \log\sigma^2 + E(\log e_t^2)$ and $\eta_t = \log e_t^2 - E(\log e_t^2)$ is an iid mean-zero process with finite variance $\sigma_\eta^2$ independent of $\{X_t\}$.

The autocovariance function of $\{Y_t\}$ is simply the sum of the covariances of the long-memory process and the noise given by

$$\gamma_Y(s) = \gamma_X(s) + \sigma_\eta^2 I_{\{s=0\}}. \tag{6}$$

Hence, the spectral density is

$$S_Y(f) = S_X(f) + \sigma_\eta^2, \tag{7}$$

where the constant $\sigma_\eta^2$ is the spectral density of the iid process $\{\eta_t\}$, that is $S_\eta(f) = \sigma_\eta^2$.

## COVARIANCES OF WAVELET COEFFICIENTS

We present some properties of the discrete wavelet packet transform (DWPT) coefficients of the log-squared GLMSV process. The ensuing nomenclature is based on [10].

Let $\{h_{1,l}\}_{l=0}^{L-1}$ denote a Daubechies' compactly supported wavelet filter of even length *L*. The scaling filter $\{g_{1,l}\}_{l=0}^{L-1}$ is defined by $g_{1,l} = (-1)^{l+1}h_{1,L-l-1}$. From [11], the filter $\{h_{1,l}\}$ has squared gain function defined by

$$|H_{1,L}(f)|^2 = 2\sin^L(\pi f)\sum_{l=0}^{\frac{L}{2}-1}\binom{\frac{L}{2}-1+l}{l}\cos^{2l}(\pi f) \tag{8}$$

such that

$$|H_{1,L}(f)|^2 + |G_{1,L}(f)|^2 = 2, \quad \text{and}$$

$$|G_{1,L}(f)|^2 = |H_{1,L}(0.5-f)|^2 \tag{9}$$

for $|f| \leq ½$.

From [10], we write the DWPT coefficients $\{D_{j,n,t} \mid j=0,\ldots,J, n=0,\ldots,2^j-1, t=0,\ldots,N2^{-j}-1\}$ of the signal $\{Y_t\}_{t=0}^{N-1}$ in the form

$$D_{j,n,t} = \sum_{l=0}^{L_j-1} u_{j,n,l} Y_{2^j[t+1]-1-l \bmod M}, \tag{10}$$

where $L_j = (2^j-1)(L-1)+1$, and $\{u_{j,n,l}\}$ is the filter corresponding to the node (*j,n*) in the wavelet packet table. The filter $\{u_{j,n,l}\}$ can be computed from { $h_{1,\ell}$ } and { $g_{1,\ell}$ } by letting $u_{1,0,l} = g_{1,l}$ and $u_{1,1,l} = h_{1,l}$. Then for *j*>1 and for each node *(j,n),* we recursively obtain $u_{j,n,l}$ using the equation

$$u_{j,n,l} = \sum_{l=0}^{L-1} u_{n,k} u_{j-1,\lfloor n/2 \rfloor, l-2^{j-1}k}, \quad l = 0,\ldots,L_j-1, \tag{11}$$

where

$$u_{n,l} = \begin{cases} g_{1,l}, & \text{if n mod(4)} = 0 \text{ or } 3 \\ h_{1,l}, & \text{if n mod(4)} = 1 \text{ or } 2 \end{cases} \tag{12}$$

For each node (*j,n*) of the wavelet packet table, we consider $\mathbf{c}_{j,n}$ a vector whose components are 0's and 1's as defined in [10], p. 215. We then write the transfer function of $\{u_{j,n,l}\}$ as

$$U_{j,n}(f) = \prod_{m=0}^{j-1} M_{c_{j,n,m}}(2^m f), \tag{13}$$

where $c_{j,n,m}$ is the $m$th element of $\mathbf{c}_{j,n}$, and $M_0(f)=G_{1,L}(f)$ and $M_1(f)=H_{1,L}(f)$.

From (10), the covariance of the DWPT coefficients of $\{Y_t\}_{t=0}^{N-1}$ is given by

$$\mathrm{cov}(D_{j,n,t}{}^{Y}, D_{j',n',t'}{}^{Y}) = \sum_{\ell=0}^{L_j-1}\sum_{\ell'=0}^{L_{j'}-1} u_{j,n,\ell}u_{j',n',\ell'}\gamma_Y(2^j(t+1)-2^{j'}(t'+1)+\ell'-\ell), \quad (14)$$

where $\gamma_Y(s)$ is the autocovariance function of $Y_t$ at lag $s$. As in [10] p. 348, (14) may be written as

$$\mathrm{cov}(D_{j,n,t}{}^{Y}, D_{j',n',t'}{}^{Y}) = \int_{-1/2}^{1/2} e^{i2\pi f(2^{j'}(t'+1)-2^{j}(t+1))}U_{j,L}(f)\overline{U}_{j',L}(f)S_Y(f)df, \quad (15)$$

where $\overline{U}$ denotes the complex conjugate of $U$. From (15), for *j=j'*, the within-scale autocovariance function of nonboundary DWPT coefficients is given by

$$\mathrm{cov}(D^Y_{j,n,t}, D^Y_{j,n,(t+s)}) = \int_{-1/2}^{1/2} e^{i2\pi f 2^j s}\,|U_{jn}(f)|^2\,S_Y(f)df, \quad (16)$$

where $U_{j,n}(f)$ is the transfer function of the filter $\{u_{j,n,l}\}$. $U_{j,n}(f)$ depends only on $G(\cdot)$ and $H(\cdot)$ such that the squared gain function $\left|U_{j,n}(f)\right|^2$ is nominally bandpass over the frequency interval $I_{j,n}=\left(\frac{n}{2^{j+1}},\frac{n+1}{2^{j+1}}\right]$ for large wavelet filter $L$. Since these intervals are mutually exclusive, then the between-scale covariance in (15) is approximately zero for large $L$. Moreover, since $S_Y(f)$ has a poles at some frequencies in the interval [0, 0.5] and the discrete wavelet packet transform (DWPT) adaptively partitions the interval into dyadic subintervals of different sizes, appropriate choice of the nodes (*j,n*) from the wavelet packet table will ensure that (16) will be close to zero, thereby decorrelating the wavelet packet coefficients.

## MAXIMUM LIKELIHOOD ESTIMATION

We provide an alternative to maximizing the time-domain likelihood function by maximizing the likelihood function in the wavelet domain. We consider an approximate maximum likelihood estimator by diagonalizing the variance-covariance matrix employing the decorrelation property of the wavelet packet coefficients, which is carried out by appropriately selecting wavelet packet coefficients that best decorrelate the signal using only the minimum number of levels in the wavelet packet table thereby minimizing the number of variances to compute. We select wavelet basis by testing for white noise starting with smallest level of *j*. We apply Portmanteau test to the wavelet packet coefficients of the log-squared process to obtain the orthonomal basis with the minimum number of elements.

We denote by the vector $\mathbf{D}_j := \mathbf{D}_{j,n} = \left(D_{jn0},\ldots,D_{jn(2^{J-j}-1)}\right)$ the DWPT coefficients of $\{Y_t\}_{t=0}^{N-1}$ at scale *j*, and $\left\{D_{j,n,t}\mid j=0,\ldots,J, n=0,\ldots,2^j-1, t=0,\ldots,M_j\right\}$ be the nonboundary DWPT coefficients, where $M_j$ denotes the number of nonboundary wavelet packet coefficients at level *j*. Since DWPT is a linear transform, then the wavelet packet coefficient of (7) may be written in the form

$$D_{j,n,t}{}^{Y} = D_{j,n,t}{}^{\mu} + D_{j,n,t}{}^{X} + D_{j,n,t}{}^{\eta}. \quad (17)$$

At each level *j*, $D_{j,n,t}{}^{\eta}$ is asymptotically normal, that is, $D_{j,n,t}{}^{\eta} \sim N\left(0,\sigma_\eta^2\right)$ [12], and $D_{j,n,t}{}^{\mu}=0$ by the bandpass property of wavelets. When $\{X_t\}$ is Gaussian, $\left\{D_{j,n,t}{}^{X}\right\}$ is normally distributed by the linearity of DWPT. Hence, we proceed as if $D_{j,n,t}{}^{Y}$ has a Gaussian distribution, that is, $D_{j,n,t}{}^{Y} \sim N\left(0,\sigma_j^2\right)$, where $\sigma_j^2$ is the variance of $D_{j,n,t}{}^{Y}$.

We will collect all parameters of $S_X(f)$ in the vector $\boldsymbol{\theta}_X = \left(\sigma_\varepsilon^2, \mathbf{d}, \mathbf{v}, \boldsymbol{\Theta}, \boldsymbol{\Phi}\right)$, and the associated parameter vector of $S_Y(f)$ is $\boldsymbol{\theta}_X$ augmented by $\sigma_\eta{}^2$ and is denoted by $\boldsymbol{\theta}_Y = \left(\sigma_\varepsilon^2, \sigma_\eta{}^2, \mathbf{d}, \mathbf{v}, \boldsymbol{\Theta}, \boldsymbol{\Phi}\right)$, where $\mathbf{d}=(d_1,\ldots,d_k)$, $\mathbf{v}=(v_1,\ldots,v_k)$, $\Theta=(\theta_1,\ldots,\theta_q)$, and $\Phi=(\phi_1,\ldots,\phi_p)$. Then the variance $\sigma_j^2$ is a function of these parameters, so we write $\sigma_j^2 := \sigma_j^2(\Theta_Y)$

We decorrelate the wavelet packet coefficients by choosing the wavelet filters $\{u_{j,n,l}\}$ appropriately. For this purpose, we consider the algorithm proposed in [13], which successively selects the DWPT coefficients at each level by performing a white noise test on wavelet packet subbands at each level. We use the Portmanteau test with test statistic in the form

$$Q = N_j(N_j+2)\sum_{s=1}^{K}\frac{\hat{\rho}^2(s)}{r-s}, \quad (18)$$

where $N_j$ is the length of the vector $\mathbf{D}_{\mathbf{j,n}}$, $K=\max\{2,\min\{20, N_j/10\}\}$, the summation is the weighted sum of the first $K$ sample autocorrelations given by

$$\hat{\rho}(s) = \frac{\mathrm{cov}(D_{j,n,t}, D_{j,n,t+s})}{\mathrm{var}(D_{j,n,t})}. \quad (19)$$

We reject white noise if $P(\chi_m^2 > Q) < \alpha$, for some value of the level of significance α.

We apply this white noise test to obtain a decorrelated wavelet packet coefficients as outlined in [13]. Suppose that $\mathbf{D}_{j,n}, n=0,\ldots,2^j-1$, are the vectors of wavelet packet coefficients in the *j*th row of the wavelet packet table. We define $\mathbf{D}_{0,0}$ to be the given input signal. For *j<J*, starting with j=1, we test the vector $\mathbf{D}_{j,n}$ for white noise. If the test fails to reject, we retain $\mathbf{D}_{j,n}$. If the test rejects, we split $\mathbf{D}_{j,n}$ into $\mathbf{D}_{j+1,2n}$ and $\mathbf{D}_{j+1,2n+1}$, and test both the resulting subbands for

white noise. We repeat this process until $j=J$ in which we retain $\mathbf{D}_{J,n}$. We denote the resulting vector of DWPT coefficients by $\mathbf{D}=\left(\mathbf{D}_{j,n},(j,n)\in B\right)$, which is approximately uncorrelated.

We diagonalize the $N$x$N$ variance-covariance matrix of a high-frequency data by choosing appropriately large wavelet filter length $L$ to decorrelate the between-scale wavelet packet coefficients in (15), and the preceding white noise test to decorrelate the within-scale coefficients. In the latter case, large $N_j$ entails small $j$. Hence, the proposed basis selection method allows us to choose the smallest levels $j$ such that the nodes $(j,n)$ satisfy the Portmanteau test for white noise. This provides a univariate density with the least number of variances $\sigma_j^2(\Theta_Y)$ to compute. From (16), the variance of $D_{j,n,t}^{\;\;Y}$ is given by

$$\sigma_j^2(\Theta_Y)=2\int_0^{1/2}|U_{jn}(f)|^2\,S_Y(f)df\;, \tag{20}$$

where $U_{j,n}(f)$ is the transfer function of the wavelet filter $\{u_{j,n,l}\}$. We use numerical integration methods for discrete valued functions to accommodate wavelet filters that do not have squared gain functions in closed form. We divide the interval of integration [0,0.5] into $n$ subintervals of width $h=\dfrac{1}{2n}$ such that no subinterval has an endpoint at a Gegenbauer frequency, which can easily be satisfied by a choice of $n$. There are several quadratures applicable for this purpose. For the trapezoidal rule, the approximate value of the integral is given by

$$\sigma_j^2(\Theta_Y)\approx\quad\frac{1}{2n}\left[f(0)+2\left(\sum_{i=1}^{n-1}f(ih)\right)+f(0.5)\right], \tag{21}$$

with $f(x)=|U_{jn}(x)|^2\,S_Y(x)$.

Let $\mathbf{D}=\left(\mathbf{D}_i\mid i=(j,n)\in B, i=1,2,\ldots,m,\exists m\in Z^+\right)$ be the vector DWPT coefficients of the signal $Y_t(t = 0,1,..,2^J-1)$, where B consists of nodes selected based on the preceding Portmanteau test for white noise. The approximate likelihood function may now be written as a univariate density given by

$$L(\mathbf{\Theta}_Y\mid\mathbf{D})=\left(\prod_{i=1}^{m}\left(2\pi\sigma_i^2(\Theta_Y)\right)^{-n_i/2}\right)\exp\left[\frac{-1}{2}\sum_{i=1}^{m}\frac{\mathbf{D}_i^T\mathbf{D}_i}{\sigma_i^2(\Theta_Y)}\right], \tag{22}$$

where $|\mathbf{D}|=n$, $n_i=|\mathbf{D}_i|$, and $m$ is the number of nodes in B. The approximate maximum likelihood estimator is given by

$$\widehat{\Theta}_Y=\arg\max_{\Theta_Y}\left(L(\Theta_Y\mid\mathbf{D}\right). \tag{23}$$

To determine the number of long-memory parameters $k$ and the orders $p$ and $q$, we use some information criteria, such as the Akaike Information Criterion (AIC) for model selection.

## APPLICATION

We applied the proposed method to a high-frequency simulated data. We first simulated a $k$-GARMA process by truncating the general linear process as in [8] using the TSWGE Package in R. The log-squared process $\{Y_t\}$ is generated by adding a simulated iid mean-zero process, $\{\eta_t\}$, with variance $\sigma_\eta^2$. We considered a standard Gaussian distribution for $e_t$, so that $E\left(\log e_t^{\;2}\right)=-\gamma-\log(2)$, where $\gamma\approx 0.5772$ is the Euler constant. Hence, $\eta_t=\log e_t^{\;2}-E\left(\log e_t^{\;2}\right)$ is distributed $\log\chi_{(1)}^2$ with variance $\sigma_\eta^2=\pi^2/2$ [2].

The simulated signal consists of $N= 2^{15}$ data points with the following parameters: v=0.3, d=0.2, $\phi$=0.4, $\theta$=0.1, $\sigma_\varepsilon^2=1, \sigma_\eta^2=\pi^2/2$. We used the minimum-bandwidth discrete-time wavelets MB(16), which were found adequate to provide between-scale decorrelation for the wavelet packet coefficients. Applying the Portmanteau test starting with nodes along $j$=1, the nodes that admitted white noise were as follows: $(j,n)$=(2,0), (2,1), (2,2), and (2,3). We considered 1024 subintervals in the numerical evaluation of the integrals to compute the variances. This resulted to a relatively fast computation of the estimates. An increase in the number of subintervals for numerical integration did not significantly affect computing time.

Initial estimates showed that the presence of short-memory parameters of AR and MA components ($\phi$=0.4, $\theta$=0.1) affected the accuracy of the estimates of the other parameters. For this matter, we first eliminated the short-memory components from the series, before the long-memory and the rest of the GLMSV parameters were estimated. The results were very close to the actual values as follows:

| | d | v | $\sigma_\varepsilon^2$ | $\sigma_\eta^2$ |
|---|---|---|---|---|
| Estimates | 0.2978 | 0.3031 | 0.9968 | 4.9322 |
| Absolute Deviation | 0.0022 | 0.0031 | 0.0032 | 0.0026 |

## CONCLUSION

We proposed a wavelet-based approximate maximum likelihood estimator of the generalized long-memory stochastic volatility model that is suitable for the analysis of high-frequency data owing to its computational efficiency. The likelihood function is a univariate density obtained by diagonalizing the variance-covariance matrix, whose dimension is the square of the signal length. Moreover, the variances in the diagonal are mostly identical by appropriately selecting the wavelet packet nodes using a white noise test. An application of the proposed method to high-frequency simulated data showed that computation converges fast, but the accuracy of the estimates is affected by the presence of the short-memory parameters in AR and MA components, which must be excised out first before the long-memory and the rest of the parameters of the GLMSV model are estimated.

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