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New Quadrature Method using Least Squire Interpolation Polynomials

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Abstract

In this paper, we are developing a Quadrature Method (Numerical Integration method) of a continuous function $f(x)$ on the compact interval [*a,b*] and deriving a polynomial $P_m(x)$ of degree m such that integration of $P_m(x)$ from *a* to *b* is equal to the integration of $f(x)$ from *a* to *b*. We are using the least square method to fit the polynomial $P_m(x)$. Also derive Newton-Cotes formulas and composite formula from this method, estimate errors and given.

Keywords: Numerical Integration, Newton-Cotes method, Quadrature method, Least Squire Interpolation Polynomials.

INTRODUCTION

With the advent of the modern high speed electronic digital computers, the Numerical Integration have been successfully applied to study problems in Mathematics, Engineering, Computer Science and Physical Sciences. Numerical integration, also called **Quadrature**, is the study of how the numerical value of an integral can be found. The purpose of this paper is quadrature methods for approximate calculation of definite integrals

$$I[f] = \int_a^b f(x)dx \tag{1.1}$$

where $f(x)$ is integrable, in the Riemann sense on [a,b]. The limit of the integration may be finite. Numerical integration is always carried out by mechanical quadrature and its basic scheme is as follows:

$$\int_a^b f(x) = \sum_{i=0}^{n-1} A_i f_i + R[f], \tag{1.2}$$

where $f_i = f(x_i)$. A_i and x_i are called **Coefficients (Weights)** and **nodes** for Numerical Quadrature, respectively, and R[f] is error of Quadrature method. Once the coefficients and nodes are set down, the scheme (1.1) can be determined.

PRELIMINARIES

Order of Numerical Integration

Order of accuracy, or precision, of a Quadrature formula, is the largest positive integer n such that the formula is exact for x^k, for each $k = 0,1,...,n.$

Error of Quadrature method

The Integration (1.1) is approximated by a finite linear combination of the value of $f(x)$ in the form (1.2). The error of approximation of (1.2) is given as

$$R_n = \frac{C}{(m+1)!} f^{(m+1)}(\xi), \tag{2.1}$$

where $\xi =$ (a,b), $m \geq n$ is order of (2) and error constant of (2) is

$$C = \int_a^b x^{m+1} - \sum_{i=0}^{n-1} A_i x_i^{m+1} \tag{2.2}$$

Interpolation Polynomial

Let $f(x)$ be a continuous function defined on some interval [*a*, *b*], and be prescribed at $n+1$ distinct tabular points $x_0, x_1,..., x_n$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. The distinct tabular points $x_0, x_1,..., x_n$ are equispaced, that is $x_{k+1} - x_k = h$, $k = 0,1,2,$ The problem of polynomial approximation is to find a polynomial $P_n(x)$, of degree ≤ n, which fits the given data exactly, that is,

$$P_n(x_i) = f(x_i), i = 0,1,2, \tag{2.3}$$

The polynomial $P_n(x)$ is called the interpolating polynomial. The conditions given in (2.5)are called the interpolating conditions.

Least Squares Interpolation Polynomial

Let the polynomial of the m^{th} degree $P_m(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$

be fitted to the data points $(x_i, f(x_i))$ $i = 0,1,2,...,n$, where $m < n$ and $a_i's$ are satisfy the system of equations

$$(n+1)a_0 + a_1\sum_{i=0}^{n} x_i + a_2\sum_{i=0}^{n} x_i^2 + ... + a_m\sum_{i=0}^{n} x_i^m = \sum_{i=0}^{n} f(x_i) \quad (2.4)$$

$$a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=0}^{n} x_i^2 + ... + a_m\sum_{i=0}^{n} x_i^{m+1} = \sum_{i=0}^{n} x_i f(x_i)$$

:

$$a_0\sum_{i=0}^{n} x_i^m + a_1\sum_{i=0}^{n} x_i^{m+1} + ... + a_m\sum_{i=0}^{n} x_i^{2m} = \sum_{i=0}^{n} x_i f(x_i).$$

There are $m+1$ equations in $m+1$ unknowns.

Lemma 1 $P_m(x)$ *is least squares interpolation equation of* $f(x)$ *on[a,b].then*

$$\sum_{i=0}^{n} P_m(x_i) \approx \sum_{i=0}^{n} f(x_i) \quad (2.5)$$

where $x_0 = a, x_n = b$

proof:-Let the $P_m(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$ is least squares interpolation equation of $f(x)$ on[a,b] then $P_m(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_m x_0^m$ and so on $P_m(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + ... + a_m x_n^m$. adding all we get

$$\sum_{i=0}^{n} P_m(x_i) = (n+1)a_0 + a_1\sum_{i=0}^{n} x_i + a_2\sum_{i=0}^{n} x_i^2 + ... + a_m\sum_{i=0}^{n} x_i^m$$

apply Equation (2.4), we get

$$\sum_{i=0}^{n} P_m(x_i) \approx \sum_{i=0}^{n} f(x_i)$$

Lemma 2 Let $P_m(x)$ is least squares interpolation equation of the integrable function $f(x)$ on finite interval [a,b] and

$$\sum_{i=0}^{n} P_m(x_i) \cong \sum_{i=0}^{n} f(x_i),$$

where $x_0 = a, x_n = b$ if and only if

$$\int_{x_0}^{x_n} P_m(x)dx \cong \int_{x_0}^{x_n} f(x)dx. \quad (2.6)$$

Proof:- Multiplying with $h = (b-a)/n$ and take the limit $h \to 0$ in (2.5), we get

$$\lim_{h\to 0} h\sum_{n=0}^{n} P_m(x_n) = \lim_{h\to 0} h\sum_{n=0}^{n} f(x_n)$$

It gives lemma 2.

LEAST SQUARE QUADRATURE METHOD

Consider the integral in the form (1.2) for each $i = 0,1,2...,n-1$. Now we are dividing the interval [*a, b*] into n (finite) equal sub-interval and take the nodes $x's$ are equispaced points such that $x_i = x_0 + ih \in$ [a,b], i=0,1,2,...n-1, where $x_0 = a, x_n = b$ and $h = (b-a)/(n)$. So we have data points $(x_i, f(x_i))$ $i = 0,1,2,...,n$ for fit a polynomial $P_m(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$, we have

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} P_m(x)dx = a_0(x_n - x_0) + a_1\frac{x_n^2 - x_0^2}{2} + a_2\frac{x_n^3 - x_0^3}{3} + ... + a_m\frac{x_n^{m+1} - x_0^{m+1}}{m+1}. \quad (3.1)$$

This method is called L_m^n -Quadrature method($L_m^n - rule$), where m is donated degree of polynomial and n is donate a number of data points. To solve the least square Quadrature method we have at least m+1 points. Order of this method is greater than or equal to m since it's exact for a polynomial of degree m. The error constant of (3.1) is

$$C = \int_{x_0}^{x_n} x^k - a_0 + \sum_{i=1}^{n} \frac{x_n^i - x_0^i}{i} a_i$$

and error

$$R = \frac{C}{k!} f^{(k)}(\xi)$$

where $k \le m, a \le \xi \le b$. Now following cases arise.

Case(i), ***m*=0** that is P_0 is a constant function. From (2.4) we have $a_0(n+1) = \sum_{i=0}^{n} f(x_i)$ and $a_1 = a_2 = ... = a_m = 0$, substituting this values in (3.1) we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{(x_n - x_0)}{n+1}\sum_{i=1}^{n} f(x_i) \quad (3.2)$$

Case(ii), ***m*=1** that is P_1 is a linear polynomial. From (2.4) we have

$$a_0(n+1) + a_1\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} f_i, a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} x_i f_i$$

and $a_2 = a_3 = ... = a_m = 0$, Solve for and, we get

$$a_0 = \frac{\sum_{i=0}^{n} f_i \sum_{i=0}^{n} x_i^2 - \sum_{i=0}^{n} x_i \sum_{i=0}^{n} x_i f_i}{(n+1)\sum_{i=0}^{n} x_i^2 - (\sum_{i=0}^{n} x_i)^2}$$

$$a_1 = \frac{(n+1)\sum_{i=0}^{n} x_i f_i - \sum_{i=0}^{n} x_i \sum_{i=0}^{n} f_i}{(n+1)\sum_{i=0}^{n} x_i^2 - (\sum_{i=0}^{n} x_i)^2}$$

After simplification we get

$$a_0 = \frac{2}{nh(n+1)(n+2)}\left[n(3x_0 + h(n+1))\sum_{i=0}^{n} f_i - 3(x_0 + x_n)\sum_{i=0}^{n} if_i\right]$$

$$a_1 = \frac{6}{nh(n+1)(n+2)}\left[2\sum_{i=0}^{n} if_i - i\sum_{i=0}^{n} f_i\right]$$

substituting this values in (3.1), simplification we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{nh}{n+1}\sum_{i=1}^{n} f(x_i)$$

this is same as m=0. The method (3.2) is called L_1^n - Quadrature method and the error constant of (3.2) is

$$C = \int_{x_0}^{x_n} x^2 dx - \frac{nh}{n+1}\sum_{i=0}^{n}(x+ih)^2 = \frac{-h^3 n^2}{6} = \frac{-(x_n - x_a)^3}{6n} = -\frac{(b-a)^3}{6n}$$

and error of (10) is

$$R = \frac{-(b-a)^3}{6n \cdot 2!} f^{(2)}(\xi) = \frac{-(b-a)^3}{12n} f^{(2)}(\xi) \qquad (3.3)$$

where $x_0 \le \xi \le x_n$ To solve this method, we have at least 2 data points and the order of (3.2) is 2.

Case(iii), m=2 that is P_2 is a polynomial of degree two. From (2.4) we have

$$(n+1)a_0 + a_1\sum_{i=0}^{n} x_i + a_2\sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} f_i = A$$

$$a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=0}^{n} x_i^2 + a_2\sum_{i=0}^{n} x_i^3 = \sum_{i=0}^{n}(x_0 + ih) f_i = Ax_0 + hB \quad a_0\sum_{i=0}^{n} x_i^2 + a_1\sum_{i=0}^{n} x_i^3 + a_2\sum_{i=0}^{n} x_i^4 = \sum_{i=0}^{n}(x_0 + ih)^2 f_i = Ax_0^2 + 2Bhx_0 + Ch^2$$ where

$A = \sum_{i=0}^{n} f_i$, $B = \sum_{i=0}^{n} if_i$, and $C = \sum_{i=0}^{n} i^2 f_i$. we have. Solve the three linear system of equation for a_0, a_1 and a_2 (using MATLAB or any programs) we get

$$a_0 = \frac{3}{(n+1)(n^3 + 4n^2 + n - 6)h^2 n}(3Ah^2n^4 + 12Ahn^3x_0 - 12Bh^2n^3 - Ah^2n^2 - 6Ahn^2x_0 + 10An^2x_0^2$$
$$+ 6Bh^2n^2 - 64Bhn^2x_0 + 10Ch^2n^2 - 2Ah^2n - 6Ahnx_0 - 10Anx_0^2$$
$$+ 6Bh^2n - 8Bhnx_0 - 60Bnx_0^2 - 10Ch^2n + 60Chnx_0 + 12Bhx_0 + 60Cx_0^2)$$

$$a_1 = -\frac{6(6Ahn^3 - 3Ahn^2 + 10An^2x - 32Bhn^2 - 3Ahn - 10Anx - 4Bhn - 60Bnx + 30Chn + 6Bh + 60Cx)}{h^2 n(n^2 + 3n + 2)(n^2 + 2n - 3)}$$

and

$$a_2 = \frac{30(An^2 - An - 6Bn + 6C)}{h^2 n(n^4 + 5n^3 + 5n^2 - 5n - 6)}$$

substituting this values in (3.1), simplification we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{hn(An^3 - An^2 + 6An + 30Bn - 6A - 30C)}{(n-1)(n+3)(n+2)(n+1)}$$

substituting *A*, *B* and *C* we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{hn}{(n-1)(n+3)(n+2)(n+1)}\sum_{i=0}^{n}(n^3 - n^2 + 6n - 6 + 30ni - 30i^2) f_i \qquad (3.4)$$

This method is called L_2^n-Quadrature method. To solve this method, we have at least 3 data points.

case(iv), *m*=3 that is P_3 is a polynomial of degree three. following the previous case, we get the same as (3.3). The error constant of (3.4) is

$$C=\int_{x_0}^{x_n} x^4 dx-\frac{hn}{(n-1)(n+3)(n+2)(n+1)}\sum_{i=0}^{n}(n^3-n^2+6n-6+30ni-30i^2)(x+ih)^4$$

$$=-\frac{(3n^2-8n+18)n^2h^5}{210}=-\frac{(3n^2-8n+18)(x_n-x_0)^5}{210n^3}=-\frac{(3n^2-8n+18)(b-a)^5}{210n^3}.$$

the error of (12) is

$$R=-\frac{(3n^2-8n+18)(b-a)^5}{210n^3\cdot 4!}f^{(4)}(\xi) \quad (3.5)$$

where $a\le\xi\le b.$ The order of (3.4) is 4.

Note. It m_0 is even number then L_m^n method same as a method.

NEWTON-COTES FORMULAS FROM LEAST SQUARE METHOD

We can derive trapezoidal rule, Simpson 1-3rd rule and Simpson 3-8th rule from the least square method.

take $n=1$ in (3.2) we get

$$\int_{x_0}^{x_1} f(x)dx=\frac{h}{2}(f_0+f_1)$$

this formula is called trapezoidal rule. The error of trapezoidal rule is, from (11)

$$R=\frac{-(b-a)^3}{12}f^{(2)}(\xi), a\le\xi\le b$$

take $n=2$ in (3.4) we get

$$\int_{x_0}^{x_2} f(x)dx=\frac{2h}{1\cdot 5\cdot 4\cdot 3}\sum_{i=0}^{2}(10+60i-30i^2)f_i$$
$$=\frac{h}{30}(10f_0+40f_1+10f_2)=\frac{h}{3}(f_0+4f_1+f_2)$$

this formula is called Simpson 1-3rd rile rule. The error Simpson 1-3rd rule of is, from (3.5)

$$R=\frac{-(b-a)^5}{90}f^{(4)}(\xi), a\le\xi\le b.$$

similarly, Simpson 3-8th rule come from (3.4) with $n=3$, that is

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)$$

and error come from (13), $R=-(3/80)h^5 f^{(4)}(\xi), a\le\xi\le b.$

The weights of the integration method of (12) with an equispaced point for $n\le 6$ are given in Table 1.

Table 1: Weight of Newton-cote rules and Weights of L_2^n Quadrature Method

n	Common ratio	Newton-Cotes weight	Common ratio	L_2^n Method
1	1/2	1 1	—	—
2	1/3	1 4 1	1/3	1 4 1
3	3/8	1 3 3 1	3/8	1 3 3 1
4	2/45	7 32 12 32 7	4/105	11 26 31 26 11
5	5/288	19 75 50 50 75 19	5/336	31 61 78 78 61 31
6	1/140	41 216 27 272 27 216 41	1/14	7 12 15 16 15 12 7

GRAPHICALLY MEANING OF LEAST SQUARE INTEGRATION METHOD

Let the polynomial $P_m(x)$ of degree m is fitted by least square interpolation method by using data points (x_i, f_i) $i=0,1,2...,n,$. If m=1, take n is a large number then the polynomial $P_1(x)$ is going to exact fit polynomial such that the area A+C=B ($fib:1(a)$). That's why the integration of $P_1(x)$ on [*a*, *b*] gives exactly.

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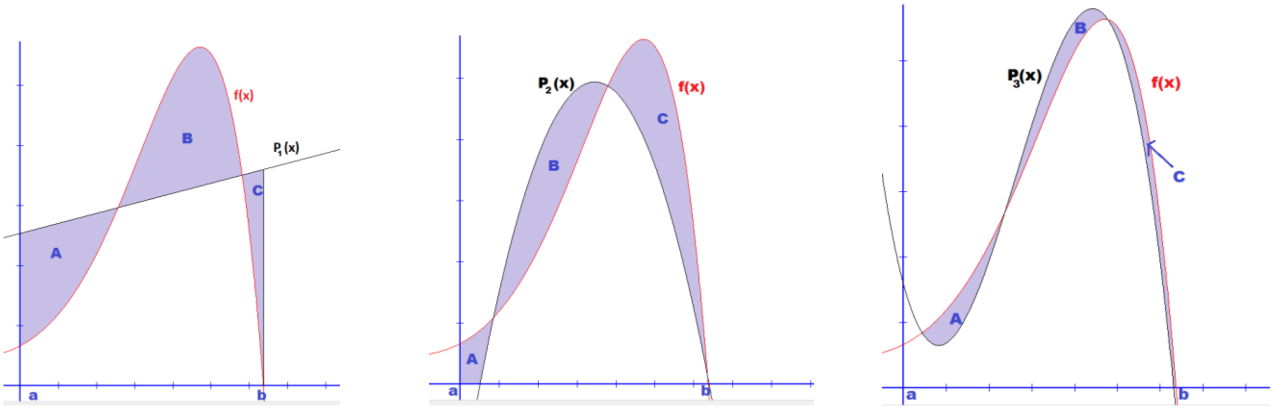

Figure 1 : a,b,c

PROBLEMS

Problem 1 Find approximate value of

$$I = \int_1^3 sin(x)e^x dx$$

fit a straight line $y(x)$ such that $\int_1^3 y(x)dx = I$.

Solution:Let $f(x) = sin(x)e^x$ and y_n be the straight line by fit n+1 data points $(x_i, f(x_i))$, i=0,1,2,..,n. Now we divide the interval [1 3] into two equal subintervals, that is $n = 2$ or $h = 1$. then 3 data points are $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$. we fit a straight line y_2 by normal equation (5) we get

$$y_2 = 0.27x + 3.4$$

following this we get

$$y_4 = 0.78x + 3.15$$

$$y_8 = 1.17x + 2.77$$

$$y_{16} = 1.39x + 2.51$$

$$y_{32} = 1.51x + 2.36$$

and

$$y_{64} = 1.57x + 2.28.$$

But we know if $n \to \infty$ then $\int_1^3 y_n(x)dx \to \int_1^3 f(x)dx$.

therefore $I = \int_1^3 (1.57x + 2.28)dx = 10.84$.

Problem 2 Fit quadratic equation $P_2(x)$ such that

$$\int_0^1 P_2(x)dx = \int_0^1 x\sqrt{x+1}dx$$

and find the approximate value of $\int_0^1 x\sqrt{x+1}dx$

Solution Let P_{2_n} be the quadratic equation by fit n equal space data points in [0 1]. By least square method we have

$$P_{2_3}(x) = 0.37893738x^2 + 1.03527618x + 3.61400724(E-20)$$

$$P_{2_{11}}(x) = 0.37892845x^2 + 1.03956285x - 0.00227848$$

$$P_{2_{51}}(x) = 0.37839273x^2 + 1.04141576x - 0.00304322$$

$$P_{2_{101}}(x) = 0.3783134x^2 + 1.0416701x - 0.00314653.$$

Let $I_n = \int_0^1 P_{2_n}(x)dx$ then $I_3 = 0.643950551$, $I_{11} = 0.643812428$, $I_{51} = 0.643795564$ and $I_{101} = 0.643792992$. The exact value of $\int_0^1 x\sqrt{x+1}dx$ up to five decimal is 0.64379.

$$I = \int_0^1 \frac{1}{2+x}dx,$$

using L_1^n and L_2^n rules with different equal subintervals. Using the exact solution, find the absolute errors.

Solution:Results for the L_1^n and L_2^n rules to estimate the integral of $f(x) = 1/(2+x)$ from x = 0 to 1. The exact value is

$$I_{exact} = \int_0^1 1/(2+x)dx = ln(x+2)]_0^1 = ln(3) - ln(2)$$

=0.4054651.

We get

Table1: Exact solution and errors of the problem (1) and (2), respectively.

n	$I_1^n = L_1^n$ method	Error= $I_1^n - I_{exact}$
1	0.4167	0.0112
2	0.4111	0.0056
4	0.4083	0.0028
8	0.4069	0.0014
16	0.4062	0.0007
32	0.4058	0.0003
64	0.4056	0.0001

n	$I_2^n = L_2^n$ method	Error= $I_2^n - I_{exact}$
2	0.4055556	0.0000905
4	0.4054930	0.0000279
8	0.4054801	0.0000150
16	0.4054735	0.0000084
32	0.4054696	0.0000045
64	0.4054675	0.0000024
128	0.4054663	0.0000012

CONCLUSION

We develop this new method for easy to solve Definite Integral of the finite interval with equispaced nodes. We have derived Simpson 1/3rd rule and Simpson 3/8th rule from L_2^n Quadrature Method. In this method (L_2^n) weights are increasing from a to midpoint of interval and decreasing from the midpoint to b . So this Method (L_2^n) is used for the closed area bounded by $x=0$ and. i.e The limits of integration a and b are intersected points of $x=0$ and $f(x)$. We have given MATLAB code also, give any bonded continuation function $f(x)$ on [a,b] that will be given an approximation integration value of $f(x)$ from a to b. Also, I'm developing this concept to high degree polynomials and high dimension.

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