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Inventory Policies for Deteriorating Items under Stock-Dependent Demand with Variable Holding Cost

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Abstract

As per the practical scenario for the inventory models dealing with demand rate based on stock availability, the higher stock availability enhances the trading of the product. An assumption of constant holding cost in earlier derived stock dependent demand models was overcome in this paper. This article deals with the computation of total inventory cost for deteriorating inventory along with the investigation of combined optimal replenishment cycle length, optimal ordered quantity and preservation technology investment for inventory models with the holding cost depending on storage time period and rate of market demand is assumed to fluctuate as a function, based on level of stock. The product in this article is considered as the mozzarella cheese used on pizza. Four inventory models with storage time-dependent holding cost are considered which includes Retroactive holding cost increase, and incremental holding cost increase with and without the utilization of preservation technology in each case. By developing algorithms for each case and utilizing the classical optimization technique for calculating the optimal values and fulfilling the objective of minimizing the inventory cost. Thereafter, using the concept of eigen-values of a Hessian matrix, we have proved the convex nature of the cost function for the case where minimum cost is obtained. Finally, in order to validate the derived models, numerical examples along with sensitivity analysis is undertaken, which extracts the fruitful managerial insights that incremental holding cost increase with preservation case yields minimum cost. Therefore, it can be concluded that preservation investment plays a major role in lowering the inventory cost.

Keywords: Deterioration, Incremental holding cost, Inventory-level based demand rate, Inventory model with storage time dependent holding cost, Preservation technology**,** Retroactive holding cost.

INTRODUCTION:

Basically, the consumer's demand for fresh produce products like, bread, milk, milk products like cheese, butter etc. are dependent on the age of the inventory which plays a major role in decision making of consumer's purchase, can be negatively impacted due to the damage of consumer's confidence on the product quality. Hence, the measurement of freshness of the product and the size of its shelf space for displaying the products which obviously fascinate more and more customers to purchase the product.

In many inventory system the deterioration in products is commonly observed, resulting in extreme damages in terms of quality as well as quantity of items. Various steps have been taken to reduce the deterioration effect. The preservation technology investment way is commonly preferred way to reduce deterioration rate. Various literature work on inventory control are done on the basis of assuming fixed rate of demand over entire inventory cycle. But, practically, there are many factors affecting the rate of demand such as the price associated with selling of the items and the obtainability of items.

Baker and Urban (1988) derived an inventory model with stock-dependent demand rate expressed as a polynomial function. The reorder point and the optimal order size are computed using non-linear programming algorithm. An inventory model consists of demand rate during stock-out periods differs from the in-stock period demand by a given amount, where the demand rate depends on both the initial stock and the instantaneous stock by formulating a profit maximizing model by Urban (1995).

Sarker *et al.* (1997) proposed an inventory model demonstrating the negative effect of ageing of stock on demand. Further an inventory model by Hsu *et al.* (2006) was constructed for perishable products including the expiration date. Bai and Kendall (2008) developed an inventory model dealing with the fresh produce products with stock- freshness condition dependent demand rate. A model for deteriorating products by considering the measurement of product freshness in terms of the time remained until the expiration date was estimated by Herbon (2014). Hwang and Hahn (2000) constructed an inventory model for an item with an inventory-level dependent demand rate and a fixed expiry date. Those units which are not sold by their expiry date are regarded as useless and therefore discarded. Separable programming is utilized to determine the optimal order level and order cycle length. Many other researchers contributed significantly in the same field like Wang *et al.* (2014), Wu and Chan (2014), and Wu *et al.* (2016).

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In practical situation, there are many aspects which influences the market demand rate like; selling price, stock, quality, time, and efforts in terms of either service and/or advertisement. Levin *et al.* (1972) highlighted on maintaining greater displayed stock level could motivate to purchase more and more. An inventory model was demonstrated by Baker and Urban (1988) expressing the demand pattern as a power function of displayed stock level, which occurs due to the product's popularity and/or variety and its visibility to the consumer. Sana and Chaudhari (2004) developed an inventory model with EOQ concept in which the total profit is maximized on fulfilling the constraints like; budget and capacity of storage with the rate of demand based on item obtainability and expenditures on advertisements. Min and Zhou (2009) proposed an inventory model using the condition of partial back-logging for deteriorating items for stock dependent demand with a limitation on maximum inventory level.

Yang *et al.* (2010) derived an inventory model with stock dependent demand for deteriorating items including partial backlogging and the inflation effect. Lee and Dye (2012) proposed an EOQ model with partial backlogging with stock dependent demand, utilizing preservation technology determining optimum ordered quantity by maximizing the total profit.

An optimum quality target for a manufacturing process was proposed by Shao *et al.* (2000). The rejected items are stored and traded in future by considering the variable holding costs. Beltran and Krass (2002) derived an optimum quantity for an inventory dynamic problem by developing efficient dynamic programming algorithm, with positive or negative demands with an assumption of time based demand and concave holding costs.

Alfares (2007) proposed inventory models with the storage time into a number of distinct periods with successively increasing holding costs. As the storage time extends to the next time period, the new holding cost can be applied either retroactively, or incrementally.

The uniqueness of this article is the symmetric incremental variation of the holding cost function, reflecting the practical scenarios of storage times bifurcated into various arrays, separately with its individual unit holding cost, truly demonstrating the storage of deteriorating items such as food products like; cheese, butter etc., along with the stock dependent demand. More and more sophisticated storage facilities and services are needed for the deteriorating food products kept in storage for longer time; up-lifting the holding cost. The preservation technology investment is utilized in each case with demand rate dependent on stock availability.

The objective of this article is to estimate the minimum inventory cost for an inventory system with rate of demand dependent on inventory level and holding cost based on time. With the assumption of stock dependent demand, the rate of demand is higher for greater level of inventory. On the basis of assumption of per unit holding cost dependent on storage time period, the inventory models are developed based on allowing unit holding cost values to only change throughout the various storage time periods in case if it exceeds definite discrete value; for obtaining the optimal inventory policies. In development of the inventory models there are two types of holding cost step functions in this article: 1. Retroactive increase type, and 2. Incremental increase type. In first case, the rate of unit holding cost of the last storage period is applicable throughout to each storage periods. In second case, the rate of each period, along with the last period, is functional only to units stored in that specific period.

NOTATIONS AND ASSUMPTIONS:

Notations:

Parameters

Q	Ordered quantity (in units)
Dt	Rate of market demand
α	Basic market scale demand
n	Number of distinct time periods with different holding cost rates
t	Time period from the start of the cycle at $t=0$ (in years)
t_i	End time of period i, where $i=1,2,...,n, t_0=0$ *and* $t_n=\infty$ (in years)
A	Ordering cost per order (in dollars)
h_i	Holding cost of the item in period i (in dollars)
T	Cycle Time (in years)
β	Demand parameter indicating elasticity in relation to the inventory level
ξ	Co-efficient of preservation constant

Decision variables

Q	Ordered quantity (in units)
T	Replenishment cycle length (in years)
u	Preservation investment (in dollars)
TIC	Total inventory cost (in dollars)

Functions

$Dt(I(t))$	Demand rate per unit inventory stock at any time t
$h(t)$	Holding cost of the item at time t, $h(t)=h_i$, if $t_{i-1}\le t\le t_i$ (in dollars)

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$I(t)$ Inventory level at time t (in units).

$TIC(Q)$ Total inventory cost in each period (in dollars)

Optimum Values

Q^* Optimum ordered quantity (in units)

T^* Optimum replenishment cycle length (in years)

u^* Optimum preservation investment (in dollars)

TIC* Optimum total inventory cost (in dollars)

Assumptions:

1. The rate of market demand is based on level of stock, stated as,
   $$Dt(I(t)) = \alpha + \beta(I(t)), \quad I \geq 0 \tag{1}$$
2. The variation of holding cost is expressed as an increasing step function of time in storage.
3. The item starts deteriorating as soon as it enters the system.
4. Let the deterioration rate co-efficient be $\theta u = \theta_0$ where $0 \leq \theta_0 < 1$, represents the deterioration under natural conditions and with preservation investment $\theta u = \theta_0 e^{-\xi u}$.
5. Replenishments are instantaneous.
6. Shortages and discounts are not allowed.
7. Inventory system deals with single deteriorating item.

FORMULATION OF MATHEMATICAL MODEL:

The objective is to minimize the Total Inventory Cost (TIC) per unit time, consists of two components: The ordering cost and the holding cost and preservation investment cost (only in case of utilization of preservation technology).

a. Ordering cost: With one order per cycle, the ordering cost per unit time is $\frac{A}{T}$

b. Holding cost: The total holding cost per cycle is obtained by integrating the product of holding cost $h(t)$ and inventory level $I(t)$ over the whole cycle is given by, $\frac{1}{T}\int_0^T h(t)I(t)\,dt$

Therefore, the total inventory cost is given by,

$$TIC = \frac{A}{T} + \frac{1}{T}\int_0^T h(t)I(t)\,dt \tag{2}$$

Let $[0,T]$ be the period of replenishment cycle where a firm tends to sell a single product, as the mozzarella cheese used on pizza which is deteriorating in nature. The firm regulates the level of inventor l to fluctuate market demand $Dt(I(t))$. Let the process of deterioration of cheese begins with the entry of the cheese in the system. Assuming that the inventory deterioration is directly proportional to the level of inventory. $ie.(\theta u)I(t)$ For the period of scheduling horizon $[0,T]$, the level of inventory declines due to the collective influences of rate of market demand, and the inventory level at the end of replenishment cycle reaches zero. This inventory level scenario can be represented by the differential equation (1), with boundary condition, $I(T)=0$

$$\frac{dI(t)}{dt} = -(\theta u)I(t) - Dt \qquad ,0 \leq t \leq T \tag{3}$$

$$\frac{dI(t)}{dt} = -(\theta u)I(t) - \left(\alpha + \beta(I(t))\right) ,0 \leq t \leq T \tag{4}$$

Equation (3) demonstrates that the level of inventory sustains non-negative nature for all time, *i.e.* $\text{for all } t \in [0,T],\ I(t) \geq 0$ without backordering throughout the scheduling horizon.

On solving equation (3), we get,

$$I(t) = \frac{\alpha}{(\theta u + \beta)}\left[e^{(\theta u+\beta)(T-t)} - 1\right] \tag{5}$$

Now, by using the initial condition, $I(0)=Q$ we get,

$$Q = I(0) = \frac{\alpha}{(\theta u + \beta)}\left[e^{(\theta u+\beta)T} - 1\right] \tag{6}$$

$$\Rightarrow \frac{\alpha}{(\theta u + \beta)} e^{(\theta u+\beta)T} = Q + \frac{\alpha}{(\theta u + \beta)} \tag{7}$$

On solving equation (5), we get,

$$I(t) = \frac{\alpha}{(\theta u + \beta)} e^{(\theta u+\beta)T} e^{-(\theta u+\beta)t} - \frac{\alpha}{(\theta u + \beta)} \tag{8}$$

On substituting equation (7) in equation (8), we have,

$$I(t) = \left[Q + \frac{\alpha}{(\theta u + \beta)}\right] e^{-(\theta u+\beta)t} - \frac{\alpha}{(\theta u + \beta)} \tag{9}$$

$$\therefore I(t) = \begin{pmatrix} Qe^{-(\theta u+\beta)t} \\ +\left(e^{-(\theta u+\beta)t} - 1\right)\left(\frac{\alpha}{(\theta u + \beta)}\right) \end{pmatrix} \tag{10}$$

Now using boundary condition, $I(T)=0$ in equation (10), the replenishment cycle length is,

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$$T = -\frac{1}{(\theta u+\beta)}\left[\ln\left(\frac{\frac{\alpha}{(\theta u+\beta)}}{Q+\frac{\alpha}{(\theta u+\beta)}}\right)\right] \quad (11)$$

OR $$Q = \frac{\left(1-e^{-(\theta u+\beta)T}\right)\left(\frac{\alpha}{(\theta u+\beta)}\right)}{e^{-(\theta u+\beta)T}} \quad (12)$$

SOLUTION METHODOLOGY

Algorithm for case 1: Retroactive holding cost increase without preservation:

As per the former assumption, the holding cost is expressed as an increasing step function of storage time $h_1 < h_2 < h_3 < ..... < h_n$ .Case-1 consists of a uniform holding cost depending on the length of storage. Precisely, the holding cost of the last storage period applies retroactively to all previous periods. Therefore, if the cycle ends in period m $(t_{m-1} \le T \le t_m)$. Then the rate of holding cost h_m is applied to all periods $1,2,...,m.$ In this case, the total inventory cost (TIC) per unit time can be expressed as,

$$TIC = \frac{A}{T}+\frac{h_i}{T}\int_0^T I(t)dt, \qquad t_{i-1} \le T \le t_i \quad (13)$$

Substituting the value of $I(t)$ from equation (10) and T from equation (11) and then equating the first order derivative of TIC with respect to Q as zero, we obtain the value of Q as optimum value $Q*$ for $t_{m-1} \le T \le t_m$ from equation (A1) in appendix-1

The optimum solution can be determined by using steps in flowchart demonstrated in Figure-1.

Example-1: (Case-1: Retroactive holding cost increase without preservation)

Considering following parametric values:
$\alpha = 400, \beta = 0.1, \theta o = 40\%, A = \$300 / unit,$

$t_1 = 0.4 years \approx 146\, days, t_2 = 0.5 years \approx 182\, days,$

$t_3 = \infty, h_1 = \$5 / unit / year, 0 < T \le 0.4,$

$h_2 = \$6 / unit / year, 0.4 < T \le 0.5,$

$h_3 = \$7 / unit / year, 0.5 < T \le \infty$

Solution:

Step-1: Starting with $h_1 = 5$,
then $Q = 228\, units, T = 0.5032\, year$.

It is not realizable as $T = 0.5032\, year$ is not satisfying the range $0 < T \le 0.4$

Substituting $h_2 = 6$, then $Q = 208\, units, T = 0.4625\, year$.

It is realizable as $T = 0.4625\, year$ is satisfying the range $0.4 < T \le 0.5$

Therefore $Q_R = 208\, units$.

Step-2: $Q_1 (T = 0.4) = 177\, units$ and $Q_2 (T = 0.5) = 227\, units$ **.**

Step-3: $TIC(Q = Q_R = 208) = 1249.0221$ at $h_i = 6$

$TIC(Q = Q_1 = 177) = 1178.0551$ at $h_i = 5$

Step-4: The optimum solution is: $Q* = 177\, units,\ T* = 0.4\, year,\ TIC = \$1178.0551 / year$

Algorithm for case 1: Retroactive holding cost increase with preservation:

The algorithm process in this case, also remains similar to case-1 without preservation as shown in Figure-1 flowchart. The deterioration rate is $\theta u = \theta o e^{-\xi u}$. But the total inventory cost (TIC) includes the preservation technology investment cost along with ordering cost and holding cost as shown in equation (14),

$i.e. TIC = OC + HC + PTI$

$$TIC = \frac{A}{T}+\frac{h_i}{T}\int_0^T I(t)dt + uT,\ t_{i-1} \le T \le t_i \quad (14)$$

Equating the first order partial derivatives of TIC with respect to Q as well as with respect to u as zero, we obtain the value of Q *and* u as optimum values $Q*$ *and* $u*$ for $t_{m-1} \le T \le t_m$ from equation (A2) in appendix-2.

Example-2: (Case-1: Retroactive holding cost increase with preservation)

Considering following parametric values:
$\alpha = 400, \beta = 0.1, \theta o = 10\%, A = \$300 / unit,$

$t_1 = 0.4 years \approx 146\, days, t_2 = 0.5 years \approx 182\, days,$

$t_3 = \infty, \xi = 0.9, h_1 = \$5 / unit / year, 0 < T \le 0.4,$

$h_2 = \$6 / unit / year, 0.4 < T \le 0.5,$

$h_3 = \$7 / unit / year, 0.5 < T \le \infty$

Solution

Step-1: Starting with $h_1 = 5$,
then $Q = 219\, units, u = 3.1101, T = 0.5452\, year$.

It is not realizable as $T = 0.5452\, year$ is not satisfying the range $0 < T \le 0.4$

Substituting $h_2 = 6$,
then $Q = 199\, units, u = 3.2114, T = 0.4980\, year$.

It is realizable as $T = 0.4980\, year$ is satisfying the range $0.4 < T \le 0.5$

Therefore $Q_R = 199\, units$.

Step-2:
$Q_1\left(T = 0.4, u = 3.33\right) = 160\, units$ and
$Q_2\left(T = 0.5, u = 3.1\right) = 200\, units$.

Step-3: $TIC\left(Q = Q_R = 199, u = 3.21\right) = 1203.1592 \;\; at \;\; h_i = 6$

$TIC\left(Q = Q_1 = 160, u = 3.33\right) = 1152.1384 \;\; at \;\; h_i = 5$

Step-4: The optimum solution is:
$Q^* = 160\, units,\; T^* = 0.4 year,\; u = 3.33,\; TIC = \$1152.1384 / \, year$

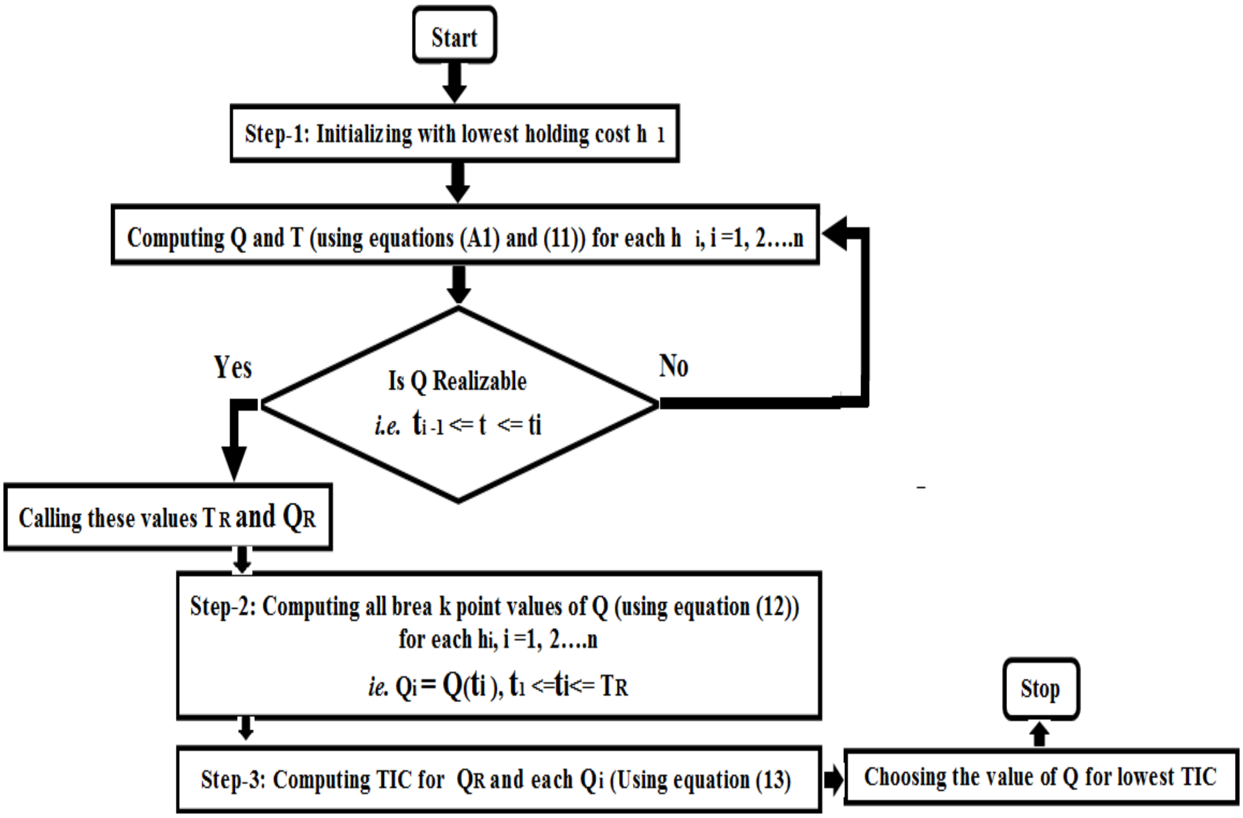

Figure 1: Flowchart for case-1: Retroactive holding cost increase without preservation

Algorithm for case 2: Incremental holding cost increase without preservation:

In this case, assuming that the holding cost is expressed as an incremental step function of storage time. Following this function, the storage in later periods have higher storage cost rates. Thus, if the cycle ends in period m, $\left(t_{m-1} \le T \le t_m\right)$ then the holding cost h_1 is applied to period 1, rate h_2 is applied to period 2, and so on; thus rate h_m is applied to period m from time t_{m-1} up-to time T, and so on. For this case, we first reset the value of t_m as $\left(t_m = T\right)$, and then express the TIC per unit time as,

$$TIC = \frac{A}{T} + \left(\frac{h_1}{T}\int_0^{t_1} I(t)\,dt + \frac{h_2}{T}\int_{t_1}^{t_2} I(t)\,dt + \ldots + \frac{h_m}{T}\int_{t_{m-1}}^{t_m = T} I(t)\,dt \right) \quad (15)$$

Substituting the values of $I(t)$ from equation (5) and T from equation (11) and then equating the first order derivative of TIC with respect to Q as zero, we obtain the value of Q as optimum value Q^* for $t_{m-1} \le T \le t_m$ from equation (A3) in appendix-3 In case, if the entire inventory cycle falls in the period 1 only when $m=1$ then the optimum solution is obtained by substituting h_1 into (A1) to calculate Q^* and then substituting Q^* into (11) to calculate T. In general, the optimum solution must be determined by the algorithm as shown in Figure-2.

Example-3: (Case-2: Incremental holding cost increase without preservation)

Considering following parametric values:
$\alpha = 400, \beta = 0.1, \theta o = 40\%, A = \$300 / unit,$

$t_1 = 0.4 year \approx 146\,days, t_2 = 0.5 year \approx 182\,days,$

$t_3 = \infty, h_1 = \$5 / unit / year, 0 < T \le 0.4,$

$h_2 = \$6 / unit / year, 0.4 < T \le 0.5,$

$h_3 = \$7 / unit / year, 0.5 < T \le \infty$

Solution:

Step-1: Using the same data as in example-1, computing $Q_{\max} = 228\,units \;\; and \;\; T_{\max} = 0.5032\,year,$

$As\ such\ T_{\max} > t_1,\ we\ must\ continue.$

Step-2: Solving equation (A1), $Q_{\min} = 192$ *units* and by using equation (11), $T_{min} = 0.4305$ *year*.

Step-3: Since, $T_{\min} = 0.4305$ is in period 2 and $T_{\max} = 0.5032$ is in period 3, we need to develop a total cost expression only for two possible end periods, m =2 and m =3.

Step-4: (a) Assuming $m = 2$

First, the cycle is assumed to end in the second period ($m = 2$). Thus, the cycle length T is assumed to be between $T_{\min} = 0.4305$ and $t_2 = 0.5 year$, i.e. the range of T in year is ($0.4305 \le T \le 0.5$). Using (12), the corresponding Q range in units is ($192 \le Q \le 279$). Substituting the given values in (A3), we obtain $Q_R = 222$ *units*.

Q_R =222 units coming in the range ($192 \le Q \le 279$).

So, relizable.

(b) Assuming $m = 3$,

Now, the cycle is assumed to end in the third period ($m = 3$). Thus, the cycle length T is assumed to be between $t_2 = 0.5 year$ and $T_{\max} = 0.5032$ year ($0.5 \le T \le 0.5032$). The corresponding Q range is ($227 \le Q \le 228$) from equation (A3), we obtain, $Q = 197\ units\ (It\ is\ not\ relizable)$

Step-5: The TIC should now be calculated for the two values of Q corresponding to the break points $\left(Q_1(T=0.4) = 177\,units\ and\ Q_2(T=0.5) = 227\,units\right)$

Since Q1 corresponds to $(m=1)$,

So, by equation (13) $TIC(Q=177) = 1178.0551$.

Now using equation (15) to calculate TIC for $Q_R = 222$ and $Q_2 = Q(T = t_2 = 0.5) = 227$

(Both corresponding to $m = 2$)

TIC(Q_2=227)=1148.4741 and TIC(Q_R=222)=1148.1937

Step-6: The optimum solution is given by

Q*=Q_R=222 units, T*=0.4902 year and TIC=$1148.1937/year

Algorithm for case 2: Incremental holding cost increase with preservation:

The algorithm process in this case, also remains similar to case-2 without preservation as shown in Figure-2 flowchart. The deterioration rate is $\theta u = \theta o e^{-\xi u}$. But the total inventory cost (TIC) includes the preservation technology investment cost along with ordering cost and holding cost as shown in equation (16),

$i.e.\ TIC = OC + HC + PTI$

$$TIC = \begin{pmatrix} \dfrac{A}{T} + \dfrac{h_1}{T}\displaystyle\int_0^{t_1} I(t)\,dt + \dfrac{h_2}{T}\int_{t_1}^{t_2} I(t)\,dt \\ +\ldots+ \dfrac{h_m}{T}\displaystyle\int_{t_{m-1}}^{t_m=T} I(t)\,dt + uT \end{pmatrix} \quad (16)$$

Equating the first order partial derivatives of TIC with respect to Q as well as with respect to u as zero, we obtain the value of Q as optimum value $Q*$ for $t_{m-1} \le T \le t_m$ from equation (A4) in appendix-4.

Example-4: (Case-2: Incremental holding cost increase with preservation)

Considering following parametric values:
$\alpha = 400, \beta = 0.01, \theta o = 10\%, A = \$300 / unit,$

$t_1 = 0.4 year \approx 146\,days, t_2 = 0.5 year \approx 182\,days,$

$t_3 = \infty, h_1 = \$5 / unit / year, 0 < T \le 0.4,$

$h_2 = \$6 / unit / year, 0.4 < T \le 0.5,$

$h_3 = \$7 / unit / year, 0.5 < T \le \infty$

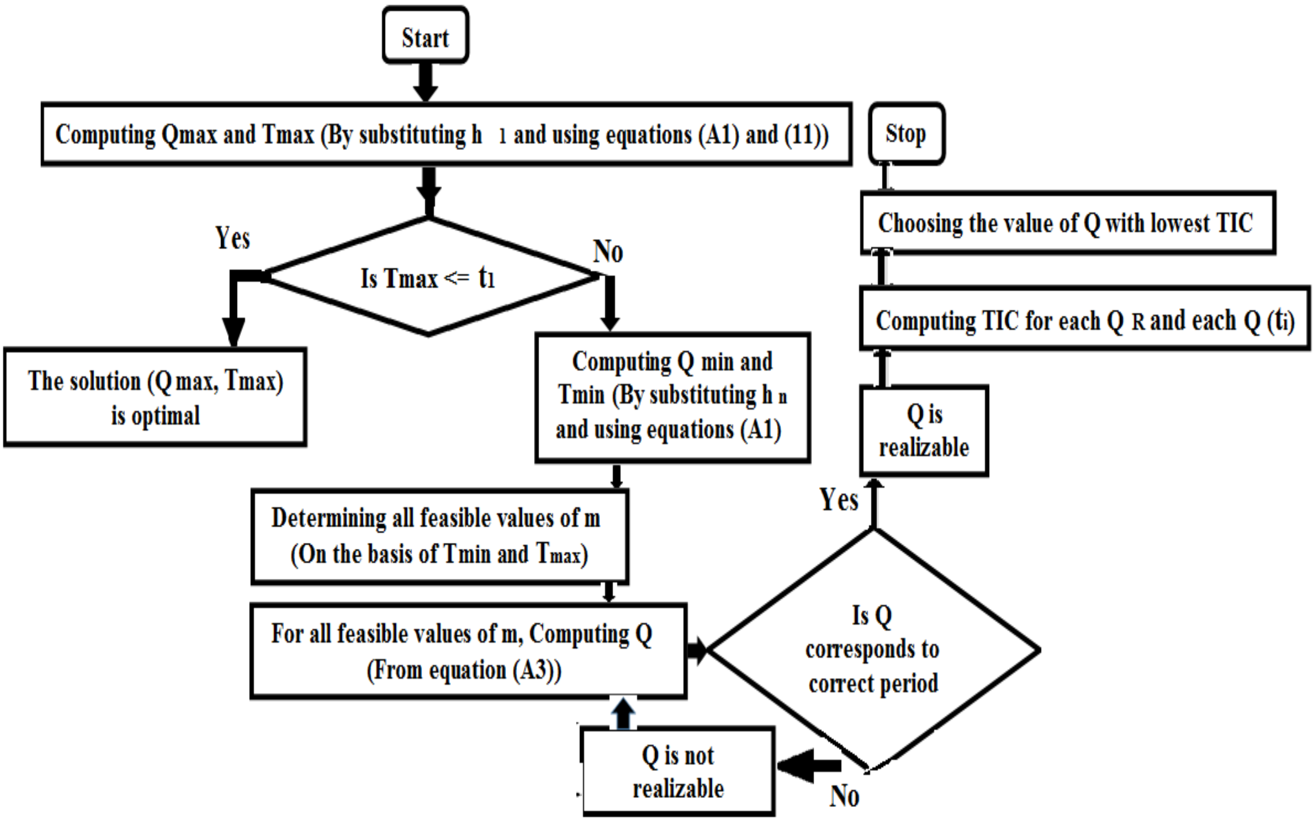

Figure 2: Flowchart for incremental holding cost increase without preservation

<u>**Solution:**</u>

Step-1: Computing

$Q_{\max} = 219\, units, u = 3.1101 \;\; and \;\; T_{\max} = 0.5452\; year,$

As such $T_{\max} > t_1$*, we must continue.*

Step-2: Solving equation (A4),

$Q_{\min} = 185$ *units*, $u = 3.2971$ and by using equation (11), $T_{min} = 0.4612$ *year*

Step-3: Since, $T_{\min} = 0.4612$ is in period 2 and $T_{\max} = 0.5452$ is in period 3, we need to develop a total cost expression only for two possible end periods, $m = 2$ and $m = 3$.

Step-4: (a) Assuming $m = 2$

First, the cycle is assumed to end in the second period ($m = 2$). Thus, the cycle length T is assumed to be between $T_{\min} = 0.4612$ and $t_2 = 0.5 year$, i.e. the range of T in year is ($0.4612 \le T \le 0.5$). Using (12), the corresponding Q range in units is ($185 \le Q \le 200$). Substituting the given values in (A3), we obtain $Q_R = 210$ *units*.

Q_R=210 units not coming in the range ($185 \le Q \le 200$).

So, not relizable.

(b) Assuming $m = 3$,

Now, the cycle is assumed to end in the third period ($m = 3$). Thus, the cycle length T is assumed to be between $t_2 = 0.5 year$ and $T_{\max} = 0.5452$ year ($0.5 \le T \le 0.5452$). The corresponding Q range is ($201 \le Q \le 219$) from equation (A4), we obtain, $Q = 209$ *units*, $u = 3.0615$ $(It\; is\; relizable)$

Step-5: The *TIC* should now be calculated for the two values of *Q* corresponding to the break points:

$Q_1(T = 0.4) = 177\; units\; and\; Q_2(T = 0.5) = 227\; units$

since Q_1 corresponds to $(m = 1)$,

So, by equation (13), $TIC(Q = 177) = 1178.0551$.

Now using equation (15) to calculate *TIC* for $Q_R = 209$ and $Q_2 = Q(T = t_2 = 0.5) = 200$

(Both corresponding to $m = 2$)

TIC(Q_2=200)=1106.8965 and

TIC(Q_R=209)=1105.6845,

TIC(Q_R=219)=1098.7479

Step-6: The optimum solution is given by

$Q^*=Q_R=209$ units, $u=3.0615$,

$T^*=0.5452$ year and

TIC=$1098.7479/year

The comparative analysis of total cost of the four prescribed examples for various cases with/without preservation is demonstrated in Figure-3.Now, to maximize the total cost stated in equation (A4), we apply the below stated necessary and sufficient condition:

$$\frac{\partial TC}{\partial Q}=0, \frac{\partial TC}{\partial u}=0 \qquad (17)$$

To check the convexity of the total cost function of obtained solution, we adopt the below stated algorithm,

Step 1: Assigning the inventory parameters some specific hypothetical values.

Step 2: Obtaining the solutions by solving simultaneous equations stated in equation (17), utilizing the mathematical software Maple XVIII.

Step 3: Computing all the Eigen values of below stated hessian matrix H at the optimal point obtained from equation (17),

$$H=\begin{bmatrix} \frac{\partial^2 TC}{\partial Q^2} & \frac{\partial^2 TC}{\partial Qu} \\ \frac{\partial^2 TC}{\partial uQ} & \frac{\partial^2 TC}{\partial u^2} \end{bmatrix}$$

- If all of the eigenvalues are positive, it is said to be a positive-definite matrix. Then the cost function is convex then stop.

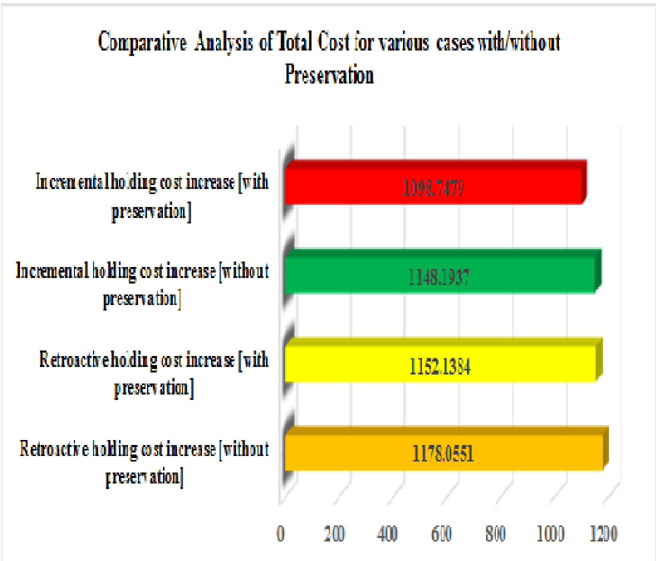

Figure 3: Comparative Analysis of Total Cost

For the case-2, in incremental holding cost increase with preservation, the inventory cost is minimum. Therefore, the hessian matrix for case-2 under preservation technology is given by,

$$H=\begin{bmatrix} 0.0227 & 0.005 \\ 0.005 & 0.4626 \end{bmatrix}$$

The two eigenvalues are computes as, $\lambda_1=0.0227>0$, $\lambda_2=0.4627>0$. So the cost function is convex in nature as shown in the Figure-4.

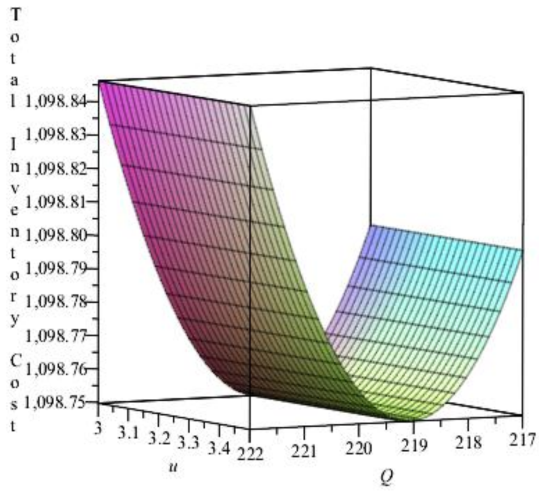

Figure 4: Convexity of Cost function for case-2

Sensitivity analysis on the optimal inventory policy:

In this part, the sensitivity analysis of the decision variables with respect to various inventory parameters is carried out. Table-1 demonstrates the values of decision variables on varying the various inventory parameters from case-2 with utilization of preservation technology in the range -20% to 20%. From table-1 the below stated observations are extracted;

a. Sensitivity analysis of basic market scale demand (α):

With respect to increase in basic market demand α, the ordered quantity of deteriorating items increases by rising the preservation constant. Hence, the replenishment cycle length declines as the stock is deteriorating which has to be cleared at the earliest. The total cost of the system increases by the variation in basic market demand.

b. Sensitivity analysis of co-efficient of the inventory level in market demand (β):

The variation in co-efficient of the inventory level in market demand β, leads to the rise in deteriorating inventory by increasing the preservation constant. Hence the replenishment cycle length declines as the stock is deteriorating which has to be cleared at the earliest. The total cost of the system increases by the variation in the co-efficient of the inventor level in market demand.

c. Sensitivity analysis of the deterioration rate co-efficient (θo):

With the variation in the deterioration rate co-efficient θo, the ordered quantity declines as the stock starts deteriorating, with shortening replenishment cycle length in order to clear the stock as quickly as possible and raising the preservation constant to maintain the freshness of the product. Hence the total cost of the system uplifts in this case.

d. Sensitivity analysis of the Ordering cost per order (A):

With th-e increment in the ordering cost per order A, each decision variable demonstrates an increment. The rise in ordered quantity, the preservation constant is observed with lengthening of the replenishment cycle length. Hence the total cost of the system uplifts in this case also.

Table 1: Sensitivity Analysis of various inventory parameters

Inventory Parameters	**Decision Variables**	**Percentage variation of Decision Variables**				
		-20%	**-10%**	**0**	**10%**	**20%**
α	Q	195.9315	207.8237770	219.0715	229.7695	239.9912
	T	0.6091	0.5745	0.5452	0.5200	0.4980
	u	2.9861	3.0515	3.1101	3.1631	3.2114
	TC	983.2934	1042.6251	1098.7479	1152.1407	1203.1531
β	Q	219.0315	219.0515	219.0715	219.0916	219.1116
	T	0.5454	0.5453	0.5452	0.5451	0.5450
	u	3.1099	3.110	3.1101	3.1102	3.1103
	TC	1098.5457	1098.6495	1098.7479	1098.8515	1098.9478
θo	Q	219.0985	219.0843	219.0715	219.0600	219.0495
	T	0.5453	0.5453	0.5452	0.5452	0.5452
	u	2.8623	2.9931	3.1101	3.2159	3.3126
	TC	1098.6136	1098.6846	1098.7479	1098.8085	1098.8575
A	Q	195.9367	207.8258	219.0715	229.7684	239.9897
	T	0.4878	0.5173	0.5452	0.5718	0.5971
	u	2.9861	3.0515	3.1101	3.1631	3.2114
	TC	982.5944	1042.2905	1098.7479	1152.4615	1203.7813
ξ	Q	219.0512	219.0617	219.0715	219.0807	219.0892
	T	0.5450	0.5451	0.5452	0.5453	0.5454
	u	3.5776	3.3255	3.1101	2.9237	2.7606
	TC	1099.1546	1098.9349	1098.7479	1098.5890	1098.4615

e. Sensitivity analysis of the co-efficient of preservation constant (ξ):

With the increment in the co-efficient of preservation constant ξ, the rise in ordered quantity with lengthening the replenishment cycle length is seen. The preservation constant increases. Hence the total cost of the system declines in this case.

RESULT AND DISCUSSION:

This article deals with the computation of total inventory cost for deteriorating inventory along with the investigation of combined optimal replenishment cycle length, optimal ordered quantity and preservation technology constant for inventory models with the storage time dependent holding cost and rate of market demand is assumed to fluctuate as a function, based on level of stock as practically, the larger product availability leads to greater sales. The holding cost depending on storage time period and rate of market demand is assumed to fluctuate as a function, based on level of stock as practically, the larger product availability leads to greater sales. The product in this article is considered as the mozzarella cheese used on pizza. . Four inventory models with storage time-dependent holding cost are considered which includes Retroactive holding cost increase, and incremental holding cost increase with and without the utilization of preservation technology in each case.

By developing algorithms for each case and utilizing the classical optimization technique for calculating the optimal values. Thereafter, using the concept of eigen-values of a Hessian matrix, we have proved the convex nature of the cost function for the case with minimum cost is obtained. Finally, in order to validate the derived models, numerical examples along with sensitivity analysis is undertaken, which extracts that incremental holding cost increase with preservation case yields minimum cost.

Only the parameter ξ , lowers the total cost of the system rest other parameters like α , β , θo and A increases the total inventory cost. Therefore, it can be concluded that preservation investment plays a major role in lowering the inventory cost.

The derived model can be further extended by utilizing the concept of trade credit and/or including the constraint of shortages, partial backlogging.

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Appendix1: (A1)

$$\frac{dTIC}{dQ} = \left[-\frac{-\frac{H(\theta u+\beta)\alpha}{Q\beta+Q\theta u+\alpha}+H\beta+H\theta u}{(\theta u+\beta)\ln\left(\frac{\alpha}{Q\beta+Q\theta u+\alpha}\right)} - \frac{\left(\begin{array}{l} A\beta^2+2A\beta\theta u+A\theta u^2 \\ +H\ln\left(\frac{\alpha}{Q\beta+Q\theta u+\alpha}\right)\alpha+HQ\beta+HQ\theta u\end{array}\right)}{\ln\left(\frac{\alpha}{Q\beta+Q\theta u+\alpha}\right)^2 (Q\beta+Q\theta u+\alpha)} \right]$$

Appendix-2: (A2)

$$\frac{\partial TIC}{\partial Q} = -$$

$$\left\{\left(\begin{array}{l}
Ae^{-2\xi u}\alpha\theta o^5 - 4\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 \\
e^{2\xi u}Q\beta^3 u\theta o - \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 e^{3\xi u}Q\beta^4 u \\
+50Q^2\beta^2\theta o^3 + 5e^{3\xi u}Q^2\beta^5 + 5e^{-2\xi u}Q^2\theta o^5 \\
+5Ae^{-\xi u}\alpha\beta\theta o^4 - 6\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 e^{\xi u}Q\beta^2 u\theta o^2 \\
-3\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 e^{\xi u}\alpha\beta u\theta o^2 \\
+30\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{\xi u}Q\alpha\beta^2\theta o^2 \\
+30\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right) e^{\xi u}Q\alpha\beta^2\theta o^2 \\
-3\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 e^{2\xi u}\alpha\beta^2 u\theta o
\end{array}\right.\right.$$

$$\left.\begin{array}{l}
+20\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{2\xi u}Q\alpha\beta^3\theta o \\
+5\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{3\xi u}Q^2\beta^5 \\
+Ae^{3\xi u}\alpha\beta^5 + 5\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) \\
e^{-2\xi u}Q^2\theta o^5 + 5e^{-\xi u}Q\alpha\theta o^4 \\
+5\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{-\xi u}Q\alpha\theta o^4 \\
+15\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right) e^{2\xi u}\alpha^2\beta^2\theta o \\
+20e^{2\xi u}Q\alpha\beta^3\theta o - \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 \alpha u\theta o^3 \\
+5\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{3\xi u}Q\alpha\beta^4 \\
+25\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{2\xi u}Q^2\beta^4\theta o \\
+5\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right) e^{3\xi u}Q\alpha\beta^4 \\
+5Ae^{2\xi u}\alpha\beta^4\theta o + 6Ae^{2\xi u}Q\beta^5\theta o \\
+6Ae^{-2\xi u}Q\beta\theta o^5 - \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 e^{3\xi u}\alpha\beta^3 u \\
+5\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right) e^{3\xi u}\alpha^2\beta^3 \\
+5e^{3\xi u}Q\alpha\beta^4 + 25e^{2\xi u}Q^2\beta^4\theta o \\
+Ae^{3\xi u}Q\beta^6 + Ae^{-3\xi u}Q\theta o^6 + 10A\alpha\beta^2\theta o^3 \\
+20Q\alpha\beta\theta o^3 + 50\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) Q^2\beta^2\theta o^3 \\
+50e^{\xi u}Q^2\beta^3\theta o^2 + 25e^{-\xi u}Q^2\beta\theta o^4 + 20AQ\beta^3\theta o^3 \\
+15Ae^{\xi u}Q\beta^4\theta o^2 + 15Ae^{-\xi u}Q\beta^2\theta o^4 \\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{-\xi u}Qu\theta o^4 \\
+50\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) e^{\xi u}Q^2\beta^3\theta o^2
\end{array}\right)$$

$$\left.\begin{pmatrix} +25\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-\xi u}Q^2\beta\theta o^4 \\ +10Ae^{\xi u}\alpha\beta^3\theta o^2-4\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 \\ Q\beta u\theta o^3+5\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right)e^{-\xi u}Q\alpha\theta o^4 \\ +20\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\alpha\beta\theta o^3 \\ +15\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right)e^{\xi u}\alpha^2\beta\theta o^2 \\ +20\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right)Q\alpha\beta\theta o^3 \\ +30e^{\xi u}Q\alpha\beta^2\theta o^2+5\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right)\alpha^2\theta o^3 \\ +20\ln\left(\frac{e^{\xi u}\alpha}{e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha}\right)e^{2\xi u}Q\alpha\beta^3\theta o \end{pmatrix}\right\}$$

$$\overline{\begin{pmatrix} \left(e^{\xi u}Q\beta+Q\theta o+e^{\xi u}\alpha\right)\left(e^{-\xi u}Q\theta o+Q\beta+\alpha\right) \\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2\left(\beta^2e^{2\xi u}+2\beta e^{\xi u}\theta o+\theta o^2\right) \\ \left(\theta o e^{-\xi u}+\beta\right) \end{pmatrix}}$$

$$\frac{\partial TIC}{\partial u}=\frac{1}{\begin{pmatrix} \left(e^{-\xi u}Q\theta o+Q\beta+\alpha\right) \\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2 \\ \left(\theta o e^{-\xi u}+\beta\right)^2 \end{pmatrix}}.$$

$$\left\{\begin{pmatrix} A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-4\xi u}Q\xi\theta o^4 \\ +3A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-3\xi u}Q\beta\xi\theta o^3 \\ +Ae^{-4\xi u}Q\xi\theta o^4+A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) \\ e^{-3\xi u}\alpha\xi\theta o^3+3A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right) \\ e^{-2\xi u}Q\beta^2\xi\theta o^2+3Ae^{-3\xi u}Q\beta\xi\theta o^3 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} -\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3e^{-2\xi u}Qu\xi\theta o^2 \\ +2A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-2\xi u}\alpha\beta\xi\theta o^2 \\ +A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-\xi u}Q\beta^3\xi\theta o \\ +3Ae^{-2\xi u}Q\beta^2\xi\theta o^2-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3 \\ e^{-\xi u}Q\beta u\xi\theta o-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2e^{-2\xi u}Qu\xi\theta o^2 \\ +A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-\xi u}\alpha\beta^2\xi\theta o \\ +Ae^{-\xi u}Q\beta^3\xi\theta o-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3 \\ e^{-2\xi u}Q\theta o^2-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3e^{-\xi u}\alpha u\xi\theta o \\ -\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2e^{-\xi u}Q\beta u\xi\theta o \\ -2\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3e^{-\xi u}Q\beta\theta o \\ -\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3e^{-\xi u}\alpha\theta o \\ -\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3Q\beta^2 \\ -\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^3\alpha\beta \end{pmatrix}\right\}$$

Appendix-3: (A3)

$$\frac{dTIC}{dQ}=-\left(\frac{1}{\ln\left(\frac{\alpha}{Q\beta+Q\theta o+\alpha}\right)^2\left(Q\beta+Q\theta o+\alpha\right)}\right).$$

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$$\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\begin{array}{l} H_i e^{-(\theta o+\beta)t_i}\ln\left(\frac{\alpha}{Q\beta+Q\theta o+\alpha}\right)\\ (Q\beta+Q\theta o)-H_i e^{-(\theta o+\beta)t_{i+1}}\\ \ln\left(\frac{\alpha}{Q\beta+Q\theta o+\alpha}\right)(Q\beta+Q\theta o)\\ +A\beta^2+A\theta o^2\\ +k_{ij}\left(\left(t_iH_j\right)(\alpha\beta+\alpha\theta o)\right)\\ +k_{ij}H_i e^{-(\theta o+\beta)t_j}\\ \left(\begin{array}{l}\ln\left(\frac{\alpha}{Q\beta+Q\theta o+\alpha}\right)\alpha\\ +Q\beta+Q\theta o\end{array}\right)\\ +2A\beta\theta o+k_{ij}H_i e^{-(\theta o+\beta)t_j}\alpha \end{array}\right)\right)$$

$$where,\ k_{ij}=\begin{cases}1, & i=j\\ -1, & i\neq j\end{cases}$$

Appendix-4: (A4)

$$\frac{\partial TIC}{\partial Q}=-\frac{1}{\left(\begin{array}{l}\left(e^{-\xi u}Q\theta o+Q\beta+\alpha\right)\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2\\ \left(\theta o e^{-\xi u}+\beta\right)\end{array}\right)}.$$

$$\left\{\sum_{i=1}^{m}\left(\sum_{j=1}^{m}k_{ij}\left(\begin{array}{l} H_it_je^{-2\xi u}\alpha\theta o^2+H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-2\xi u}Q\theta o^2\\ +2H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\beta\theta o\\ +2H_it_je^{-\xi u}\alpha\beta\theta o\\ +H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-2\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\theta o^2\\ +Ae^{-3\xi u}\theta o^3-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2\beta u\\ +H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)\alpha\theta o \end{array}\right)\right)\right.$$

$$\left.+\sum_{i=1}^{m}\left(\sum_{j=1}^{m}kij\left(\begin{array}{l} -2H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}Q\beta\theta o\\ +H_ie^{-\left(\theta oe^{-\xi u}+\beta\right)t_j}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)\alpha\beta\\ +H_ie^{-\left(\theta oe^{-\xi u}+\beta\right)t_j}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\beta^2\\ +A\beta^3+H_ie^{-\left(\theta oe^{-\xi u}+\beta\right)t_j}Q\beta^2\\ -H_it_j\alpha\beta^2-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2\\ e^{-\xi u}u\theta o\\ +H_ie^{-\left(\theta oe^{-\xi u}+\beta\right)t_j}\alpha\beta+3Ae^{-2\xi u}\beta\theta o^2\\ +H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\alpha\theta o+3Ae^{-\xi u}\beta^2\theta o \end{array}\right)\right)\right\}$$

$$where,\ k_{ij}=\begin{cases}1, & i=j\\ -1, & i\neq j\end{cases}$$

$$\frac{\partial TIC}{\partial u}=-\frac{1}{\left(\begin{array}{l}\left(e^{-\xi u}Q\theta o+Q\beta+\alpha\right)\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^2\\ \left(\theta o e^{-\xi u}+\beta\right)^2\end{array}\right)}.$$

$$\left\{\sum_{i=1}^{m}\left(\sum_{j=1}^{m}k_{ij}\left(\begin{array}{l} H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_jQ^2\beta^3\xi\theta o\\ +H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_j\alpha^2\beta\xi\theta o\\ +H_it_je^{-\xi u}Q\alpha\beta^2\xi\theta o+H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\alpha\beta\xi\theta o\\ +H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-3\xi u}Q^2\xi\theta o^3\\ -2H_ie^{-t_je^{-\xi u}\theta o-t_j\beta-\xi u}\\ \ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_jQ\alpha\beta^2\xi\theta o \end{array}\right)\right)\right.$$

$$
+\sum_{i=1}^{m}\left(\sum_{j=1}^{m} kij\left(\begin{array}{l}
+2A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-2\xi u}\alpha\beta\xi\theta o^{2}\\
+3A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-3\xi u}Q\beta\xi\theta o^{3}\\
-H_{i}t_{j}e^{-3\xi u}Q\alpha\xi\theta o^{3}+3A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)\\
e^{-2\xi u}Q\beta^{2}\xi\theta o^{2}\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}Q\beta^{2}\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}\alpha\beta\\
+2H_{i}e^{-t_{j}e^{-\xi u\theta o-t_{j}\beta-2\xi u}}Q^{2}\beta\xi\theta o^{2}\\
+3H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-3\xi u}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)\\
t_{j}Q^{2}\beta\xi\theta o^{3}+2H_{i}e^{-t_{j}e^{-\xi u\theta o-t_{j}\beta-3\xi u}}\\
\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_{j}Q\alpha\xi\theta o^{3}\\
+3H_{i}e^{-t_{j}e^{-\xi u\theta o-t_{j}\beta-2\xi u}}\\
\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_{j}Q^{2}\beta^{2}\xi\theta o^{2}\\
+2H_{i}t_{j}e^{-2\xi u}Q\alpha\beta\xi\theta o^{2}\\
+Ae^{-4\xi u}Q\xi\theta o^{4}-2\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}\\
e^{-\xi u}Q\beta\theta o+A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-3\xi u}\alpha\xi\theta o^{3}\\
+3Ae^{-3\xi u}Q\beta\xi\theta o^{3}\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}e^{-2\xi u}Qu\xi\theta o^{2}\\
+3Ae^{-2\xi u}Q\beta^{2}\xi\theta o^{2}-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{2}\\
e^{-2\xi u}Qu\xi\theta o^{2}\\
+A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-4\xi u}Q\xi\theta o^{4}
\end{array}\right)\right)
$$

$$
+\sum_{i=1}^{m}\left(\sum_{j=1}^{m} kij\left(\begin{array}{l}
Ae^{-\xi u}Q\beta^{3}\xi\theta o-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}e^{-\xi u}\alpha u\xi\theta o\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{2}e^{-\xi u}Q\beta u\xi\theta o\\
+A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-\xi u}Q\beta^{3}\xi\theta o\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}e^{-\xi u}Q\beta u\xi\theta o\\
+A\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)e^{-\xi u}\alpha\beta^{2}\xi\theta o\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-\xi u}Q\alpha\beta\xi\theta o\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-2\xi u}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_{j}\alpha^{2}\xi\theta o^{2}\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-4\xi u}\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_{j}Q^{2}\xi\theta o^{4}\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}e^{-2\xi u}Q\theta o^{2}\\
-\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)^{3}e^{-\xi u}\alpha\theta o\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-2\xi u}Q\alpha\xi\theta o^{2}\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-\xi u}Q^{2}\beta^{2}\xi\theta o\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-\xi u}\\
\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)\alpha^{2}\xi\theta o\\
+4H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-2\xi u}\\
\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)t_{j}Q\alpha\beta\xi\theta o^{2}\\
+H_{i}e^{-t_{j}e^{-\xi u}\theta o-t_{j}\beta-2\xi u}\\
\ln\left(\frac{\alpha}{e^{-\xi u}Q\theta o+Q\beta+\alpha}\right)Q\alpha\xi\theta o^{2}
\end{array}\right)\right)\Bigg\}
$$