International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 12 (2018) pp. 10804-10818

Equilibrium Problem and Convergence Results for a New Implicit Iteration Process in Banach Space

Anju Panwar[1], Ravi Parkash Bhokal[2]

[1,2] *Department of Mathematics, M.D.U. Rohtak–124001 (Haryana), India.*

Abstract

The aim of the paper is to introduce a new implicit iteration process for three generalized I-asymptotically quasi non expansive mappings in Banach space. Under appropriate conditions on the mappings, we prove weak and strong convergence results. The solution of the equilibrium problem is also provided for the newly introduced implicit iteration process.

Keywords: generalized I– asymptotically quasi non expansive mapping, Opial's condition, semi closed mapping, semi compact mapping, implicit iteration.

2010 AMS Subject Classification: 47H09, 47H10.

INTRODUCTION

Let H be a non empty subset of a real Banach space K. Let $\delta : H \to H$ be a mapping and the set $F(\delta) = \{t \in H : \delta(t) = t, \forall t \in H\}$ denotes the set of fixed points of mapping δ.

Definition 1.1: Let H be a non empty subset of a real Banach space K. Then mapping $\delta : H \to H$ is called

(i) non expansive if

$$\| \delta u - \delta v \| \le \| u - v \| \ \forall u, v \in H.$$

(ii) asymptotically non expansive if for any sequence $\{a_n\} \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, we have

$\| \delta^n u - \delta^n v \| \le (1+a_n) \| u - v \|$ for all $u, v \in H$, $n \ge 1$

(iii) quasi non expansive if

$\| \delta^n u - q \| \le \| u - q \|$ for all $u \in H$, $q \in F(\delta)$ and $n \ge 1$.

(iv) asymptotically quasi non expansive if for any sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, we have

$\| \delta^n u - q \| \le (1+a_n) \| u - q \|$ for all $u \in H$, $q \in F(\delta)$ and $n \ge 1$.

The class of I–non expansive mappings was introduced by Rhoades and Temir [15]. In this direction Yao and Wang [16] introduced the class of I–quasi–non expansive mappings and Shahzad [6] gave generalized I-non expansive mappings. Further these class of mappings are generalized by the introduction of generalized I–asymptotically quasi non expansive mappings by Temir [14].

Definition 1.2: Let H be a non empty subset of a Banach K and I be a self map on H. Mapping $\delta : H \to H$ is said to be

(i) I–non expansive on H. If

$$\| \delta u - \delta v \| \le \| Iu - Iv \|$$

for all $u, v \in H$ and I be a self map on H.

(ii) I–asymptotically non expansive on H if for some sequence $\{a_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} a_n = 0$ such that

$$\| \delta^n u - \delta^n v \| \le (1+a_n) \| I^n u - I^n v \|$$

for all $u, v \in H$, $n = 1, 2, 3, \ldots$ and I be a self map on H.

(iii) I–asymptotically quasi non expansive on H if there exists a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ such that

$$\| \delta^n u - q \| \le (1+a_n) \| I^n u - q \|$$

for all $u \in H$, $q \in F(T) \cap F(I)$ and $n = 1, 2, 3, \ldots$

(iv) generalized I-asymptotically quasi non expansive mapping if $F = F(\delta) \cap F(I) \ne \phi$ and there exists a sub sequences $\{a_n\}$, $\{b_n\}$in K with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and

$$\| \delta^n u - q \| \le \| I^n u - q \| + a_n \| I^n u - q \| + b_n$$

for all $u, q \in F$ and $n \ge 1$.

Remark 1.3 : If we take $b_n = 0$ in the definition of generalized I-asymptotically quasi non expansive mapping then it becomes I asymptotically quasi non expansive mapping.

[1] anjupanwar15@gmail.com

[2] ravibhokal08@gmail.com

The study of implict iteration process gain popularity with the work of Xu and Ori [8]. In this direction Sun [7] extended the work of Xu and Ori [8] to the case of asymptotically quasi non expansive mappings in Banach space. Since then various papers had been published on the convergence of implicit iterative process including the papers of Brinde [10] in 2011, Chidume and Shahzad [11] in 2005, Khan, Fukhar–ud–din and Khan [12] in 2012 etc. In recent work of [9, 18, 13] the convergence of implict iteration process of asymptotically quasi–I non expansive mapping is studied.

Inspired by all above works, we introduce a new implict iteration process for three generalized I asymptotically quasi non expansive mappings :

For $x_0 \in H$ let

$$x_n = \alpha_n x_{n-1} + \beta_n T_1^n y_n + \gamma_n I_1^n x_n,$$

$$y_n = \alpha'_n x_n + \beta'_n T_2^n z_n + \gamma'_n + I_2^n x_n,$$

$$z_n = \alpha''_n x_n + \beta''_n T_3^n x_n + \gamma''_n + I_3^n x_n \qquad \ldots(1.1)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}, \{\alpha''_n\}, \{\beta''_n\}$ and $\{\gamma''_n\}$ are nine sequences in (0, 1) satisfying

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$$

and $T_i : H \to H$, $i \in \{1, 2, 3\}$ be generalized I-asymptotically quasi non expansive mappings and $I_i : H \to H$, $i \in \{1, 2, 3\}$ be asymptotically quasi non expansive mappings.

Definition 1.3 : Let H be a non empty subset of a Banach space K. The mapping $T : H \to H$ is called

(i) Semi closed at zero if for every bounded sequence $\{u_n\}$ in H, if u_n converges weakly to $u \in H$ and Tu_n converges strongly to 0 implies $Tu = 0$.

(ii) Semi compact if for any bounded sequence $\{u_n\}$ in H such that $\|u_n - Tu_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{u_{n_m}\} \subset \{u_n\}$ s.t.

$$u_{n_m} \to u^* \in H \text{ strongly.}$$

Definition 1.4[3] : A Banach space K is said to satisfy Opial's condition if for each sequence $\{u_n\}$ converges weakly to u implies

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - \omega\|$$

for all $\omega \in K$ and $u \neq \omega$.

Let H be a non empty subset of a Banach space K and ψ be a bifunction of $H \times H$ in to R (set of real numbers). The equilibrium problem for $\psi : H \times H \to R$ deals with finding $u \in H$ such that

$$\psi(u, v) \geq 0 \ \forall\ v \in H \qquad \ldots(1.2)$$

The set of solutions of (1.2) is denoted by $EP(\psi)$. For the solution of equilibrium problem we assume that the function ψ has following properties :

(P1) $\psi(u, u) = 0$ for all $u \in H$

(P2) ψ is monotone, that is $\psi(x, y) + \psi(y, x) \leq 0$, for all $x, y \in H$.

(P3) for each $u, v, w \in H$

$$\lim_{t\downarrow 0} \psi(tw + (1-t)u, v) \leq \psi(u, v),$$

(P4) for each $u \in H$, $v \to \psi(u, v)$ is convex and lower semi continuous.

Lemma 1.5 [1]: Let H be a non empty subset of a Banach space K and J be a duality mapping defined on H. Let $\psi : H \times H \to R$ (set of real numbers) is a bifunction satisfying the properties (P1) to (P4). Let $t > 0$ and $u \in H$. Then there exists $v \in H$ such that

$$\psi(v, \omega) + \frac{1}{t} < \omega - v, Jv - J\omega > \geq 0, \forall \omega \in H.$$

Lemma 1.6[4] : let H be a non empty subset of a uniformly convex Banach space K and ψ be a bifunction of $H \times H \to R$ satisfying (P1) to (P4). Let J be a duality mapping defined on H. For $t > 0$ and $u \in H$ consider a mapping $\delta_t : H \to K$ as

$$\delta_t(u) : \{\omega \in H : \psi(\omega, u) + \frac{1}{t} < v - \omega, J\omega - Jv > \forall\ v \in K$$

Then the following hold :

(i) δ_t is single valued,

(ii) δ_t is firmly non–expansive mapping i.e. for all $u, v \in H$

$$< \delta_t u - \delta_t v, J\delta_t u - J\delta_t v > \leq < \delta_t u - \delta_t v, J_u - J_v >$$

(iii) $\psi(\delta_t) = EP(\psi)$

(iv) $EP(\psi)$ is closed and convex.

Lemma 1.7[17] : let $\{a_n\}$ and $\{b_n\}$ be two sequences of non–negative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following condition is satisfied:

(i) $a_{n+1} \leq a_n + b_n$, $n \geq 1$

(ii) $a_{n+1} \leq (1 + b_n)a_n$, $n \geq 1$

then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.8[5] : Let H be a non empty subset of a Banach space K and let $0 < b < a < 1$. Suppose that $\{u_n\}$ is a sequence [b, a] and $\{v_n\}$, $\{w_n\}$ are two sequences in H such that

$$\lim_{n\to\infty} \|u_n v_n + (1-u_n)\omega_n\| = d, \limsup_{n\to\infty} \|v_n\| \leq d,$$

$$\limsup_{n\to\infty} \|\omega_n\| \leq d \qquad \ldots(1.3)$$

holds for some $d \geq 0$. Then $\lim_{n\to\infty} \|v_n - \omega_n\| = 0$.

Lemma 1.9[2] : Let K be a Banach space satisfying Opial's condition and H be a non empty subset of K. Let $\delta : H \to H$ be an asymptotically non expansive mapping. If the sequence $\{u_n\} \subset H$ such that $u_n \to u^* \in H$ and if $\lim_{n\to\infty} \| u_n - \delta u_n \| = 0,$ then $\delta u^* = u^*$.

MAIN RESULTS

Lemma 2.1 : Let H be a non empty subset of a Banach space K. Let $T_i : H \to H$, $i \in \{1, 2, 3\}$ be three generalized I_i asymptotically quasi non expansive mappings with the sequence $\{a_n\}$, $\{a'_n\},\{b_n\},\{b'_n\}$ and $\{c_n\},\{c'_n\} \subset (0,\infty]$. Let $I_i : H \to H$, $i \in \{1, 2, 3\}$ be asymptotically quasi non expansive mappings with the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$.

Let $\mu_n = \max_{n\in N}\{a_n, a'_n, b_n, b'_n, c_n, c'_n, u_n, v_n, w_n\}$.

Let $F = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(I_1) \cap F(I_2) \cap F(I_3)$ is non empty. Let $\{x_n\}$ be the sequence defined by the iterative process (H). Also let

$$R_1 = \sup_{n\in N}\{1-\alpha_n\}, \qquad R_2 = \sup_{n\in N}\{1-\alpha'_n\} \text{ and } M = \sup \mu_n^2 .$$

(1) $R_1 (M + M^2) < 1$, $\qquad R_2 (M + M^2) < 1$

(2) $$\sum_{n=1}^{\infty}(1-\alpha_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$$

Then the sequence $\{x_n\}$ converges strongly to common fixed point in F if and only if $\lim_{n\to\infty}\inf d(x_n, F) = 0$.

Proof : Let $p^* \in F$. Now it follows from (1.1) that

$$\begin{aligned}
\| x_n - p^* \| &= \| \alpha_n x_{n-1} + \beta_n T_1^n y_n + \gamma_n I_1^n x_n - p^* \| \\
&\le \alpha_n \| x_{n-1} - p^* \| + \beta_n \| T_1^n y_n - p^* \| + \gamma_n \| I_1^n x_n - p^* \| \\
&\le \alpha_n \| x_{n-1} - p^* \| + \beta_n [\| I_1^n y_n - p^* \| + a_n \| I_1^n y_n - p^* \| + a'_n] + \gamma_n u_n \| x_n - p^* \| \\
&\le \alpha_n \| x_{n-1} - p^* \| + \beta_n u_n \| y_n - p^* \| + \beta_n a_n u_n \| y_n - p^* \| \\
&\quad + \beta_n a'_n + \gamma_n u_n \| x_n - p^* \| \\
&\le \alpha_n \| x_{n-1} - p^* \| + \beta_n u_n (1 + a_n) \| y_n - p^* \| + \gamma_n u_n \| x_n - p^* \| + \beta_n a'_n \\
&\le \alpha_n \| x_{n-1} - p^* | + (1 - \alpha_n - \gamma_n) u_n (1 + a_n) \| y_n - p^* \| \\
&\quad + (1 - \alpha_n - \beta_n) u_n \| x_n - p^* \| + \beta_n a'_n \\
&\le \alpha_n \| x_{n-1} - p^* \| + (1-\alpha_n)\mu_n^2 \| y_n - p^* \| + (1-\alpha_n)\mu_n^2 \| x_n - p^* \| + \beta_n a'_n
\end{aligned} \qquad \ldots(2.1)$$

Again using (1.1), we get

$$\begin{aligned}
\| y_n - p^* \| &= \| \alpha'_n x_n + \beta'_n T_2^n z_n + I_2^n x_n - p^* \| \\
&\le \alpha'_n \| x_n - p^* \| + \beta'_n \| T_2^n z_n - p^* \| + \gamma'_n \| I_2^n x_n - p^* \| \\
&\le \alpha'_n \| x_n - p^* \| + \beta'_n [\| I_2^n z_n - p^* \| + b_n \| I_2^n z_n - p^* \| \\
&\quad + b'_n] + \gamma'_n v_n \| x_n - p^* \|
\end{aligned}$$

$$\leq \alpha'_n \| x_n - p^* \| + \beta'_n v_n \| z_n - p^* \| + \beta'_n b_n v_n \| z_n - p^* \|$$

$$+ \beta'_n b'_n + \gamma'_n v_n \| x_n - p^* \|$$

$$\leq \alpha'_n \| x_n - p^* \| + \beta'_n v_n (1 + b_n) \| z_n - p^* \| + \gamma'_n v_n \| x_n - p^* \| + \beta'_n b_n$$

$$\leq \alpha'_n \| x_n - p^* \| + (1 - \alpha'_n - \gamma'_n) v_n (1 + b_n) \| z_n - p^* \|$$

$$+ \gamma'_n v_n \| x_n - p^* \| + \beta'_n b_n$$

$$\leq \alpha'_n \| x_n - p^* \| + (1 - \alpha'_n) \mu_n^2 \| z_n - p^* \| + (1 - \alpha'_n) \mu_n^2 \| x_n - q^* \| + \beta'_n b'_n \quad \dots(2.2)$$

Again using (1.1) we obtain

$$\| z_n - p^* \| = \| \alpha''_n x_n + \beta''_n T_3^n x_n + \gamma''_n I_3^n x_n - p^* \|$$

$$\leq \alpha''_n \| x_n - p^* \| + \beta''_n \| T_3^n x_n - p^* \| + \gamma''_n \| T_3^n x_n - p^* \|$$

$$\leq \alpha''_n \| x_n - p^* \| + \beta''_n [\| I_3^n x_n - p^* \| + c_n \| I_3^n x_n - p^* \|$$

$$+ c'_n] + \gamma''_n w_n \| x_n - p^* \|$$

$$\leq \alpha''_n \| x_n - p^* \| + \beta''_n [w_n \| x_n - p^* \| + c_n w_n \| x_n - p^* \|$$

$$+ c'_n] + \gamma''_n w_n \| x_n - p^* \|$$

$$\leq \alpha''_n \| x_n - p^* \| + \beta''_n w_n \| x_n - p^* \| + \beta''_n c_n w_n \| x_n - p^* \|$$

$$+ \beta''_n c'_n + \gamma''_n w_n \| x_n - p^* \|$$

$$\leq (\alpha''_n + \beta''_n w_n + \beta''_n c_n w_n + \gamma''_n w_n) \| x_n - p^* \| + \beta''_n c'_n$$

$$\leq \mu_n^2 \| x_n - p^* \| + \beta''_n c'_n \quad \dots(2.3)$$

Using (2.3) and (2.2), we get

$$\| y_n - p^* \| \leq \alpha'_n \| x_n - p^* \| + (1 - \alpha'_n) \mu_n^2 [\mu_n^2 \| x_n - p^* \| + \beta''_n c'_n]$$

$$+ (1 - \alpha'_n) \mu_n^2 \| x_n - p^* \| + \beta'_n b'_n$$

$$\leq (\alpha'_n + (1 - \alpha'_n) \mu_n^4 + (1 - \alpha'_n) \mu_n^2) \| x_n - p^* \|$$

$$+ (1 - \alpha'_n \mu_n^2 c'_n \beta''_n + \beta'_n b'_n)$$

$$\leq [\alpha'_n + (1 - \alpha'_n) \mu_n^4 + (1 - \alpha'_n) \mu_n^2)] \| x_n - p^* \|$$

$$+ (1 - \alpha'_n \mu_n^2 \beta''_n c'_n + \beta'_n b'_n)$$

$$\leq [\alpha'_n + (1 - \alpha'_n)(\mu_n^2 + \mu_n^4)] \| x_n - p^* \| + (1 - \alpha'_n \mu_n^2 \beta''_n c'_n + \beta'_n b'_n)$$

Let $\delta_n = (1 - \alpha'_n) \mu_n^2 \beta''_n c'_n + \beta'_n b'_n$

Since $b'_n, \beta'_n, \alpha'_n, c'_n, \beta''_n, \mu_n < \infty$ implies $\delta_n < \infty$

Hence

$$\| y_n - p^* \| \leq [\alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)] \| x_n - p^* \| + \delta_n \qquad \ldots(2.4)$$

Using (2.4) in (2.1), we get

$$\| x_n - p^* \| \leq \alpha_n \| x_{n-1} - p^* \| + (1-\alpha_n)\mu_n^2[\alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)] \ \| x_n - p^* \|$$

$$+ \ (1-\alpha_n)\mu_n^2\delta_n + (1-\alpha_n)\mu_n^2 \| x_n - q^* \| + \beta_n a'_n$$

$$\leq \alpha_n \| x_{n-1} - p^* \| + [(1-\alpha_n)\mu_n^2[\alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)$$

$$+ \ (1-\alpha_n)\mu_n^2)] \| x_n - p^* \| + \beta_n a'_n$$

$$\leq \alpha_n \| x_{n-1} - p^* \| + [(1-\alpha_n)\mu_n^2)(1 + \alpha'_n + (1-\alpha'_n)$$

$$(\mu_n^2 + \mu_n^4)] \| x_n - p^* \| + \beta_n a'_n$$

$$\Rightarrow \qquad [1 - ((1-\alpha_n)\mu_n^2)(1 + \alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)]] \| x_n - p^* \|$$

$$\leq \alpha_n \| x_{n-1} - p^* \| + \beta_n a'_n$$

Using the given assumptions, we have

$$[1 - ((1-\alpha_n)\mu_n^2) + (1 + \alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)]$$

$$\leq \sup[1 - ((1-\alpha_n)\mu_n^2)(1 + \alpha'_n + (1-\alpha'_n)(\mu_n^2 + \mu_n^4)]$$

$$= [1 - R_1 M(1 + \alpha'_n + R_2(M + M^2)]$$

Now

$$\| x_n - p^* \| \leq \frac{\alpha_n \| x_{n-1} - p^* \|}{1 - (R_1 M(1 + \alpha'_n + R_2(M + M^2)))} + \frac{\beta_n a'_n}{1 - (R_1 M(1 + \alpha'_n + R_2(M + M^2)))}$$

Let $\delta''_n = \dfrac{\beta_n a'_n}{[1 - (R_1 M(1 + \alpha'_n + R_2(M + M^2))}$

So $\| x_n - p^* \| \leq \left[\dfrac{1 + a_n + (1 - R_1 M(1 + \alpha'_n) + R_2(M + M^2))}{[1 - R_1 M(1 + \alpha'_n) + R_2(M + M^2)]} \right] \| x_{n-1} - p^* \| + \delta''_n$

$$\leq (1 + \delta'''_n) \| x_{n-1} - p^* \| + \delta''_n \qquad \ldots(2.5)$$

where $\delta'''_n = \dfrac{a_n[1 - R_n M(1 + \alpha'_n) + R_2(M + M^2)]}{1 - (R_1 M(1 + \alpha'_n) + R_2(M + M^2)]}$

Clearly by the given assumptions we can easily conclude that $\sum_{n=1}^{\infty} \delta'''_n < \infty$.

Let $d_n = \|x_{n-1} - p^*\|$. Then $d_{n+1} = (1 + \delta'''_n) d_n + \delta''_n$ .Hence by lemma 1.7 $\lim_{n\to\infty} d_n$ exists.

Let $\lim_{n\to\infty} \| x_n - p^* \| = \rho$. W.L.O.G. let $\rho \geq 0$. Also we can write $\lim_{n\to\infty}$ (x_m, F) exists.

Since $\lim_{n\to\infty} \inf d(x_n, F) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, F) = 0$

Now we claim that $\{x_n\}$ is Cauchy sequence in D. Let $\in > 0$ by any arbitrary number. Since $\lim_{n\to\infty} d(x_n, F) = 0$ so there exists a positive number no s.t. $d(x_n, F) < \dfrac{\varepsilon}{2} \ \forall n \geq n_0$.

$$\Rightarrow \qquad \text{Inf } \{\| x_{n_0} 0 - p \| : p \in F\} < \frac{\varepsilon}{2}$$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 12 (2018) pp. 10804-10818
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$\Rightarrow \qquad \| x_{n_0} - p \| < \frac{\varepsilon}{2}$

Now for all $n_1, n_2 \geq n_0$, we have

$$\| x_{n_1+n_2} - x_{n_2} \| \leq \| D_{n+n_2} - p \| + \| x_{n_2} - p \|$$
$$\leq 2 \| x_{n_0} - p \|$$
$$\leq 2. \frac{\varepsilon}{2} = \varepsilon$$

Hence $\{x_n\}$ is a Cauchy sequence. So it must converge to some $p^* \in H$. Also $\lim_{n\to\infty} d(x_n, F) = 0$ implies that $\lim_{n\to\infty} d(p^*, F) = 0$. Thus $p^* \in F$. This completes the proof.

Theorem 2.2 : Let H be a non empty subset of a Banach space K. Let $\psi : H \times H \to R$ be a bifunction satisfying conditions (C1) - (C4). Let $T_i : H \to H$, $i \in \{1, 2, 3\}$ be three L_i, $i \in \{1, 2, 3\}$ uniformly Lipschitzian generalized I_i asymptotically quasi non expansive mappings with sequences $\{a_n, a'_n, b_n, b'_n, c_n, c'_n\} \subset [0, \infty)$ and I_i, $i \in \{1, 2, 3\}$ be the three $\{L_3, L_4, L_5\}$ Lipschitzian and asymptotically quasi non expansive mappings with sequence $\{u_n, v_n, w_n\} \subset [0, \infty)$.

Let $\quad \mu_n = \max_{n\in N}\{a_n, b_n, c_n, a'_n, b'_n, c'_n, u_n, v_n, w_n\}$

$M = \sup \mu_n^2$, $\quad F_1 = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(I_1) \cap F(I_2) \cap F(I_3)$

and let $F \neq \phi$.

Let $\{x_n\}$ be the sequence generated by the iterative scheme

For some $h_n \in H$

$$\psi(h_n, z) + \frac{1}{s_n} < z - h_n, J_{h_n} - J_{x_n} > \geq 0 \quad \forall z \in H$$

$$x_n = \alpha_n x_{n-1} + \beta_n T_1^n y_n + \gamma_n I_1^n x_n$$
$$y_n = \alpha'_n x_n + \beta'_n T_2^n y_n + \gamma'_n I_2^n x_n \qquad \ldots(2.6)$$
$$z_n = \alpha''_n x_n + \beta''_n T_3^n y_n + \gamma''_n I_3^n x_n$$

where $\{\alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n, \alpha''_n, \beta''_n, \gamma''_n\}$ are the sequences in (0, 1) satisfying

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$$

and $\gamma_n \in [s, \infty)$ for some $s > 0$. Here J is a duality mapping defined on H and let $R_1 = \sup \{1 - \alpha_n\}$ and $R_2 = \sup \{1 - \alpha'_n\}$. Also let $s_n \subset (\psi, \infty)$ for some $\gamma > 0$.

Now consider the following assumptions :

(i) $\quad R_1(M + M^2) < 1, \; R_2(M + M^2) < 1$

(ii) $\quad \sum_{n=1}^{\infty} (1 - \alpha_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$

$\quad \sum_{n=1}^{\infty} (1 - \alpha'_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$

(iii) $\quad \lim_{n\to\infty} \operatorname{Inf} s_n > 0, \; \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha'_n = \lim_{n\to\infty} \beta'_n = 0$.

(iv) $\quad F = F_1 \cap EP(\psi) \neq \phi$.

Then the sequence $\{x_n\}$ generated by the iterative process (2.6) satisfies the following

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \qquad i \in \{1, 2, 3\}$$

and $\quad \lim_{n\to\infty} \|x_n - I_i x_n\| = 0, \qquad i \in \{1, 2, 3\}$

Proof : We shall divide the proof of the theorem in two parts

Part–1

First we shall prove that

(i) $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \ i \in \{1,2,3\}$

(ii) $\lim_{n\to\infty} \|x_n - I_i^n x_n\| = 0, \ i \in \{1,2,3\}$

By lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p^*\|$ exists for all $p^* \in F$.

Let $\lim_{n\to\infty} \|x_n - p^*\| = \xi$.

Now using (2.6), we have

$$\lim_{n\to\infty} \|x_n - p^*\| = \lim_{n\to\infty} \|\alpha_n x_{n-1} + \beta_n T_1^n y_n + \gamma_n I_1^n x_n - p^*\|$$

$$= \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p^*) + \beta_n (T_1^n y_n - p^*) + \gamma_n (I_1^n x_n - p^*\|$$

$$\xi = \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p^*) + (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n}(T_1^n y_n - p^*) + \frac{\gamma_n}{1-\alpha_n}(I_1^n x_n - p^*)\right]\| \quad \ldots(2.7)$$

Now $\lim_{n\to\infty} \|x_n - p^*\| = \xi$ implies that $\lim_{n\to\infty} \|x_{n-1} - p^*\| = \xi$

Taking lim sup we have $\limsup_{n\to\infty} \|x_{n-1} - p^*\| = 0$ …(2.8)

Now, we obtain the following estimate

$$\limsup_{n\to\infty} \|\frac{\beta_n}{1-\alpha_n}(T_1^n y_n - p^*) + \frac{\gamma_n}{1-\alpha_n}(I_1^n x_n - p^*)\|$$

$$\leq \limsup_{n\to\infty}\left[\frac{\beta_n}{1-\alpha_n}\|T_1^n y_n - p^*\| + \frac{\gamma_n}{1-\alpha_n}\|T_1^n x_n - p^*\|\right]$$

$$\leq \limsup_{n\to\infty}\left[\frac{\beta_n}{1-\alpha_n}[\|I_1^n y_n - p^*\| + a_n\|I_1^n y_n - p^*\| + a'_n] + \frac{\gamma_n}{1-\alpha_n}u_n\|x_n - p^*\|\right]$$

$$\leq \limsup_{n\to\infty}\left[\frac{\beta_n}{1-\alpha_n}[u_n\|y_n - p^*\| + a_n u_n\|y_n - p^*\| + a'_n] + \frac{\gamma_n}{1-\alpha_n}u_n\|x_n - p^*\|\right]$$

$$\leq \limsup_{n\to\infty}\left[\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)\|y_n - p^*\| + a'_n] + \frac{\gamma_n}{1-\alpha_n}u_n\|x_n - p^*\|\right] \quad \ldots(2.9)$$

Using (2.1) in (2.9), we get

$$\limsup_{n\to\infty} \|\frac{\beta_n}{1-\alpha_n}(T_1^n y_n - p^*) + \frac{\gamma_n}{1-\alpha_n}(I_1^n x_n - p^*)\|$$

$$\leq \limsup_{n\to\infty}[\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)[\alpha'_n\|x_n - p^*\| + (1-\alpha'_n)\mu_n^2\|z_n - p^*\|$$

$$+ (1-\alpha'_n)\mu_n^2 \| x_n - p^* \| + \beta'_n b'_n + a'_n] + \frac{\gamma_n}{1-\alpha_n} u_n \| x_n - p^* \|]$$

$$\leq \limsup_{n\to\infty} [\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)[(\alpha'_n + (1-\alpha'_n)\mu_n^2) \| x_n - p^* \|$$

$$+ (1-\alpha'_n)\mu_n^2 \| z_n - p^* \| + \beta'_n b'_n + a'_n] + \frac{\gamma_n}{1-\alpha_n} u_n \| x_n - p^* \|$$

$$\leq \limsup_{n\to\infty} [\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)(\alpha'_n + (1-\alpha'_n)\mu_n^2) \| x_n - p^* \|$$

$$+ (1-\alpha'_n)\mu_n^2 (u_n^2 \| x_n - p^* \| + \beta''_n c'_n) + \beta'_n b'_n) + a'_n]$$

$$+ \frac{\gamma}{1-\alpha_n} u_n \| x_n - p^* \|]$$

$$\leq \limsup_{n\to\infty} [\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)(\alpha'_n + (1-\alpha'_n)\mu_n^2) + (1-\alpha'_n)\mu_n^4)$$

$$+ \frac{\gamma_n}{1-\alpha_n} u_n) \| x_n - p^* \|]$$

$$+ \limsup_{n\to\infty} [\frac{\beta_n}{1-\alpha_n}[u_n(1+a_n)((1-\alpha'_n)\mu_n^2 \beta''_n c'_n + \beta'_n b'_n) + a'_n]$$

$$\leq \limsup_{n\to\infty} \left[\frac{\beta_n}{1-\alpha_n}((\alpha'_n + (1-\alpha'_n)\mu_n^2) + (1-\alpha'_n)\mu_n^4) + \frac{\gamma_n}{1-\alpha_n} \right] \| x_n - p^* \|$$

$$\leq \limsup_{n\to\infty} [\beta_n(\mu_n^2 + \mu_n^4) + \gamma_n] \| x_n - p^* \|$$

$$\leq \limsup_{n\to\infty} \gamma_n \| x_n - p^* \| \leq \xi \qquad \text{…(2.10)}$$

Also $$\lim_{n\to\infty} \| x_{n-1} - q^* \| \leq \xi \qquad \text{…(2.11)}$$

Now making use of (2.7), (2.10), (2.11) and lemma 1.7, we have

$$\lim_{n\to\infty} \| x_n - x_{n-1} \| = 0 \qquad \text{…(2.12)}$$

Again from (2.6), we have

$$\lim_{n\to\infty} \| x_n - p^* \| = \lim_{n\to\infty} \| \beta_n (T_1^n y_n - p^*) + (1-\beta_n)[\frac{\alpha_n}{1-\beta_n}(x_{n-1} - p^*)$$

$$+ \frac{\gamma_n}{1-\beta_n}(I_1^n x_n - p^*)] \| = \xi \qquad \text{…(2.13)}$$

Now

$$\lim_{n\to\infty} \| T_1^n y_n - p^* \| \leq \lim_{n\to\infty} [\| y_n - p^* \| + \| I_1^n y_n - p^* \| + a'_n]$$

$$\leq \lim_{n\to\infty} \| y_n - p^* \|$$

$$\leq \lim_{n\to\infty}[(a'_n \| x_n - p^*\| + (1-\alpha'_n)\mu_n^4 \| z_n - p^*\| + (1-\alpha'_n)\beta''_n + c'_n)$$

$$+ (1-\alpha'_n)\mu_n^2 \| x_n - p^*\| + \beta'_n b'_n]$$

Taking lim sup, we get

$$\limsup_{n\to\infty} \| T_1^n y_n - p^*\| \leq \xi \qquad \ldots(2.14)$$

Also

$$\limsup_{n\to\infty} \| \frac{\alpha_n}{1-\beta_n}(x_n - p^*) + \frac{\gamma_n}{1-\beta_n}(I_1^n x_n - p^*) \|$$

$$\leq \limsup_{n\to\infty}\left[\frac{\alpha_n}{1-\beta_n}\mu_n^2 \| x_n - p^*\| + \frac{\gamma_n}{1-\beta_n}\mu_n \| I_1^n x_n - p^*\|\right]$$

$$\leq \limsup_{n\to\infty}\left[\frac{\alpha_n \mu_n^2}{1-\beta_n}(\| x_n - p^*\|) + \frac{\gamma_n}{1-\beta_n}\mu_n^2 \| x_n - p^*\|\right]$$

$$\leq \limsup_{n\to\infty}\left[\frac{\mu_n^2}{1-\beta_n}(\alpha_n + \gamma_n)x_n - p^*\|\right]$$

$$\leq \xi \qquad \ldots(2.15)$$

Now from (2.13), (2.14), (2.15) and lemma 1.7, we have

$$\lim_{n\to\infty} \| x_n - T_1^n y_n \| = 0 \qquad \ldots(2.16)$$

Also, we can write by similar procedure

$$\lim_{n\to\infty} \| x_n - I_1^n x_n \| = 0 \qquad \ldots(2.17)$$

Now from (11) and (15), we have

$$\lim_{n\to\infty} \| x_{n-1} - T_1^n y_n \| \leq \lim_{n\to\infty} \| x_{n-1} - x_n \| + \lim_{n\to\infty} \| x_n - T_1^n y_n \| = 0 \qquad \ldots(2.18)$$

Now consider,

$$\| x_{n-1} - p^*\| \leq \| x_{n-1} - T_1^n y_n \| + \| T_1^n y_n - p^*\|$$

$$\leq \| x_{n-1} - T_1^n y_n \| + u_n \| y_n - p^*\|$$

$$\leq \| x_{n-1} - T_1^n y_n \| + \mu_n \| y_n - p^*\|$$

$$\Rightarrow \qquad \| x_{n-1} - p^*\| - \| x_{n-1} - T_1^n y_n \|$$

$$\leq \mu_n[\alpha'_n \| x_n - p^*\| + (1-\alpha'_n)\mu_n^4(\| x_n - p^*\| + (1-\alpha'_n)\mu_n^2\beta''_n c'_n$$

$$+ (1-\alpha'_n)\mu_n^2 \| x_n - p^*\| + \beta'_n b''_n]$$

$$\leq \mu_n[(\alpha'_n + (1-\alpha'_n)\mu_n^4 + (1-\alpha'_n)\mu_n^2) \| x_n - p^*\|$$

$$+ (\mu_n(1-\alpha'_n \mu_n^2\beta''_n c'_n) + \mu_n\beta'_n b'_n)$$

Using (2.7), (2.18) along with the conditions of theorem and squeeze theorem, we have

$$\lim_{n\to\infty} \| y_n - p^* \| = \xi \qquad \ldots(2.19)$$

Now using (2.6), we have

$$\lim_{n\to\infty} \| y_n - p^* \| = \lim_{n\to\infty} \| \alpha'_n x_n + \beta'_n T_2^n z_n + \gamma'_n I_2^n x_n - p^* \|$$

$$= \lim_{n\to\infty} \| \alpha'_n (x_n - p^*) + \beta'_n (T_2^n z_n - p^*) + \gamma'_n (I_2^n x_n - p^*) \|$$

$$= \lim_{n\to\infty} \| \alpha'_1 (x_n - p^*) + (1-\alpha'_n)[\frac{\beta'_n}{1-\alpha'_n}(T_2^n z_n - p^*)$$

$$+ \frac{\gamma'_n}{1-\alpha'_n}(I_2^n x_n - p^*)] \| = \xi \qquad \ldots(2.20)$$

Now,

$$\lim_{n\to\infty} \sup \| \frac{\beta'_n}{1-\alpha'_n}(T_2^n z_n - p^*) + \frac{\gamma'_n}{1-\alpha'_n}(I_2^n x_n - p^*) \|$$

$$\leq \lim_{n\to\infty} \sup \left[\frac{\beta'_n}{1-\alpha'_n} \| T_2^n z_n - p^* \| + \frac{\gamma'_n}{1-\alpha'_n} \| I_2^n x_n - p^* \| \right]$$

$$\leq \lim_{n\to\infty} \sup [\frac{\beta'_n}{1-\alpha'_n}(\| z_n - p^*) + b_n \| I_2^n z_n - p^* \| + b'_n)$$

$$+ \frac{\gamma'_n}{1-\alpha'_n} v_n \| x_n - p^* \|]$$

$$\leq \lim_{n\to\infty} \sup [\frac{\beta'_n}{1-\alpha'_n}(\mu_n^2 \| x_n - p^* \| + \beta''_n c'_n + b_n v_n \| z_n - p^* \|$$

$$+ b'_n) + \frac{\gamma'_n}{1-\alpha'_n} V_n \| x_n - p^* \|]$$

$$\leq \lim_{n\to\infty} \sup [\frac{\beta'_n}{1-\alpha'_n} \| u_n^2 + b_n v_n \mu_n^2) \| x_n - p^* \|$$

$$+ (\beta''_n c'_n + b_n v_n \beta''_n c'_n + b'_n) + \frac{\gamma'_n}{1-\alpha''_n} v_n \| x_n - p^* \|]$$

$$\leq \lim_{n\to\infty} \sup \left[\left(\frac{\beta'_n}{1-\alpha'_n}(\mu_n^2 + b'_n v_n \mu_n^2 + \frac{\gamma'_n}{1-\alpha'_n} v_n \right) + \| x_n - p^* \| \right]$$

$$\leq \lim_{n\to\infty} \sup (\beta'_n + \gamma'_n) \| x_n - p^* \|$$

$$\leq \lim_{n\to\infty} \sup (1-\alpha'_n) \| x - p^* \|$$

$$\leq \xi \qquad \ldots(2.21)$$

Since

$$\lim_{n\to\infty} \| x_n - p^* \| = \xi \text{ implies } \lim_{n\to\infty} \sup \| x_n - p^* \| \leq \xi \qquad \ldots(2.22)$$

Using (2.20), (2.21), (2.22) and lemma 1.7, we have

$$\lim_{n\to\infty} \| (x_n - \beta^*) - [\frac{\beta'_n}{1-\alpha'_n}(T_2^n z_n - p^*) + \frac{\gamma'_n}{1-\alpha'_n}(I_2^n x_n - p^* \|$$

$$= \lim_{n\to\infty} \frac{1}{1-\alpha'_n}[(1-\alpha'_n)\| x_n - p^*\| - \beta'_n \| T_2^n z_n - p^*\| - \gamma'_n \| I_2^n x_n - p^*]$$

$$= \lim_{n\to\infty} \frac{1}{1-\alpha'_n}[(1-\alpha'_n)\| x_n - p^*\| - \beta'_n (\| z_n - p^*\| + a'_n \| I_2^n z_n - p^*\| + b'_n)$$

$$- \gamma'_n v_n \| x_n - p^*\|]$$

$$= \lim_{n\to\infty} \frac{1}{1-\alpha'_n}[(1-\alpha'_n)\| x_n - p^*\| - \beta'_n (1 + a'_n v_n)\| z_n - p^*\| + b'_n)$$

$$- \gamma'_n v_n \| x_n - p^*\|]$$

$$= \lim_{n\to\infty} \frac{1}{1-\alpha'_n}[(1-\alpha'_n - \gamma'_n v_n)\| x_n - p^*\|$$

$$- \beta'_n (1 + a'_n v_n)\| z_n - p^*\| + b'_n) = 0 \qquad \ldots(2.23)$$

Using given conditions of the theorem, we have

$$\lim_{n\to\infty} \| x_n - z_n \| = 0 \qquad \ldots(2.24)$$

Similarly, $$\lim_{n\to\infty} \| x_n - y_n \| = 0 \qquad \ldots(2.25)$$

Now,

$$\lim_{n\to\infty} \| x_n - T_1^n x_n \| \leq \lim_{n\to\infty} [\| x_n - x_{n-1} \| + \| x_{n-1} - T_1^n y_n \| + \| T_1^n y_n - T_1^n x_n \|]$$

$$\leq \lim_{n\to\infty} [\| x_n - x_{n-1} \| + \| x_{n-1} - T_1^n y_n + L_1 \| y_n - x_n \|]$$

$$= 0 \qquad \ldots(2.26)$$

Also, we can write

$$\lim_{n\to\infty} \| x_{n-1} - T_2^n x_n \| = \lim_{n\to\infty} \| x_{n-1} - I_1^n x_n \| = 0 \qquad \ldots(2.27)$$

and

$$\lim_{n\to\infty} \| x_{n-1} - I_2^n x_n \| = \lim_{n\to\infty} \| x_{n-1} - I_3^n x_n \| = 0 \qquad \ldots(2.28)$$

Now

$$\| x_n - T_1 x_n \| \leq \| x_n - T_1^n x_n \| + \| T_1^n x_n - T_1 x_n \|$$

$$\leq \| x_n - T_1^n x_n \| + L_1 \| T_1^{n-1} x_n - x_n \|$$

$$\leq \| x_n - T_1^n x_n \| + L_1 [\| T_1^{n-1} x_n - T_1^{n-2} x_n \| + \| T_1^{n-2} x_n - x_n \|]$$

$$\leq \| x_n - T_1^n x_n \| + L_1 [(1 + L_1)\| T_1^{n-2} x_n - x_n \|]$$

Continuing in the similar way and using (2.26), 2.12), we can write

$$\lim_{n\to\infty} \| x_n - T_1 x_n \| = 0 \qquad \ldots(2.29)$$

Similarly, we have

$$\lim_{n\to\infty} \| x_n - T_2 x_n \| = 0 = \lim_{n\to\infty} \| x_n - T_3 x_n \| \qquad \ldots(2.30)$$

Now, $\| x_n - I_1 x_n \| \leq \| x_n - I_1^n x_n \| + \| I_1^n x - I_1 x_n \|$

$$\leq \| x_n - I_1^n x_n \| + L_4 \| T_1^{n-1} x_{n-1} - x_n \|$$

By using above arguments, we have

$$\lim_{n\to\infty} \| x_n - I_i x_n \| = 0,\ i \in \{1,2,3\} \qquad \ldots(2.31)$$

Part 2 :

Let p* ∈ F. By using the definition of the notation G_r as in lemma 1.6, we have

$$h_n = G_{\gamma_n} x_n$$

and

$$\| h_n - p^* \| = \| G_{\gamma_n} x_n - G_{\gamma_n} p^* \| \leq \| x_n - p^* \| \qquad \ldots(2.32)$$

Since $\{h_n\}$ bounded so there exists a subsequence $\{G_{hk}\}$ of $\{G_n\}$ such that $\{G_{n_k}\}$ converges weakly to g* ∈ H and g* = J^{-1}w* for some w* ∈ J(H). now using lemma 2.1 along with (2.24), we have that $\{x_{n_k}\}$ and $\{z_{n_k}\}$ converges weakly to g* ∈ H. Also by the first part of the proof we can easily conclude that g* ∈ F.

Now we claim that g* ∈ FP(ψ) i.e. Jg* = w * J(EP(ψ)). Since J is norm to norm continuous on the bounded subset of H, we have $\lim_{n\to\infty} \| Jx_n - Jh_n \| = 0$ …(2.33)

Since $s_n \in (\rho, 0)$, ρ > 0, we have

$$\lim_{n\to\infty} \frac{\| Jx_n - Jh_n \|}{s_n} = 0 \qquad \ldots(2.34)$$

Since $\{x_n\}$ is bounded and hence $\{Jx_n\}$ is also bounded. So there exists $\{Jx_{n_k}\} \subset \{Jx_n\}$ s.t. Jx_n converges weakly to g*.

Now $$\psi(h_n, z) + \frac{1}{s_n} < z - h_n, Jh_n - Jx_n > \geq 0$$

$$\Rightarrow \quad \psi(h_{n_k}, z) + < z - h_{n_k}, \frac{Jh_{n_k} - Jx_{n_k}}{s_{n_k}} > \geq 0 \ \forall z \in H \qquad \ldots(2.35)$$

Also, $$\lim_{n\to\infty} \left[\frac{Jh_{n_k} - Jx_{n_k}}{s_{n_k}} \right] = 0$$

Using condition (P2), we have

$$\frac{1}{s_n} < z - h_n, Jh_n - Jx_n > \geq \psi(h_n, z)$$

$$\Rightarrow \quad < z - h_n, \frac{Jh_{x_n} - Jx_n}{s_n} > \geq \psi(z, h_n)$$

Since $\lim_{n\to\infty} \frac{\| Jx_n - Jh_n \|}{s_n} = 0$ and $\{Jh_n - w^*\}$, we conclude

$$\psi(z, w^*) \le 0 \,\forall \in H \qquad \ldots(2.36)$$

Now for some $0 \le s \le 1$ and $t \in H$

Let $s_t = s_t + (1 - t)w^*$

Now $s_t \in H$ and hence $\psi(s_t, w^*) \le 0$. Using condition (P1) and (P2), we have

$$0 = \psi(s_t, s_t) \le t\,\psi(s_t, t) + (1 - t)\,\psi(s_t, w^*)$$

$$\le t\,\psi(s_t, t)$$

Hence $0 \le \psi(s_t, t)$. Using (P3), $0 \le \psi(w^*, s)$ $\forall s \in H$. Hence $w^* \in EP(\psi)$. This completes the proof.

Theorem 2.3 : Let K be a uniformly convex Banach space satisfying Opial's condition and let H be a non empty closed convex subset of it. Let $\{T_i\}$, $\{I_i\}$ and $\{x_n\}$ be defined as in lemma 2.1 and satisfying all the conditions of lemma 2.1. Let $S : H \to H$ be an identity mapping. If $S - T_i$, $i \in \{1, 2, 3\}$ and $\delta - T_i$, $i \in \{1, 2, 3\}$ are semi closed at zero, then $\{x_n\}$ converges weakly to common fixed point in F.

Proof : Lemma 2.1 ensures the convergence of $\{x_n\}$ to some $p^* \in F$. This implies that $\{x_n\}$ is bounded. We know that in a uniformly convex Banach space every bounded subset is weakly compact and hence H is weakly compact. So for any bounded sequence $\{x_n\} \subset H$, there exists a subsequence $\{x_{n_k}\} \subset \{x_k\}$ such that $x_{n_k} \to p \in F$. now using lemma 2.2, we can write

$$\lim_{n\to\infty} \| x_{n_k} - T_i x_{n_k} \| = 0,\ i \in \{1,2,3\} \qquad \ldots(2.37)$$

$$\lim_{n\to\infty} \| x_{n_k} - I_i x_{n_k} \| = 0,\ i \in \{1,2,3\} \qquad \ldots(2.38)$$

Since the mappings $\delta - T_i$, $i \in \{1, 2, 3\}$. $\delta - I_i$, $i = \{1, 2, 3\}$ are semi compact at zero so we can find some $p \in H$ s.t.

$$T_i p = p,\ I_i p = p,\ i \in \{1, 2, 3\}$$

and hence $p \in F$

Now we prove weak convergence of the sequence $\{x_n\}$ to $P \in F$. On the contrary let there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to p_1 \in H$ and $p_1 \ne p$. Using the above arguments we can show that $p_1 \in F$.

Now by lemma 2.1 $\lim_{n\to\infty} \| x_n - p \|$ and $\lim_{n\to\infty} \| x_n - p_1 \|$ both exists

Let

$$\lim_{n\to\infty} \| x_n - p \| = d_1,\quad \lim_{n\to\infty} \| x_n - p_1 \| = d_2$$

Using Opial condition

$$d_1 = \lim_{n_k\to\infty} \| x_{n_k} - p \| < \limsup_{n_k\to\infty} \| x_{n_k} - p_1 \| = d_2$$

and

$$d_2 = \lim_{n_k\to\infty} \| x_{n_k} - p_1 \| < \limsup_{n\to\infty} \| x_{n_k} - p \| = d_1$$

which implies that $d_1 = d_2$.

Hence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof.

Theorem 2.4 : Let H be a non empty subset of a uniformly convex Banach space K and T_i, I_i, $\{x_n\}$ be same as in Theorem 2.2, If at least one of the mappings T_i, I_i, $i \in \{1, 2, 3\}$ are semi compact then $\{x_n\}$ converges to a common fixed point in F.

Proof : W.L.O.G. Let T_i, I_i, $i \in \{1, 2, 3\}$ are semi compact. Now using Theorem 2.2, we have

$$\lim_{n\to\infty} \| x_n - T_i x_n \| = \lim_{n\to\infty} \| x_n - I_i x_n \| = 0,\quad i \in \{1, 2, 3\}$$

Now by semi–compactness property there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in H$.

Again using Theorem 2.2, we can write

$$\lim_{n\to\infty} \| x_{n_k} - T_i x_{n_k} \| = 0 \Rightarrow \| \phi - T_i p \| = 0,\ i \in \{1,2,3\}$$

Hence $T_i p = p$, $i \in \{1, 2, 3\}$ …(2.39)

Also by Theorem 2

$$\lim_{n\to\infty} \| x_{n_k} - I_i x_{n_k} \| = 0 \Rightarrow \| P - I_i P \| = 0,\ i \in \{1,2,3\}$$

Therefore $I_i p = p$, $i \in \{1, 2, 3\}$

Hence $p \in F$.

Now by lemma 2.1, $\lim_{n\to\infty} \| x_n - p \|$ exists and so

$\lim_{n\to\infty} \| x_{n_k} - p \|$ exists and equal to 0. Hence $\{x_n\} \to p \in F$.

This completes the proof.

Now we produce an example related to equilibrium problem.

Let K =R , H = [–10, 10]. Then H is a closed compact subset of K. Consider the mapping

$$\psi : H \times H \to R$$

The equilibrium problem deals with finding some $x \in H$ such that $\psi(x, y) \geq 0 \; \forall \; y \in H$.

Then ψ satisfies the conditions

(P1) $\psi(x, x) = -(x - x)^2 = 0 \; \forall \; x \in [-10, 10]$

(P2) $\psi(x, y) + \psi(y, x) = -(x- y)^2 + (-(y-x)^2)$

$$= -[(x - y)^2 + (y - x)^2]$$

$$= - [x^2 + y^2 - 2xy + y^2 + x^2 - 2xy]$$

$$= -2[x - y]^2$$

$$\leq 0 \; \forall \; (x, y) \in [-10, 10]$$

(P3) $\lim_{t\downarrow 0} \psi(tz + (1-t)x, y) = \lim_{t\downarrow 0} \psi(x + t(z-x), y)$

$$= \lim_{t\downarrow 0} -[(x + t(z-x)^2 - y^2]$$

$$= \lim_{t\downarrow 0} -[x^2 + t^2(z-x)^2 + 2x + (z-x) - y^2]$$

$$= \lim_{t\downarrow 0} -[x^2 + t^2(z^2 + x^2 - 2zx) + 2x + z - 2x^2t - y^2]$$

$$= \lim_{t\downarrow 0} -[x^2t^2z^2 + t^2x^2 - 2t^2zx + 2xtz - 2x^2t - y^2]$$

$$= -[x^2 - y^2]$$

$$\leq \psi(x, y)$$

(P4) $\psi(x, y) \; \forall \; x, y \in [-10, 10]$ is a polynomial function. Also $\psi(x, y)$ is convex and weakly lower semi continuous.

Example : Let K = R and H = [–10, 10] then H is a closed subspace of K. Consider the mappings $T_1x = \frac{x+1}{2}, T_2x = 2x - 1, T_3x = 2x^2 - 1$ defined from H to K. Then T_1, T_2, T_3 are generalized I_i asymptotically quasi non expansive mappings with sequences $\{a_n, a'_n, b_n, b'_n, c_n, c'_n\} = \left\{\frac{1}{n}\right\}$. Also let $I_i : H \to H$, $i \in \{1, 2, 3\}$ is defined by $I_i(x) = x$, $i \in \{1, 2, 3\}$. Then I_i, $i \in \{1, 2, 3\}$ are asymptotically quasi non expansive mappings with sequences

$\{u_n\} = \{v_n\} = \{w_n\} = \left\{\frac{1}{n}\right\}$. Also $F = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(I_1) \cap F(I_2) \cap F(I_3) = \{1\} \neq \phi$

Now the mappings T_i, I_i, $i \in \{1, 2, 3\}$ are semi compact. So all the hypothesis of Theorem 2.4 are satisfied and hence the sequence $\{x_n\}$ defined by iterative process (1.1) converges to common fixed point 1 in F.

CONCLUSION

Our results extends many results in the literature including the work of [9] and [13] to more general implicit iteration process. We also have extended many common fixed results by taking more general class of mappings.

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