International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 15 (2018) pp. 12344-12352

Bayesian Approach in Assessing the Lifetime Performance Index of Exponential Lifetimes Distribution with an Adaptive Hybrid Type-II Progressive Censoring Sample

Neveen Sayed-Ahmed[1]

Department of Mathematics, Faculty of Commerce, Al-Azhar University, Nasr City, P.N. Box: 11884, Cairo, Egypt (Girls' Branch).
Mathematics Department, Faculty of Science, Taif University, Taif City 888, Saudi Arabia.

Abstract

In industrial processes, the lifetime performance index C_L is presented as a popular means to assess the performance and potential of their processes. An adaptive hybrid Type-II progressive censoring scheme as a mixture of Type-I and Type-II progressive censoring schemes. In this article, Bayesian and non-Bayesian estimation of C_L under an adaptive hybrid Type-II progressive censoring scheme when the lifetimes of products are independent exponential distribution. The maximum likelihood confidence interval and Bayes credible interval of C_L are developing. The behavior of the confidence interval and credible interval for the parameter C_L given a significance level is investigated with two illustrative examples. Monte Carlo simulation is then utilized to assess the behavior of the lifetime performance index C_L.

Keywords: Exponential distribution; Process capability indices; Adaptive hybrid Type-II progressive censored sample; Maximum likelihood estimator; Confidence interval; Bayes estimator ; Credible interval.

AMS 2000 Subject Classification: Primary 62G30; Secondary 62F15.

INTRODUCTION

In modern enterprises, assessment of quality performance for products is important, hence process capability analysis is an effective means of measuring process performance and potential capability. In the service (or manufacturing) industry, process capability indices are utilized to assess whether product quality meets the required level. Montgomery [1] proposed the process capability index C_L (or C_{PL}) for evaluating the lifetime performance of electronic components, where L is the lower specification limit, since the lifetime of electronic components exhibits the larger-the-better quality characteristic of time orientation. Tong et. al. [2] constructed a uniformly minimum variance unbiased estimator (UMVUE) of C_L under an exponential distribution. Moreover, the UMVUE of C_L is then utilized to develop the confidence interval. The purchasers can then employ the testing procedure to determine whether the lifetime of electronic components adheres to the required level. Manufacturers can also utilize this procedure to enhance process capability. All of the above process capability indices (PCIs) have been developed or investigated under normal lifetime model or exponential lifetime model. Nevertheless, in many processes including manufacturing processes and service processes, the assumption of normality is common in process capability analysis, and is often not valid. Therefore, the lifetime model of many products generally may possess a non-normal distribution including two-parameters exponential (Wu et al. [3]), Burr XII model (Lee et al. [4]), Rayleigh (Lee et al. [5]), Weibull (Ahmadi et al. [6]).

Censoring is very common in life tests. There are several types of censored tests. The most common censoring schemes are Type-I (time) censoring, where the life testing experiment will be terminated at a prescribed time T , and Type-II (failure) censoring, where the life testing experiment will be terminated upon the r^{th} (r is pre-fixed) failure. However, the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. Because of this lack of flexibility, a more general censoring scheme called progressive Type-II right censoring, for extensive reviews of the literature on progressive censoring see Balakrishnan and Aggarwala [7]. Kundu and Joarder [8] proposed a censoring scheme called Type-II progressive hybrid censoring scheme, in which a life testing experiment with progressive Type-II right censoring scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$ is terminated at a prefixed time T. However, the drawback of the Type-II progressive hybrid censoring, similar to the conventional Type-I censoring (time censoring), is that the effective sample size is random and it can turn out to be a very small number (even equal to zero), and therefore the standard statistical inference procedures may not be applicable or they will have low efficiency. Ng et al.[9] suggested an adaptive Type-II progressive censoring, in this censoring, a properly planned adaptive progressively censored life testing experiment can save both the total test time and the cost induced by failure of the units and increase the efficiency of statistical analysis.

[1]E-mail:nevensayd@yahoo.com

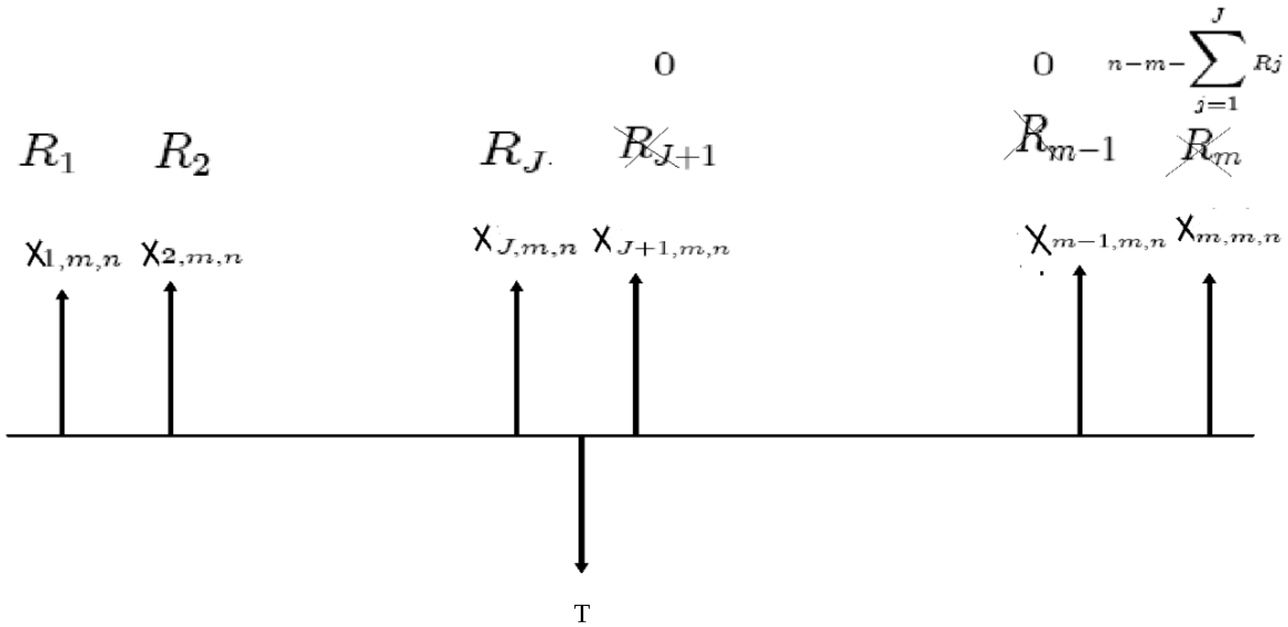
Suppose n units are placed on a life testing experiment and let $T_1, T_2, \ldots, T_n$ be their corresponding lifetimes. We assume that $T_i, \ i=1, 2, \ldots, n$ are independent and identically distributed with probability density function (pdf) $f(t)$ and cumulative distribution function (cdf) $F(t)$. Prior to the experiment, an integer $m<n$ is determined and the progressive Type-II censoring scheme $\mathbf{R}=(R_1, R_2, \ldots, R_m)$ with $R_i \geq 0$ and $n=m+\sum_{i=1}^{m} R_i$ is specified. During the experiment, the i -th failure is observed and immediately after the failure, R_i function items are randomly removed from the test. We denote the m completely observed (ordered) lifetimes by $T^{\mathbf{R}}_{i\,:\,m\,:\,n}, \ i=1,2,\ldots,m$, which are the observed progressively Type-II right censored sample. For convenience, we will suppress the censoring scheme in the notation of the $T_{i\,:\,m\,:\,n}$. We also denote the observed values of such a progressively Type-II right censored sample by $t_{1\,:\,m\,:\,n} < t_{2\,:\,m\,:\,n} < \ldots < t_{m\,:\,m\,:\,n}$.

Suppose the experimenter provides a time τ , which is an ideal total test time, but we may allow the experiment to run over time τ . If the m -th progressively censored observed failure occurs before time τ (i.e. $T_{m\,:\,m\,:\,n} < \tau$), the experiment stops at the time $T_{m\,:\,m\,:\,n}$. Otherwise, once the experimental time passes time τ but the number of observed failures has not reached m, we would want to terminate the experiment as soon as possible. This setting can be viewed as a design in which we are assured of getting m observed failure times for efficiency of statistical inference and at the same time the total test time will not be too far away from the ideal time τ. From the basic properties of order statistics (see, for example, David and Nagaraja [10], Section 4.4), we know that the fewer operating items are withdrawn (i.e., the larger the number of items on the test), the smaller the expected total test time. Therefore, if we want to terminate the experiment as soon as possible for fixed value of m, then we should leave as many surviving items on the test as possible.

Suppose J is the number of failures observed before time T , i.e.

$$T_{J\,:\,m\,:\,n} < \tau < T_{J+1\,:\,m\,:\,n}, \quad J=1,2,\ldots,m$$

where $T_{0;m,n}=0$ and $T_{m+1;m,n}=\infty$. According to the above result on stochastic ordering of first order statistics from different sample sizes, after the experiment passed time τ , we set $R_{J+1} = \ldots = R_{m-1}=0$ and $R_m = n-m-\sum_{i=1}^{J} R_i$. This formulation leads us to terminate the experiment as soon as possible if the $(J+1)$-th failure time is greater than τ for $J+1<m$. The value of τ plays an important role in the determination of the values of R_i and also as a compromise between a shorter experimental time and a higher chance to observe extreme failures.



One extreme case is when $\tau \to \infty$, which means time is not the main consideration for the experimenter, then we will have a usual progressive Type-II censoring scheme with the pre-fixed progressive censoring scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$. Another extreme case can occur when $\tau = 0$, which means we always want to end the experiment as soon as possible, then we will have $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - m$ which results in the conventional Type-II censoring scheme. Figure (1) gives the schematic representation of this situation.

Given J, the likelihood function is given by

$$f_{1,2,...,m}(t^{\mathbf{R}}_{1;m,n}, t^{\mathbf{R}}_{2;m,n}, ..., t^{\mathbf{R}}_{m;m,n}) = A\prod_{i=1}^{m} f(t^{\mathbf{R}}_{i;m,n})[1-F(t^{\mathbf{R}}_{i;m,n})]^{\delta_i R_i}, \quad (1)$$

$$0 < t^{\mathbf{R}}_{1;m,n} < t^{\mathbf{R}}_{2;m,n} < ... < t^{\mathbf{R}}_{m;m,n} < \infty,$$

where

$$A = \prod_{i=1}^{m}\left[n - i + 1 - \sum_{i=1}^{\min\{i-1, J\}} R_i\right], \quad (2)$$

and

$$\delta_i = \begin{cases} 1 & \text{if } i \le J \\ 0 & \text{if } J < i < m \\ \frac{n-m-\sum_{j=1}^{J} Rj}{R_m} & \text{if } i = m \text{ and } J < m \end{cases} \quad (3)$$

The rest of this paper is organized as follows. In Section 2, we introduces some properties of the lifetime performance index for lifetime of product with the NDBS based on the progressively Type II censored sample and discusses the relationship between the lifetime performance index and conforming rate. Section 3 then presents the ML and Bayes estimators of the lifetime performance index and its statistical properties. Section 4 develops a lower bound for the lifetime performance index C_L. Two numerical examples and concluding remarks are made in Sections 5. Sensitivity study via a Monte Carlo method are conducted in Section 6.

THE LIFETIME PERFORMANCE INDEX

Suppose that the lifetime (T) of products has the two-parameter exponential distribution with the probability density

$$f(t) = \frac{1}{\beta}\exp(-\frac{t-\theta}{\beta}), \quad t > \theta, \ \beta > 0, \quad (4)$$

where θ and β are the threshold parameter (or shift parameter, or guarantee time, or minimum life) and the scale parameter, respectively.

By using the transformation $Y = T - \theta$, and the distribution of Y has a one-parameter exponential distribution with the p.d.f. and c.d.f. as

$$f_Y(y) = \frac{1}{\beta}exp(-\frac{y}{\beta}) \text{ and } F_Y(y) = 1 - exp(-\frac{y}{\beta}), \ y \ge 0, \ \beta > 0. \quad (5)$$

Hence, if $t^{\mathbf{R}}_{1;m,n} < t^{\mathbf{R}}_{2;m,n} < ... < t^{\mathbf{R}}_{m;m,n}$ is an adaptive Type-II progressive censored sample from the two-parameter exponential distribution with p.d.f. as (4), then the new lifetimes $Y_{i;m,n} = T^{\mathbf{R}}_{i+1;m,n} - T^{\mathbf{R}}_{1;m,n}$, $i = 1, 2, ..., m-1$ can be treated as an adaptive Type-II progressive censored sample of size $m-1$ from the one-parameter exponential distribution with the p.d.f. and c.d.f. as (5), respectively. So, in this paper we shall use one-parameter exponential distribution instead of two-parameter exponential distribution.

To assess the lifetime performance of products, C_L can be defined as the lifetime performance index. Suppose that the lifetime of products may be modeled by a (4). Let T denote the lifetime of such a product and T has the two-parameter exponential distribution with the p.d.f. is given as (4). Clearly, a longer lifetime implies a better product quality. Hence, the lifetime is a larger-the-better type quality characteristic. The lifetime is generally required to exceed L unit times to both be economically profitable and satisfy customers. Montgomery [1] developed a capability index C_L for properly measuring the larger-the-better quality characteristic. C_L is defined as follows:

$$C_L = \frac{\mu - L}{\sigma} \quad (6)$$

where the process mean μ, the process standard deviation σ, and L is the lower specification limit. To assess the lifetime performance of products, C_L can be defines as the lifetime performance index.

Under T has the two-parameter exponential distribution and the data transformation $Y = T - \theta$, $\theta > 0$, the distribution of Y is an exponential distribution (5). Moreover, there are several important properties, as follows:

1. The lifetime performance index C_L can be rewritten as

$$C_L = \frac{\mu - L}{\sigma} = \frac{\beta - L}{\beta} = \left[1 - \frac{L}{\beta}\right], \ -\infty < C_L < 1, \quad (7)$$

where the process mean μ = E(Y) = β, the process standard deviation $\sigma = \sqrt{\text{Var}(Y)} = \beta$, and L is the lower specification limit.

2. The failure rate functions of Y is given by

$$r(y) = \frac{1}{\beta}. \quad (8)$$

When the mean new lifetime of products $\beta > L$, then the lifetime performance index $C_L \geq 0$. From Eqs. (6) and (8), we can see that the larger the β, the smaller the failure rate and the larger the lifetime performance index C_L. Conversely, when $\beta < L$, then $C_L < 0$, thus the smaller the β, the larger the failure rate and the smaller the lifetime performance index C_L. Therefore, the lifetime performance index C_L reasonably and accurately represents the lifetime performance of products. If the new lifetime of a product exceeds the lower specification limit L, then the product is labeled as a conforming product. Otherwise, the product is labeled as a non-conforming product.

The ratio of conforming products is known as the conforming rate which can be defined as

$$\Pr = \Pr(Y \geq L) = \exp(-\frac{L}{\beta}) = \exp(C_L - 1),\ -\infty < C_L < 1. \quad (9)$$

Obviously, a strictly increasing relationship exists between conforming rate Pr and the lifetime performance index C_L, then Table 1 lists various values of C_L and the corresponding conforming rates $\Pr$. The C_L values which are not listed in Table 1, the conforming rate $\Pr$ can be calculated by Eq. (9). Since an one-to-one mathematical relationship exists between the conforming rate Pr and the lifetime performance index C_L. Therefore, utilizing the one-to-one relationship between Pr and C_L, lifetime performance index can be a flexible and effective tool, not only evaluating product quality, but also for estimating the conforming rate Pr.

Table 1: The lifetime performance index C_L is the corresponding conforming rate $\Pr$.

C_L	Pr	C_L	Pr	C_L	Pr	C_L	Pr
$-\infty$	0.00000	-0.5	0.22313	0.35	0.52205	0.70	0.74082
-10	0.00002	0.00	0.36788	0.40	0.54881	0.75	0.77880
-8	0.00012	0.05	0.38674	0.45	0.57695	0.80	0.81873
-6	0.00091	0.10	0.40657	0.50	0.60653	0.85	0.86071
-4	0.00674	0.20	0.44933	0.55	0.63763	0.90	0.90484
-2	0.04979	0.25	0.47237	0.60	0.67032	0.95	0.95123
-1	0.13534	0.30	0.49659	0.65	0.70469	1.00	1.00000

ESTIMATION OF LIFETIME PERFORMANCE INDEX

Maximum likelihood estimator of lifetime performance index

In lifetime testing experiments of products, the experimenter may not always be in a position to observe the lifetimes of all the items on test due to time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties) on data collection. Therefore, censored samples may arise in practice. In this paper, we consider the case of an adaptive Type-II progressive censoring scheme. An adaptive Type-II progressive censoring scheme is quite useful in many practical situations. where budget constraints are in place or there is a demand for rapid testing. Let T denote the lifetime of such a product and T has the two-parameter exponential distribution with the p.d.f. is as (4). With an adaptive Type-II progressive censoring, n units are placed on test. Consider that $T^{\mathbf{R}}_{1;m,n} < T^{\mathbf{R}}_{2;m,n} < \ldots < T^{\mathbf{R}}_{m;m,n}$ is the corresponding an adaptive progressively Type II censored sample, with censoring scheme $\mathbf{R} = \{R_1, R_2, \ldots, R_m\}$. By using the transformation $Y_i = T^{\mathbf{R}}_{i+1;m,n} - T^{\mathbf{R}}_{1;m,n}$. Hence the joint p.d.f. of $Y_1 < Y_2 < \ldots < Y_{m-1}$ is given by (1). So, the likelihood function is given by

$$L(\beta \mid \mathrm{y}) = A(\frac{1}{\beta})^{m-1} \exp\left[-\sum_{i=1}^{m-1}(\delta_{i+1}R_{i+1}+1)\frac{y_i}{\beta}\right], \quad (10)$$

where A and δ_i as given by (2) and (3), we used Y_i instead of $Y^{\mathbf{R}}_{i;m,n}$. The logarithm of the likelihood function may then be written as

$$\ell(\beta \mid \mathrm{y}) = \log \mathrm{A} + (m-1)\log(\frac{1}{\beta}) - \sum_{i=1}^{m-1}(\delta_{i+1}R_{i+1}+1)\frac{y_i}{\beta} \quad (11)$$

Calculating the first partial derivatives of (11) with respect to β and equating each to zero, we get the ML estimator of β as

$$\begin{aligned}\hat{\beta} &= \frac{1}{m-1}\sum_{i=1}^{m-1}(\delta_{i+1}R_{i+1}+1)y_i = \frac{1}{m-1}\sum_{i=1}^{m-1}(\delta_{i+1}R_{i+1}+1)(t_{i+1}-t_1) \\ &= \frac{1}{m-1}\sum_{i=2}^{m}(\delta_i R_i+1)(t_i-t_1) = \frac{W}{m-1},\end{aligned} \quad (12)$$

where

$$W = \sum_{i=2}^{m}(\delta_i R_i + 1)(t_i - t_1). \quad (13)$$

By using the invariance property of ML estimators see Zehna [11], the ML estimator of C_L is given by

$$\hat{C}_L = 1 - \frac{L}{\hat{\beta}} = 1 - \frac{(m-1)L}{W} = 1 - \frac{(m-1)L}{\sum_{i=2}^{m}(\delta_i R_i+1)(t_i-t_1)}. \quad (14)$$

Theorem 1: Let $T^{\mathbf{R}}_{i;m,n}$, $i = 1, 2, \cdots, m$ be an adaptive progressive Type II censored order statistic from two-parameter exponential distribution (4) with censored scheme $\mathbf{R}$. Then

$$\frac{2W}{\beta} \sim \chi^2_{1-\alpha(2(m-1))}, \qquad (15)$$

where W given by (13)

Proof let $Z_i = \frac{T^{\mathbf{R}}_{i+1;m,n} - T^{\mathbf{R}}_{1;m,n}}{\beta}, i = 1,\ldots,m-1.$ It can be seen that $Z_1 < Z_2 < \cdots < Z_{m-1}$ is an adaptive progressive Type II censored order statistic from standard exponential distribution. Consider the following transformation

$$\begin{cases} U_1 = (n - \delta_1 R_1 - 1)Z_1, \\ U_2 = (n - \delta_1 R_1 - \delta_2 R_2 - 2)(Z_2 - Z_1), \\ \vdots \end{cases} \qquad (16)$$

The generalized spacing's $U_1, U_2, \ldots, U_{m-1}$ are independent and identically distributed as standard exponential distribution, see Tomas and Wilson [12]. Hence,

$$2\sum_{i=1}^{m-1} U_i = 2\sum_{i=1}^{m-1} (\delta_{i+1} R_{i+1} + 1) Z_i = 2\sum_{i=2}^{m} (\delta_i R_i + 1) Z_{i-1} = \frac{2W}{\beta}, \qquad (17)$$

has a chi-squared distribution with $2(m-1)$ degrees of freedom

Remark 1. The expectation of $\hat{C}_L$ can be derived as follows

$$\mathrm{E}(\hat{C}_L) = 1 - \frac{2(m-1)L}{\beta} \mathrm{E}(\frac{\beta}{2W}) = 1 - \frac{(m-1)L}{\beta(m-2)}. \qquad (18)$$

The ML estimator $\hat{C}_L$ is not an unbiased estimator of C_L. But when $m \to \infty$, E $(\hat{C}_L) \to C_L$, so the ML estimator $\hat{C}_L$ is asymptotically unbiased estimator. Moreover, we also show that $\hat{C}_L$ is consistent.

Bayes estimator of lifetime performance index

The Bayesian approach provides the methodology for incorporation of previous information with the current data. Waller et al. [13] presented a method by which engineering experiences, judgments, and beliefs can be used to assign values to the parameters of gamma prior distribution . In this paper, we considered β is a random variable having the conjugate inverted gamma prior distribution

$$\pi^*(\beta) = \frac{b^a}{\Gamma(a)\beta^{a+1}} \exp(\frac{-b}{\beta}), \qquad (19)$$

where the parameters a and b are obtained from the past history. From (10) and (19), we can derive the posterior distribution of β is given by

$$\pi(\beta) = \frac{\left[\sum_{i=1}^{m-1} (\delta_i R_{i+1} + 1) y_i + b\right]^{m+a-1}}{\Gamma(m+a-1)\beta^{m+a}} \exp\left(\frac{-\sum_{i=1}^{m-1} (\delta_i R_{i+1} + 1) y_i + b}{\beta}\right). \qquad (20)$$

By consider a squared-error loss function, $\phi(\lambda, \hat{\lambda}) = (\hat{\lambda} - \lambda)^2$, then the Bayes estimator of β is the posterior mean

$$\tilde{\beta} = \mathbf{E}(\beta \mid \pi(\beta)) = \frac{\sum_{i=1}^{m-1} (\delta_i R_{i+1} + 1) y_i + b}{m+a-1}$$

$$= \frac{\sum_{i=2}^{m} (\delta_i R_i + 1)(t_i - t_1) + b}{m+a-1} = \frac{W^*}{m+a-1},$$

where

$$W^* = \sum_{i=2}^{m} (\delta_i R_i + 1)(t_i - t_1) + b. \qquad (23)$$

Hence, the Bayes estimator $\tilde{C}_L$ of C_L can be written by using (7) and (20) as

$$\tilde{C}_L = E(C_L \mid \pi(\beta)) = 1 - \frac{(m+a-1)L}{\sum_{i=2}^{m} (\delta_i R_i + 1)(t_i - t_1) + b} \qquad (24)$$

Theorem 2: Let $T^{\mathbf{R}}_{i;m,n}$, $i = 1, 2, \ldots, m$ be an adaptive progressive Type II censored order statistic from two-parameter exponential distribution (4) with censored scheme $\mathbf{R}$. Then

$$\frac{2W^*}{\beta} \sim \chi^2_{1-\alpha(2(m+a-1))}, \qquad (24)$$

where W^* given by (22)

Proof Let Z= $\frac{2W^*}{\beta}$ by using the change of variables (see Casella and Berger [14]), then we obtain that the p.d.f. of Z is given by

$$f_Z(z) = \pi(\frac{2W^*}{z}) \parallel I_z \parallel = \frac{z^{\frac{2(m+a-1)}{2}-1}}{2^{\frac{2(m+a-1)}{2}} \Gamma\left(\frac{2(m+a-1)}{2}\right)} \exp\left(-\frac{z}{2}\right), \qquad (25)$$

hence, $\frac{2W^*}{\beta} \sim \chi^2_{1-\alpha(2(m+a-1))}$.

Remark 2. The expectation of $\tilde{C}_L$ can be derived as follows

$$\mathrm{E}(\tilde{C}_L) = 1 - \frac{2(m+a-1)L}{\beta} L\mathrm{E}(\frac{\beta}{2W^*}) = 1 - \frac{(m+a-1)\beta L}{m+a-2}. \qquad (26)$$

The Beyas estimator $\tilde{C}_L$ is not an unbiased estimator of C_L. But when $m \to \infty$, E $(\tilde{C}_L) \to C_L$, so the Beyas estimator $\tilde{C}_L$ is asymptotically unbiased estimator. Moreover, we also show that $\tilde{C}_L$ is consistent.

CONFIDENCE INTERVAL FOR C_L

In this section, to determine whether the lifetime performance index of products meets the predetermined level, we obtained $100(1-\alpha)\%$ lower bound for C_L by using the ML and Bayes estimator given by (14) and (23), then based on this lower bound, a hypothesis testing procedure is developed. Assuming that the required index value of lifetime performance is larger than c, where c denotes the target value, the null hypothesis H_0: $C_L \le c$ and the alternative hypothesis H_1: $C_L > c$ are constructed.

In the Bayesian approach, given the specified significance level α, the level 100(1 - α)% one-sided credible interval for C_L can be derived as follows:

Since the pivotal quantity $\frac{2W^*}{\beta} \sim \chi^2_{2(m+a-1)}$ and CHIINV (1 - α, 2($m+a-1$)) represents the lower 1 - α percentile of $\chi^2_{2(m+a-1)}$

$$\begin{aligned}1-\alpha &= P\left(\frac{2W^*}{\beta} \le \text{CHIINV}(1-\alpha, 2(m+a-1))\right)\\ &= P\left(\frac{1}{\beta} \le \frac{\text{CHIINV}(1-\gamma, 2(m+a-1))}{2W^*}\right)\\ &= P\left(1-\frac{L}{\beta} \ge 1-\frac{L\text{CHIINV}(1-\alpha, 2(m+a-1))}{2W^*}\right)\\ &= P\left(C_L \ge 1+\frac{(\tilde{C}_L-1)\text{CHIINV}(1-\alpha, 2(m+a-1))}{2(m+a-1)}\right),\end{aligned} \tag{27}$$

where $\tilde{C}_L$ is given by (23). From (27), we obtain that a lower bound for C_L is

$$L_{\text{BB}} = 1+\frac{(\tilde{C}_L-1)\text{CHIINV}(1-\alpha, 2(m+a-1))}{2(m+a-1)} \tag{28}$$

where $\tilde{C}_L$ denotes the Bayes estimator of C_L, α is the specified significance level, m is the number of observed failures before termination and a is a parameter of prior distribution.

In the non-Bayesian approach, by using the MLE $\hat{C}_L$. Since $\theta = T^{\mathbf{R}}_{1;m,n}$ is known and by Remark 1 and CHIINV ($1-\alpha$, $2(m-1)$) represents the lower $1-\alpha$ percentile of $\chi^2_{1-\alpha(2(m-1))}$, then

$$\begin{aligned}1-\gamma &= P\left(\frac{2W}{\beta} \le \text{CHIINV}(1-\alpha, 2(m-1))\right)\\ &= P\left(\frac{1}{\beta} \le \frac{\text{CHIINV}(1-\alpha, 2(m-1))}{2W}\right)\\ &= P\left(C_L \ge 1-\frac{L\text{CHIINV}(1-\alpha, 2(m-1))}{2W}\right)\\ &= P\left(C_L \ge 1+\frac{(\hat{C}_L-1)\text{CHIINV}(1-\alpha, 2(m-1))}{2(m-1)}\right),\end{aligned} \tag{29}$$

or a $100(1-\alpha)\%$ lower bound for C_L is

$$L_{\text{BML}} = 1+\frac{(\hat{C}_{L_y}-1)\text{CHIINV}(1-\alpha, 2m)}{2(m-1)}. \tag{30}$$

The managers can then employ the one-sided hypothesis testing to determine whether the lifetime performance index adheres to the required level. The proposed testing procedure about C_L can be organized as follows:

1) From the observed adaptive progressively Type II censored data ($t^{\mathbf{R}}_{1;m,n} < t^{\mathbf{R}}_{2;m,n} < \cdots < t^{\mathbf{R}}_{m;m,n}$) from the two-parameter exponential distribution, we can obtained $(y_1, y_2, ..., y_{m-1})$ from the one-parameter exponential distribution, by using transformation $Y_i = T^{\mathbf{R}}_{i+1;m,n} - T^{\mathbf{R}}_{1;m,n}$.
2) Determine the lower lifetime limit L with the new lifetimes, for products and performance index value c, then the testing null hypothesis H_0 : $C_L \le c$ and the alternative hypothesis H_1 : $C_L > c$ is constructed.
3) Specify a significance level α .
4) Calculate the value of test statistic $\hat{C}_L$ and $\tilde{C}_L$ using (14) and (23).
5) Calculate the value of lower bound L_{BML} and L_{BB} for C_L from (28) and (30).
6) The decision rule of statistical test is provided as follows: If $c \notin [L_{\text{BB}}, \infty)$ or $c \notin [L_{\text{BML}}, \infty)$, we reject the null hypothesis and it is concluded that the lifetime performance index of product meets the required level.

Table 2: An adaptive progressive Type II censored order statistic from (5).

		1	2	3	4	5	6	7	8	9	10	11	12
Case II	$y_{i,m,n}$	38	109	158	231	231	346	467	544	615	722	846	939
Case III	$y_{i,m,n}$	38	109	158	231	346	467	544	615	722	846	1020	1301

Table 3: Test statistic, one-sided confidence interval and cases of test

Cases	τ	$\hat{C}_L$	$[L_B, \infty)$	$H_0: C_L \leq 0.80$	$H_1: C_L > 0.80$
I	0	0.9470	$[0.9357, \infty)$	reject	Accept
II	500	0.9379	$[0.9058, \infty)$	reject	accept
III	1000	0.9493	$[0.9231, \infty)$	reject	accept

ILLUSTRATIVE EXAMPLES

In this section, we propose the new hypothesis testing procedure to a practical data set (Lawless [15] and Lawless [16]) on the mileages at which $n = 19$ military personnel carriers failed in service. There is no censoring ($n = m$), and the mileages ($t_i, i = 1, 2, \cdots, 19$) are 162, 200, 271, 320, 393, 508, 539, 629, 706, 777, 884, 1008, 1101, 1182, 1463, 1603, 1984, 2355, 2880. The data set has been checked that exponential model is correct (Wu et al. [17]). In addition, a probability plot of the values $Y_i = T_{i+1} - T_1$, $i = 1, 2, ..., n-1$ indicates that an exponential model is consistent with the data (Lawless [16]). The lower lifetime limit is assumed to be $L_y = 47.5258$. To deal with the product managers concerns regarding lifetime performance, the conforming rate $\Pr$ of products is required to exceed 80. Referring to Table 1, the value is required to exceed 0.80. Thus, the performance index value is set at c =0.80. The testing hypothesis $H_0 : C_{L_y} \leq 0.80$ and the alternative hypothesis $H_1 : C_{L_y} > 0.80$ is constructed. In our example we have used three cases.

- $\tau = 0$ and $m = 16$ (the right Type II censored random sample Wu et al. [17]).
- $\tau = 500$, $m = 13$ and $\mathbf{R} =$ {2, 0, 0, 1, 0, 0, 1, 0, 0, 0, 2, 0, 0}, the corresponding adaptive Type-II progressive censoring data are given in Table 3.
- $\tau = 1000$, $m = 13$ and $\mathbf{R} =$ {2, 0, 0, 1, 0, 0, 1, 0, 0, 0, 2, 0, 0}, the corresponding adaptive Type-II progressive censoring data are given in Table 3.

Since we do not have any prior information and to find the Bayes estimates, small values are given to the gamma hyper parameters to reflect vague prior information. Namely, we assumed that $a = b = 0.0001$. Hence the results in the Bayesian and non-Bayesian are conforming. Referring to Table 3, we reject to the null hypothesis $H_0: C_L \leq 0.80$. Thus, we can conclude that the lifetime performance index of military personnel carriers failed in service meets the required level.

SIMULATION STUDY

In this section, we conducted some of simulation study for confidence level (1 - α) based on one-sided credible and confidence intervals of the lifetime performance index C_L. We consider α = 0.05 and without loss of generality θ =3, $\beta = 5$. We used different sample sizes (n), different effective sample sizes (m), different hyperparameters (prior 1: $a = b = 1$, prior 2: $a = 2$, $b=3$ and prior 3: $a = b = 5$) and different sampling schemes (i.e., different $\mathbf{R}$ values). The lower lifetime limit L_x is assumed to be 0.2. The Monte Carlo simulation algorithm of confidence level (1 - γ) is given in the following steps:

1) Generate an adaptive Type-II progressive censored data $(t_{1;m,n}, t_{2;m,n}, \ldots, t_{m;m,n})$ from exp(1.0, 0.5) using the algorithm proposed by Ng et al. [1] then by transformation $Y_i = T_{i+1} - T_1$, the values ($y_{1;m,n}$, $y_{2;m,n}$, $\ldots$, $y_{m-1;m,n}$) are obtained.
2) The 95 % lower bounds L_{BML} and L_{BB} are calculated from (28) and (30).
3) If $C_{L_y} > L_{\text{BML}}$ then Count Q1 = 1 else Count Q1 = 0.
4) If $C_{L_y} > L_{\text{BB}}$ then Count Q2 = 1 else Count Q2 = 0.
5) Steps 2--5 are repeated 100 times.
6) The ML estimation of confidence level (1 - γ) is (1 - $\hat{\gamma}$)

= $\frac{\text{TotalCountQ1}}{100}$ for one-sided confidence interval.

7) The Bayes estimation of confidence level (1 - γ) is (1- $\tilde{\gamma}$) $\tilde{\gamma}$ = $\frac{\text{TotalCountQ2}}{100}$ for one-sided credible interval.

8) Repeat steps 2--7 1000 times, then we can get the 1000 estimations of confidence level as follows: $(1-\hat{\gamma})_1$, $(1-\hat{\gamma})_2$, ..., $(1-\hat{\gamma})_{1000}$

9) Calculate the average empirical confidence level

10) $$\text{Average}(1-\hat{\gamma}) = \frac{1}{1000}\sum_{i=1}^{1000}(1-\hat{\gamma})_i, \quad (31)$$

11) and the sample mean square error (SMSE)

12) $$\text{SMSE}(1-\hat{\gamma}) = \frac{1}{1000}\sum_{i=1}^{1000}\left[(1-\hat{\gamma})_i-(1-\gamma)\right]^2, \quad (32)$$

13) for one-sided confidence interval and one-sided credible interval, respectively.

Table 4 Average empirical confidence level (1 - γ) for C_L when γ = 0.05 and T={4,7}

			$\tau_1 = 4$		$\tau_2 = 7$	
n	m	C.S.	MLE	Bayes	MLE	Bayes
				a =0.5, b =0.5		a =0.5, b =0.5
30	15	(15, 0, ..., 0)	0.955(0.00045)	0.960(0.00044)	0.951(0.00048)	0.958(0.00044)
		(0, ..., 0, 15)	0.949(0.00050)	0.962(0.00048)	0.965(0.00049)	0.955(0.00050)
		(0, ...,0,15,0^7)	0.951(0.00048)	0.964(0.00046)	0.967(0.00056)	0.958(0.00051)
		(1^{15})	0.950(0.00057)	0.963(0.00053)	0.966(0.00045)	0.957(0.00045)
	25	(5, 0^{24})	0.950(0.00056)	0.970(0.00052)	0.975(0.00049)	0.961(0.00050)
		(0^{24}, 5)	0.950(0.00049)	0.971(0.00047)	0.977(0.00042)	0.962(0.00050)
		(0^{12},5,0^{12})	0.951(0.00050)	0.972(0.00043)	0.977(0.00045)	0.963(0.00049)
		($(1,0^4)^5$)	0.948(0.00050)	0.958(0.00045)	0.960(0.00042)	0.953(0.00049)
50	25	(25, 0^{24})	0.951(0.00044)	0.961(0.00045)	0.963(0.0005)	0.956(0.00042)
		(0^{24}, 25)	0.950(0.00050)	0.959(0.00041)	0.962(0.00050)	0.954(0.00045)
		(0^{12},25,0^{12})	0.953(0.00050)	0.963(0.00050)	0.965(0.00050)	0.958(0.00047)
		(1^{25})	0.952(0.00045)	0.961(0.00050)	0.963(0.00048)	0.957(0.00043)
	40	(10, 0^{39})	0.948(0.00045)	0.957(0.00042)	0.959(0.00040)	0.952(0.00045)
		(0^{39}, 10)	0.949(0.00047)	0.958(0.00041)	0.960(0.00045)	0.954(0.00047)
		(0^{19},10,0^{20})	0.949(0.00046)	0.958(0.00040)	0.959(0.00043)	0.953(0.00044)
		($(1,0^3)^{10}$)	0.950(0.00045)	0.958(0.00041)	0.959(0.00044)	0.953(0.00043)

CONCLUSIONS

Process capability indices are widely used to measure the potential and performance of a process. Moreover, in life testing experiments, the experimenter may not always be in a position to observe the lifetimes of all products on test. The censoring scheme which can save both the total test time and the cost induced by failure of the units and increase the efficiency of statistical analysis. So, in this paper, we conducted an adaptive Type-II progressive censoring scheme to determine whether the lifetime performance index of products meets the predetermined level. A simulation study was conducted to examine and compare the performance of the proposed methods for different sample sizes, different censoring schemes and deferent values of c . From the results, we observe the following.

The results of simulation are summarized in Table 4 for the different combination of n, m, $\mathbf{R}$ and prior parameter (a , b). From Table 6, based on L = 0.2 and γ = 0.05, the following points can be drawn.

1- For any m all of the average empirical confidence level (1 - γ) very close to confidence level (1 - γ).

2- All of the average empirical confidence levels have very small MSE.

3- The MSE for one-sided credible interval based on Bayes estimates are almost smaller than the MSE for one-sided confidence interval based on the MLE.

4- The average empirical confidence levels have very small MSE for small values of τ .

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