International Journal of Applied Engineering Research, ISSN 0973-4562 Volume 13, Number 18 (2018) pp. 13589–13595

Binary Čech Closure spaces

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Abstract: In this paper we introduce the concept of Binary Čech Closure Operator(BČCO), its induced Čech closure operators and establish the relation between them. Here we study different types of closed and open sets having nice properties with sufficient examples. Later we define the operations of Binary Čech Closure Operators.

AMS subject classification: 46A99, 54A05.
Keywords: Binary Čech Closure space, Binary Topology, b̌-closed sets, b̌-semiopen, b̌-γ-open.

INTRODUCTION

Closure spaces were introduced by E. Čech [1] and then studied by many authors like Jeeranunt Khampaladee [4], Chawalit Boonpok [2], David Niel Roth[7] etc.. Čech closure spaces, is a generalisation of the concept of topological spaces. T. A. Sunitha[8] studied on Čech closure spaces. The concepts of generalized closed sets and generalized continuous maps of topological spaces were extended to closure spaces by C. Boonpok and J. Khampaladee[2].Nithanantha Jothi and P. Thangavelu introduced the concept of binary topology [10].

In this paper we extend the concept of Čech closure spaces to Binary Čech Closure Spaces. Section 2 contains the pre-requisites.

In Section 3 we introduce the concepts of Binary Closure Operator, Binary Čech Closure Operator, induced closure operators and establish the relation between them. We also study the properties of b̌-closed sets and b̌-open sets. Section 4 describes about b̌-semiopen sets, b̌-gamma opensets and generalised b̌-semiopen sets.

Section 5 presents the operations union, composition and intersection of Binary Čech closure operators.

PRELIMINARIES

Definition 1. [1] Let X be a set and $\wp(X)$ be its powerset. A function $c : \wp(X) \to \wp(X)$ is called a Čech closure operator for X if

1. $c(\phi) = \phi$
2. $A \subseteq c(A)$
3. $c(A \cup B) = c(A) \cup c(B), \forall A, B \subseteq X$
   Then (X, c) is called Čech closure space or simply closure space.
   If in addition
4. $c(c(A)) = c(A), \forall A \subseteq X$,
   the space (X, c) is called a Kuratowski (topological) space.
   If further
5. for any family of subsets of X, $\{A_i\}_{(i \in I)}$, $c(\cup_{i\in I} A_i) = \cup_{i\in I} c(A_i)$, the space is called a total closure space.

Definition 2. [1] A function $c : \wp(X) \to \wp(X)$is called a monotone(or simply closure) operator for X if

1. $c(\phi) = \phi$
2. $A \subseteq c(A)$
3. $A \subseteq B \Rightarrow c(A) \subseteq c(B), \forall A, B \subseteq X$

Then (X, c) is called monotone(closure) space.
A subset A of a closure space (X, c) will be closed if $c(A) = A$ and open if its complement is closed, i.e. if $c(X - A) = X - A$.
If (X, c) is a closure space, we denote the associated topology on X by t. i.e. $t = \{A^c : c(A) = A\}$.

Lemma 3. [1] A Čech closure space is a monotone space.

Definition 4. [7] Let (X, c) be a Čech closure space, $c(A) = A, \forall A \subseteq X$, then c is called the discrete closure operator on X. If $c(A) = X, \forall A \subseteq X$, then c is called the trivial operator or indiscrete operator on X.

Definition 5. [10] Let X and Y be any two non-empty sets and $\wp(X)$ and $\wp(Y)$ be their power sets respectively. A binary topology from X to Y is a binary structure $M \subseteq \wp(X) \times \wp(Y)$ that satisfies the following axioms.

1. (ϕ, ϕ) and $(X, Y) \in M$
2. If (A_1, B_1) and $(A_2, B_2) \in M$, then $(A_1 \cap A_2, B_1 \cap B_2) \in M$.
3. If $\{(A_\alpha, B_\alpha) : \alpha \in \Delta\}$ is a family of members of M, then $(\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha) \in M$.

If M is a binary topology from X to Y then the triplet (X, Y, M) is called a binary topological space and the members of M are called binary open sets. (C, D) is called binary closed if $(X \setminus C, Y \setminus D)$ is binary open.

The elements of $X \times Y$ are called the binary points of the binary topological space (X, Y, M). Let (X, Y, M) be a binary topological space and let $(x, y) \in X \times Y$. The binary open set (A, B) is called a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$. If $X = Y$ then M is called a binary topology on X and we write (X, M) as a binary space.

Definition 6. [4] Let (X, c) be a closure space. A subset A of X is called a semi open set if there exists an open set G in (X, c) such that $G \subseteq A \subseteq c(G)$. A subset $A \subseteq X$ is called a semi-closed set if its complement is semi-open.

Definition 7. [3] A non-empty subset D of V will be called c-dense in (V, c) if $c(D) = X$.

Definition 8. [8] A map $f : (X, c) \to (Y, c')$ is said to be a $c - c'$ morphism or just a morphism if $f(c(A)) \subseteq c' f(A)$.

Note: $\wp(X)$ denotes the power set of a set X.

BINARY ČECH CLOSURE SPACE

Definition 9. Let X and Y be two sets. A function $\check{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ is called a binary closure (monotone) operator if
$\check{b}(\phi, \phi) = (\phi, \phi)$
$(A, B) \subseteq \check{b}(A, B)$
$(A, B) \subseteq (C, D) \Rightarrow \check{b}(A, B) \subseteq \check{b}(C, D)$.
Then $(X, Y, \check{b})$ is called a binary closure (monotone) space.

Example 10. Let $X = \{0, 1, 2\}$ and $Y = \{a, b\}$.
$\wp(X) = \{\phi, X, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$
$\wp(Y) = \{\phi, Y, \{a\}, \{b\}\}$
Let $\check{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ be defined as
$\check{b}(\phi, \phi) = (\phi, \phi)$
$\check{b}(A, \phi) = (A, \phi)$, $\forall A \subseteq X$
$\check{b}(\phi, B) = (\phi, B) \, \forall B \subseteq Y$
$\check{b}(A, B) = (X, Y)$, $A \neq \phi, B \neq \phi$ Then b̌ is a binary closure operator.

Definition 11. The binary closure operator is a Binary Čech Closure Operator(BČCO) if it satisfies the property $\check{b}[(A, B) \cup (C, D)] = \check{b}(A, B) \cup \check{b}(C, D)$. Then $(X, Y, \check{b})$ is called a Binary Čech Closure Space(BČCS).

Example 12. Let $X = \{0, 1, 2\}$ and $Y = \{a, b\}$.
Let $\check{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ be defined as
$\check{b}(\phi, \phi) = (\phi, \phi)$
$\check{b}(\{0\}, \phi) = (\{0, 1\}, \{a\})$
$\check{b}(\{1\}, \phi) = (\{1\}, \{a\})$
$\check{b}(\{2\}, \phi) = (\{2\}, \{a\})$
$\check{b}(\{0, 1\}, \phi) = (\{0, 1\}, \{a\})$
$\check{b}(\{0, 2\}, \phi) = (X, \{a\})$
$\check{b}(\{1, 2\}, \phi) = (\{1, 2\}, \{a\})$
$\check{b}(X, \phi) = (X, \{a\})$
$\check{b}(\phi, \{a\}) = (\phi, \{a\})$
$\check{b}(\phi, \{b\}) = (\{0\}, Y)$
$\check{b}(\phi, Y) = (\{0\}, Y)$
$\check{b}(A, B) = \check{b}(A, \phi) \cup \check{b}(\phi, B)$, $\forall A \subseteq X$ and $\forall B \subseteq Y$. Then $(X, Y, \check{b})$ is a BČCS.

Remark 13. Example 10 is not a binary Čech closure operator since
$\check{b}(\{0, 1\}, \{a\}) = (X, Y) \neq \check{b}(\{0, 1\}, \phi) \cup \check{b}(\phi, \{a\})$

Definition 14. A set $(A, B) \in \wp(X) \times \wp(Y)$ is b̌-closed if $\check{b}(A, B) = (A, B)$ and a set (C, D) is b̌-open if $\check{b}(X \setminus C, Y \setminus D) = (X \setminus C, Y \setminus D)$.

International Journal of Applied Engineering Research, ISSN 0973-4562 Volume 13, Number 18 (2018) pp. 13589–13595

Proposition 15. Let $(X, Y, \check{b})$ be a binary Čech closure space. Then (ϕ, ϕ) and (X, Y) are both open and closed.

Proof. $\check{b}(\phi, \phi) = (\phi, \phi)$ So (ϕ, ϕ) is b̌-closed. Hence $(X, Y) = (X \setminus \phi, Y \setminus \phi)$ is b̌-open.
Since $(A, B) \subseteq \check{b}(A, B), \forall\ (A, B) \in \wp(X) \times \wp(Y)$, $(X, Y) \subseteq \check{b}(X, Y)$. Also $\check{b}(A, B) \in \wp(X) \times \wp(Y)$, $\forall\ (A, B) \in \wp(X) \times \wp(Y)$. So $\check{b}(A, B) \subseteq (X, Y)$ and $\check{b}(X, Y) \subseteq (X, Y)$. Thus we get $(X, Y) = \check{b}(X, Y)$ i.e. (X, Y) is b̌-closed and therefore $(\phi, \phi) = (X \setminus X, Y \setminus Y)$ is b̌-open. ■

Definition 16. A binary Čech closure operator $\check{b_1}$ is said to be coarser than a binary Čech closure operator $\check{b_2}$ on the same sets X and Y if $\check{b}_2(A, B) \subseteq \check{b}_1(A, B), \forall (A, B) \in \wp(X) \times \wp(Y)$. Then we write $\check{b_1} < \check{b_2}$.

The discrete closure operator given by $\check{b}(A, B) = (A, B)$, $\forall (A, B) \in \wp(X) \times \wp(Y)$ is the finest closure between X and Y. The indiscrete closure operator given by $\check{b}(\phi, \phi) = (\phi, \phi)$ and $\check{b}(A, B) = (X, Y)$, $\forall (A, B)\{\neq (\phi, \phi)\} \subseteq (X, Y)$ is the coarsest closure between X and Y.

Definition 17. Let $(X, Y, \check{b})$ be a binary Čech closure space. Then the binary interior operator associated with b̌, $Int_{\check{b}}$ is a function from $\wp(X) \times \wp(Y)$ to itself given by $Int_{\check{b}}(A, B) = (X \setminus C, Y \setminus D)$ where $(C, D) = \check{b}(X \setminus A, Y \setminus B)$.
A binary set (A, B) is b̌-open if and only if $Int_{\check{b}}(A, B) = (A, B)$.

Definition 18. Let (Z, c) be a closure space and $(X, Y, \check{b})$ be a binary closure space. Then a mapping $f : Z \to X \times Y$ is called a $c - \check{b}$ morphism if $f(c(A)) \subseteq \check{b}[f(A)], \forall A \subseteq Z$ where $f(A) = (C, D)$ and $C = \{x : (x, y) = f(a)$ for some $a \in A\}$ and $D = \{y : (x, y) = f(a)$ for some $a \in A\}$. i.e. C is the projection of $f(A)$ to X and D is the projection of $f(A)$ to Y.

Proposition 19. Given a binary Čech closure operator, b̌ from X to Y, the function $\check{b}_X : \wp(X) \to \wp(X)$ given by $\check{b}_X(A) = C$ where $\check{b}(A, \phi) = (C, D)$ is a Čech closure operator on X. Similarly $\check{b}_Y : \wp(Y) \to \wp(Y)$ given by $\check{b}_Y(B) = D$ where $\check{b}(\phi, B) = (C, D)$ is a Čech closure operator on Y.

Proof. Since $\check{b}$ is a BČCO, $\check{b}(\phi, \phi) = (\phi, \phi)$.
$\therefore \check{b}_X(\phi) = \phi$ and $\check{b}_Y(\phi) = \phi$.
Let $A \subseteq X$ and $B \subseteq Y$. Then $(A, \phi) \subseteq \check{b}(A, \phi)$, by the property of b̌.
$\check{b}(A, \phi) = (\check{b}_X(A), D)$, for some $D \subseteq Y$.
i.e. $(A, \phi) \subseteq (\check{b}_X(A), D)$, which gives $A \subseteq \check{b}_X(A)$, $\forall A \subseteq X$.
Similarly $(\phi, B) \subseteq \check{b}(\phi, B)$ gives $B \subseteq \check{b}_Y(B)$, $\forall B \subseteq Y$.
Let $A \subseteq C \subseteq X$ and $B \subseteq D \subseteq Y$. Then $(A, \phi) \subseteq (C, \phi)$.
$\therefore \check{b}(A, \phi) \subseteq \check{b}(C, \phi) \Rightarrow \check{b}_X(A) \subseteq \check{b}_X(C)$.
Similarly $(\phi, B) \subseteq (\phi, D) \Rightarrow \check{b}_Y(B) \subseteq \check{b}_Y(D)$.
Now let $A, C \subseteq X$ and $B, D \subseteq Y$. $\check{b}(A, \phi) \cup \check{b}(C, \phi) = \check{b}(A \cup C, \phi) \Rightarrow \check{b}_X(A) \cup \check{b}_X(C) = \check{b}_X(A \cup C)$.
Also $\check{b}(\phi, B) \cup \check{b}(\phi, D) = \check{b}(\phi, B \cup D) \Rightarrow \check{b}_Y(B) \cup \check{b}_Y(D) = \check{b}_Y(B \cup D)$.
Hence $\check{b}_X$ and $\check{b}_Y$ are Čech closure operators. ■

Proposition 20. If (X, c_1) and (Y, c_2) are two Čech closure spaces, then $(X, Y, \check{c})$ where $\check{c} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ is given by $\check{c}(A, B) = (c_1(A), c_2(B))$, is a binary Čech closure operator.

Proof. $\check{c}(\phi, \phi) = (c_1(\phi), c_2(\phi)) = (\phi, \phi)$, since c_1 and c_2 are Čech closure operators.
Let $A \subseteq X$ and $B \subseteq Y$. Then $A \subseteq c_1(A)$ and $B \subseteq c_2(B)$.
$\Rightarrow (A, B) \subseteq (c_1(A), c_2(B)) = \check{c}(A, B)$.
Let $(A, B), (C, D) \in \wp(X) \times \wp(Y)$. Then $A, C \subseteq X$ and $B, D \subseteq Y$.
So $c_1(A) \cup c_1(C) = c_1(A \cup C)$ and $c_2(B) \cup c_2(D) = c_2(B \cup D)$.
Now

$$\begin{aligned}\check{c}(A, B) \cup \check{c}(C, D) &= (c_1(A), c_2(B)) \cup (c_1(C), c_2(D)) \\ &= (c_1(A) \cup c_1(C), c_2(B) \cup c_2(D)) \\ &= (c_1(A \cup C), c_2(B \cup D)) \\ &= \check{c}(A \cup C, B \cup D) \\ &= \check{c}[(A, B) \cup (C, D)]\end{aligned}$$

Thus č is a binary Čech closure operator. ■

Proposition 21. Let $(X, Y, \check{b})$ be a binary Čech closure space and $\check{b}_X$ and $\check{b}_Y$ be the associated Čech closure operators (as in Proposition 19)in X and Y respectively. Then the binary Čech closure operator $\check{b}_{XY}$ obtained from $\check{b}_X$ and $\check{b}_Y$ (as in Proposition 20) need not be same as $\check{b}$.

Proof. In Example 12, $\check{b}_X(\phi) = \phi, \check{b}_X(X) = X, \check{b}_X(\{0\}) = \{0, 1\}, \check{b}_X(\{1\}) = \{1\}, \check{b}_X(\{2\}) = \{2\}, \check{b}_X(\{0, 1\}) = \{0, 1\}, \check{b}_X(\{1, 2\}) = \{1, 2\}$, and $\check{b}_X(\{0, 2\}) = X$.
$\check{b}_Y(\phi) = \phi, \check{b}_Y(Y) = Y, \check{b}_Y(\{a\}) = \{a\}, \check{b}_Y(\{b\}) = Y$
Then $\check{b}_{XY}(\{1, 2\}, b) = (\{1, 2\}, Y)$

International Journal of Applied Engineering Research, ISSN 0973-4562 Volume 13, Number 18 (2018) pp. 13589–13595
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But $\check{b}(\{1,2\}, b) = (X, Y)$.
Thus $\check{b} \neq \check{b}_{XY}$. ■

Lemma 22. Let $(X, Y, \check{b})$ be a binary Čech closure operator. Then $(\check{b}_X(A), \check{b}_Y(B)) \subseteq \check{b}(A, B) \forall (A, B) \in \wp(X) \times \wp(Y)$.

Proof.

$$\begin{aligned}(\check{b}_X(A), \check{b}_Y(B)) &= (\check{b}_X(A), \phi) \cup (\phi, \check{b}_Y(B)) \\ &\subseteq \check{b}(A, \phi) \cup \check{b}(\phi, B), \textit{ by the definition of } \check{b}_X(A) \textit{ and } \check{b}_Y(B) \\ &= \check{b}(A, B).\end{aligned}$$

Remark 23. b̌ is coarser than $\check{b}_{XY}$ in any binary Čech closure space $(X, Y, \check{b})$.

Proof. $(\check{b}_{XY}(A, B) = (\check{b}_X(A), \check{b}_Y(B)) \subseteq \check{b}(A, B)$, by above lemma $\forall (A, B) \in \wp(X) \times \wp(Y)$
Hence the result. ■

Proposition 24. Let $(X, Y, \check{b})$ be a BČCS. If (A, B) is b̌-closed then A is $\check{b}_X$-closed and B is $\check{b}_Y$-closed.

Proof. Let $\check{b}(A, B) = (A, B)$.
$\check{b}(A, B) = \check{b}(A, \phi) \cup \check{b}(\phi, B)$.
Suppose that $\check{b}(A, \phi) = (\check{b}_X(A), D)$ for some $D \subseteq Y$ and $\check{b}(\phi, B) = (C, \check{b}_Y(B))$ for some $C \subseteq X$.
Then $(A, B) = \check{b}(A, B) = (\check{b}_X(A) \cup C, D \cup \check{b}_Y(B))$.
By Proposition 19, $A \subseteq \check{b}_X(A)$. Also $B \subseteq \check{b}_Y(B)$.
Here $A = \check{b}_X(A) \cup C$.
$\Rightarrow \check{b}_X(A) \subseteq A$.
$\Rightarrow A = \check{b}_X(A)$, since $\check{b}_X$ is a Čech closure operator by Propsition 19.
i.e. A is $\check{b}_X$-closed.
Similarly $B = \check{b}_Y(B)$. i.e. B is $\check{b}_Y$-closed. ■

Remark 25. The converse of the above prposition need not be true as shown in the following examole.
In Example 12, $\{2\}$ is $\check{b}_X$-closed and Y is $\check{b}_Y$-closed. But $\check{b}(\{2\}, Y) = (\{0, 2\}, Y)$. So $(\{2\}, Y)$ is not b̌-closed.

Proposition 26. Let (X, c_1) and (Y, c_2) be two ČCS and $\check{c} = (c_1, c_2)$ as in Proposition 20. If $A \subseteq X$ is c_1-closed and $B \subseteq Y$ is c_2-closed, then (A, B) is č-closed.

Proof. $c_1(A) = A$ and $c_2(B) = B$.
$\check{c}(A, B) = (c_1(A), c_2(B)) = (A, B)$.
Therefore (A, B) is č-closed. ■

Definition 27. Let $(X, Y, \check{b})$ be a Binary Čech closure space. A set $(A, B) \in \wp(X) \times \wp(Y)$ is said to be b̌-dense if $\check{b}(A, B) = (X, Y)$. In Example 12, $(X, \{b\}), (\{1, 2\}, Y), (\{0, 2\}, Y)$ and $(\{0, 2\}, b)$ are b̌-dense.

Proposition 28. Let $(X, Y, \check{b})$ be a binary Čech closure space. $A \subseteq X$ is $\check{b}_X$-dense and $B \subseteq Y$ is $\check{b}_Y$-dense then (A, B) is b̌-dense.

Proof. A is $\check{b}_X$-dense $\Rightarrow \check{b}_X(A) = X$ and B is $\check{b}_Y$-dense $\Rightarrow \check{b}_Y(B) = Y$ Now $(X, Y) = (\check{b}_X(A), \check{b}_Y(B)) \subseteq \check{b}(A, B)$
$\therefore \check{b}(A, B) = (X, Y)$, since $\check{b}(A, B) \in \wp(X) \times \wp(Y)$, $\forall (A, B) \in \wp(X) \times \wp(Y)$.
Thus (A, B) is b̌-dense. ■

Remark 29. Converse of the above Proposition is not true in general.
In Example 12 $(\{1, 2\}, Y)$ is b̌-dense. But $\check{b}_X(\{1, 2\}) = \{1, 2\} \neq X$. i.e. $\{1, 2\}$ is not $\check{b}_X-$ dense.

Corollary 30. Let (X, c_1) and (Y, c_2) be two Čech closure spaces. Then (A, B) is č-dense if and only if A is c_1-dense and B is c_2-dense.

Proof. A is c_1-dense and B is c_2-dense
$\Leftrightarrow c_1(A) = X$ and $c_2(B) = Y$
$\Leftrightarrow (X, Y) = (c_1(A), c_2(B)) = \check{c}(A, B)$
(AB) is č-dense. ■

Proposition 31. Let $(X, Y, \check{b})$ be a binary Čech closure space. If $(A, B) \in \wp(X) \times \wp(Y)$ is b̌-open, then A is $\check{b}_X-$open and B is $\check{b}_Y-$open.

Proof. (A, B) is b̌ -open $\Rightarrow (X \setminus A, Y \setminus B)$ is b̌-closed
$\Rightarrow X \setminus A$ is $\check{b}_X$-closed and $Y \setminus B$ is $\check{b}_Y$-closed
$\Rightarrow A$ is $\check{b}_X-$open and B is $\check{b}_Y-$open. ■

Proposition 32. Let (X, c_1) and (Y, c_2) be two ČCS and $\check{c} = (c_1, c_2)$ (as in Proposition 20). If A is c_1-open and B is c_2-open, then (A, B) is č-open.

Remark 33. If A is $\check{b}_X-$open and B is $\check{b}_Y-$open, then (A, B) need not be b̌-open.
In Example 12 $\{0, 1\}$ is $\check{b}_X$-open and Y is $\check{b}_Y$-open, but $(\{0, 1\}, Y)$ is not b̌-open.

Proposition 34. Let $(X, Y, \check{b})$ be a binary Čech closure space. Then the set of all b̌-open sets ,i.e. $M(\check{b}) := \{(A, B) \mid \check{b}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ is a binary topology.

Proof. Since (ϕ, ϕ) and (X, Y) are b̌-open (ϕ, ϕ) and $(X, Y) \in M(\check{b})$.
Let $(A, B), (C, D) \in M(\check{b})$. Then $\check{b}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ and $\check{b}(X \setminus C, Y \setminus D) = (X \setminus C, Y \setminus D)$.
$\check{b}(X \setminus (A \cap C), Y \setminus (B \cap D)) = \check{b}[(X \setminus A) \cup (X \setminus C), (Y \setminus B) \cup (Y \setminus D)]$
$= \check{b}[(X \setminus A, Y \setminus B) \cup (X \setminus C, Y \setminus D)]$
$= \check{b}(X \setminus A, Y \setminus B) \cup \check{b}(X \setminus C, Y \setminus D)$
$= (X \setminus A, Y \setminus B) \cup (X \setminus C, Y \setminus D)$
$= [(X \setminus A) \cup (X \setminus C), (Y \setminus B) \cup (Y \setminus D)]$
$= (X \setminus (A \cap C), Y \setminus (B \cap D))$.
Thus $(A \cap C, B \cap D)$ is b̌-open. Consider an arbitrary collection of sets $\{(O_\alpha, U_\alpha) : \alpha \in \Lambda\}$, each a member of $M(\check{b})$. For each $\alpha \in \Lambda, (X \setminus O_\alpha, Y \setminus U_\alpha)$ is b̌-closed and $\cap_{\alpha\in\Lambda}(X \setminus O_\alpha, Y \setminus U_\alpha)$ is contained in $(X \setminus O_\alpha, Y \setminus U_\alpha)$. So $\check{b}[\cap_{\alpha\in\Lambda} (X \setminus O_\alpha, Y \setminus U_\alpha)] \subseteq \check{b}(X \setminus O_\alpha, Y \setminus U_\alpha) = (X \setminus O_\alpha, Y \setminus U_\alpha), \forall \alpha \in \Lambda$.
Hence $\check{b}[\cap_{\alpha\in\Lambda} (X \setminus O_\alpha, Y \setminus U_\alpha)] \subseteq \cap_{\alpha\in\Lambda}(X \setminus O_\alpha, Y \setminus U_\alpha)$.
Thus $\cap_{\alpha\in\Lambda}(X \setminus O_\alpha, Y \setminus U_\alpha)$ is b̌-closed.
i.e. $\{X \setminus \cup_{\alpha\in\Lambda} O_\alpha, Y \setminus \cup_{\alpha\in\Lambda} U_\alpha\}$ is b̌-closed.
i.e. $\cup_{\alpha\in\Lambda}(O_\alpha, U_\alpha)$ is b̌-open. ■

VARIOUS OPEN SETS IN BINARY ČECH CLOSURE SPACES

Definition 35. Let $(X, Y, \check{b})$ be a binary Čech closure space. $(A, B) \in \wp(X) \times \wp(Y)$ is said to be b̌-semiopen if there exists a binary open set (U, V) such that $(U, V) \subseteq (A, B) \subseteq \check{b}(U, V)$.
$(C, D) \in \wp(X) \times \wp(Y)$ is said to be b̌-semiclosed if $(X \setminus C, Y \setminus D)$ is b̌-semiopen.

Remark 36. Let $(X, Y, \check{b})$ be a binary Čech closure space. If $(A, B) \in \wp(X) \times \wp(Y)$ is b̌-semiopen, need not imply A is $\check{b}_X$−semiopen and B is $\check{b}_Y$−semiopen.

Example 37. Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{a, b\}$.
b̌ be defined as
$\check{b}(\phi, \phi) = (\phi, \phi), \check{b}(\{0\}, \phi) = (\{0\}, \phi), \check{b}(\{1\}, \phi) = (\{0, 1\}, \phi), \check{b}(\{2\}, \phi) = (\{1, 2\}, \phi)$,
$\check{b}(\{3\}, \phi) = (\{2, 3\}, \phi), \check{b}(\{4\}, \phi) = (\{3, 4\}, \phi), \check{b}(\phi, \{a\}) = (\{1\}, \{a\}), \check{b}(\phi, \{b\}) = (\{1, 2\}, Y)$
For all other $(A, B) \in \wp(X) \times \wp(Y)$,
$\check{b}(A, B) = [\cup_{x\in A} \check{b}(\{x\}, \phi)] \cup [\cup_{y\in B} \check{b}(\phi, \{y\})]$
Here $\check{b}(\{0, 1, 2\}, \{a\}) = (\{0, 1, 2\}, \{a\})$
So $(\{0, 1, 2\}, \{a\})$ is b̌-closed and $(\{3, 4\}, \{b\})$ is b̌-open.
$(\{3, 4\}, \{b\}) = (\{1, 2, 3, 4\}, Y)$
Now $(A, B) = (\{1, 3, 4\}, \{b\})$ is semi open since
$(\{3, 4\}, \{b\}) \subseteq (\{1, 3, 4\}, \{b\}) \subseteq \check{b}(\{3, 4\}, \{b\})$.
But the set $\{1, 3, 4\}$ is not $\check{b}_X$-semi open since the open sets contained in $\{1, 3, 4\}$ are $\{3, 4\}$ and $\{4\}$.
But the closure of $\{3, 4\}$ and $\{4\}$, both, doesnot contain $\{1, 3, 4\}$.

Remark 38. Let $(X, Y, \check{b})$ be a binary Čech closure space. Let $A \subseteq X$ be $\check{b}_X$-semi open and $B \subseteq Y$ be $\check{b}_Y$-semiopen, then (A, B) need not be b̌-semiopen.

Example 39. Let $X = \{1, 2\}, Y = \{a, b\}$
$\check{b}(\phi, \phi) = (\phi, \phi), \check{b}(\{1\}, \phi) = (\{1\}, \{a\}), \check{b}(\{2\}, \phi) = (\{2\}, \{b\}), \check{b}(\phi, \{a\}) = (\{2\}, \{a\}), \check{b}(\phi, \{b\}) = (\{1\}, \{b\})$
For all other $(A, B) \in \wp(X) \times \wp(Y)$,
$\check{b}(A, B) = [\cup_{x\in A} (\{x\}, \phi)] \cup [\cup_{y\in B} (\phi, \{y\})]$
Here $\{2\}$ is $\check{b}_X$- open and $\{b\}$ is $\check{b}_Y$- open. So $\{2\}$ is $\check{b}_X$-semiopen and $\{b\}$ is $\check{b}_Y$- semiopen. But $(\{2\}, \{b\})$ is not $\check{b}$-semiopen, since the only $\check{b}$-open set contained in $(\{2\}, \{b\})$ is (ϕ, ϕ) and $\check{b}(\phi, \phi) = (\phi, \phi)$.

Definition 40. Let $(X, Y, \check{b})$ be a binary Čech closure space. A set (A, B) is b̌-γ-open if there exists a b̌-open set, (U, V) such that $(U, V) \subseteq (A, B)$ and $\check{b}(A, B) = \check{b}(U, V)$.
$(C, D) \in \wp(X) \times \wp(Y)$ is said to be b̌-γ closed if $(X \setminus C, Y \setminus D)$ is b̌-γ open.

Remark 41. (A, B) is b̌-γ-open need not imply A is $\check{b}_X - \gamma open$ and B is $\check{b}_Y - \gamma open$ and vice versa.

Definition 42. Let $(X, Y, \check{b})$ be a binary Čech closure space. A set (A, B) is generalised b̌-semiopen or g-b̌-semiopen if there exists a b̌-semiopen set (U, V) such that $(U, V) \subseteq (A, B) \subseteq \check{b}(U, V)$.
$(C, D) \in \wp(X) \times \wp(Y)$ is said to be generalised b̌-semiclosed if $(X \setminus C, Y \setminus D)$ is generalised b̌-semiopen.

Proposition 43. Let $(X, Y, \check{b})$ be a binary Čech closure space. $(A, B) \in \wp(X) \times \wp(Y)$ is
b̌-open $\Rightarrow \check{b} - \gamma open \Rightarrow \check{b} - semiopen \Rightarrow g - \check{b} - semiopen$.
The reverse implications need not hold.

Example 44. We observe that in Example 37,
$(\{2, 4\}, b)$ is b̌-γ open, but not b̌-open
$(\{1, 4\}, b)$ is b̌-semiopen, but not b̌-γ open
$(\{1, 2, 3, 4\}, \phi)$ is generalised b̌-semiopen, but not b̌-semiopen.

International Journal of Applied Engineering Research, ISSN 0973-4562 Volume 13, Number 18 (2018) pp. 13589–13595

OPERATIONS ON BINARY ČECH CLOSURE OPERATORS

Definition 45. Let $\check{b_1}$ and $\check{b_2}$ be two binary Čech closure operators from X to Y. Then $(\check{b_1} \cup \check{b_2})(A, B) = \check{b_1}(A, B) \cup \check{b_2}(A, B)$ and $(\check{b_1} \circ \check{b_2})(A, B) = \check{b_1}[\check{b_2}(A, B)]$.

Note: In the set of all binary Čech closure operators from a set X to a set Y, the operation $\cup$ is associative, commutative and has an identity. The operation $\circ$ is associative and has an identity.

Hence it becomes a monoid with respect to these operations.

Proposition 46. If $\check{b_1}$ and $\check{b_2}$ are two binary Čech closure operators from X to Y and $(A, B) \in \wp(X) \times \wp(Y)$. Then $(\check{b_1} \cup \check{b_2})(A, B) \subseteq (\check{b_1} \circ \check{b_2})(A, B)$.

Proof. Let $(A, B) \in \wp(X) \times \wp(Y)$. $(A, B) \subseteq \check{b_2}(A, B)$
$\check{b_1}(A, B) \subseteq \check{b_1}[\check{b_2}(A, B)]$
Also $\check{b_2}(A, B) \subseteq \check{b_1}[\check{b_2}(A, B)]$
Hence $\check{b_1}(A, B) \cup \check{b_2}(A, B) \subseteq \check{b_1}[\check{b_2}(A, B)]$. i.e. $(\check{b_1} \cup \check{b_2})(A, B) \subseteq (\check{b_1} \circ \check{b_2})(A, B)$. ∎

Proposition 47. Let $(X, Y, \check{b_1})$ and $(X, Y, \check{b_2})$ be two binary Čech closure spaces. Then $M(\check{b_1} \circ \check{b_2}) = M(\check{b_2} \circ \check{b_1}) = M(\check{b_1}) \cap M(\check{b_2}) = M(\check{b_1} \cup \check{b_2})$.

Proof. Let $(A, B) \in M(\check{b_1} \circ \check{b_2})$. Then $(\check{b_1} \circ \check{b_2})(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$. i.e.$\check{b_1}[\check{b_2}((X \setminus A, Y \setminus B))] = (X \setminus A, Y \setminus B) \subseteq \check{b_2}((X \setminus A, Y \setminus B))$ by the property of $\check{b_2}$.
Now by the property of $\check{b_1}$, $\check{b_2}((X \setminus A, Y \setminus B))$ is $\check{b_1}-$closed.
i.e.$\check{b_1}[\check{b_2}((X \setminus A, Y \setminus B))] = \check{b_2}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$
Then (A, B) is $\check{b_2}$-open.
$\check{b_1}(X \setminus A, Y \setminus B) = \check{b_1}[\check{b_2}((X \setminus A, Y \setminus B))] = (\check{b_1} \circ \check{b_2})(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$
Then (A, B) is $\check{b_1}$-open.
Thus we get $(A, B) \in M(\check{b_1}) \cap M(\check{b_2})$.
i.e. $M(\check{b_1} \circ \check{b_2}) \subseteq M(\check{b_1}) \cap M(\check{b_2})$ and similarly $M(\check{b_2} \circ \check{b_1}) \subseteq M(\check{b_1}) \cap M(\check{b_2})$ Conversely let $(A, B) \in M(\check{b_1}) \cap M(\check{b_2})$.
Then $(A, B) \in M(\check{b_1})$ and $(A, B) \in M(\check{b_2})$
$\check{b_1}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ and
$\check{b_2}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ $(\check{b_1} \circ \check{b_2})(X \setminus A, Y \setminus B) = \check{b_1}[\check{b_2}(X \setminus A, Y \setminus B)] = \check{b_1}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B) = \check{b_2}(X \setminus A, Y \setminus B) = \check{b_2}[\check{b_1}(X \setminus A, Y \setminus B)] = (\check{b_2} \circ \check{b_1})(X \setminus A, Y \setminus B)$. Thus $M(\check{b_1}) \cap M(\check{b_2}) \subseteq M(\check{b_1} \circ \check{b_2})$ and $M(\check{b_1}) \cap M(\check{b_2}) \subseteq M(\check{b_2} \circ \check{b_1})$.
Now suppose that $(A, B) \subseteq M(\check{b_1} \cup \check{b_2})$.
$(\check{b_1} \cup \check{b_2})(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B) = \check{b_1}(X \setminus A, Y \setminus B) \cup \check{b_2}(X \setminus A, Y \setminus B)$
if and only if $\check{b_1}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ and $\check{b_2}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$
i.e. (A, B) is $\check{b_1} - open$ and $\check{b_2} - open$.
Thus $(A, B) \in M(\check{b_1}) \cap M(\check{b_2})$ i.e. $M(\check{b_1} \cup \check{b_2}) \subseteq M(\check{b_1}) \cap M(\check{b_2})$. ∎

Remark 48. Let $\check{b_1}$ and $\check{b_2}$ be two BČCO. Then $\check{b_1} \cap \check{b_2}$ is a binary closure operator and it need not be a BČCO.

Proof. Let $\check{b} = \check{b_1} \cap \check{b_2}$.
i.e. $\check{b}(A, B) = \check{b_1}(A, B) \cap \check{b_2}(A, B)$, $\forall\, (A, B) \in \wp(X) \times \wp(Y)$
$\check{b}(\phi, \phi) = (\phi, \phi)$ since $\check{b_1}(\phi, \phi) = (\phi, \phi)$ and $\check{b_2}(\phi, \phi) = (\phi, \phi)$
$(A, B) \subseteq \check{b_1}(A, B)$ and $(A, B) \subseteq \check{b_2}(A, B)$. So $(A, B) \subseteq \check{b}(A, B)$
$(A, B) \subseteq (C, D) \Rightarrow \check{b_1}(A, B) \subseteq \check{b_1}(C, D)$ and $\check{b_2}(A, B) \subseteq \check{b_2}(C, D)$
$\Rightarrow \check{b_1}(A, B) \cap \check{b_2}(A, B) \subseteq \check{b_1}(C, D) \cap \check{b_2}(C, D)$.
i.e. $\check{b}(A, B) \subseteq \check{b}(C, D)$
Thus b̌ is a Binary Closure Operator.
But

$$\begin{aligned}
&\check{b}[(A, B) \cup (C, D)] \\
&= \check{b_1}[(A, B) \cup (C, D)] \cap \check{b_2}[(A, B) \cup (C, D)] \\
&= [\check{b_1}(A, B) \cup \check{b_1}(C, D)] \cap [\check{b_2}(A, B) \cup \check{b_2}(C, D)] \\
&= [\check{b_1}(A, B) \cap \check{b_2}(A, B)] \cup [\check{b_1}(C, D) \cap \check{b_2}(A, B)] \\
&\qquad \cup [\check{b_1}(A, B) \cap \check{b_2}(C, D)] \cup [\check{b_1}(C, D) \cap \check{b_2}(C, D)] \\
&\supseteq \check{b}(A, B) \cup \check{b}(C, D).
\end{aligned}$$

Thus $\check{b}[(A, B) \cup (C, D)]$ need not be equal to $\check{b}(A, B) \cup \check{b}(C, D)$. ∎

Acknowledgement

The author is indebted to the University Grants Commission as the work is under the Faculty Development Programme of UGC (XII plan).

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