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Weak Separation Properties as Hereditary Properties

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Abstract

The main focus of present research article is to list selective separation and weak separation properties of the bitopological spaces and to discuss that which of these weak separation properties are hereditary.

Keywords: Bitopological spaces, separation properties, weak separation properties.

INTRODUCTION AND PRELIMINARIES

In the year1963, the theory of bispaces came into existence with the introduction of a new concept of bitopological spaces by J. C. Kelly [1]. Kelly [1] used this new concept of bitopological spaces to study non-symmetric functions that introduce two arbitrary topologies on X. In the same research work, the concepts of selective separation properties of topological space are generalized to bitopological spaces in the form of pairwise Hausdorff, pairwise regular and pairwise normal. Further work related to compactness, connectedness, total disconnectedness and more detailed separation properties and weak separation properties of bitopological spaces is carried out by Kim [2], Fletcher [3] et al., Patty [4], Pervin [5], Saegrove [6] and many other researchers. In present research study, my focus is on hereditary of selective separation properties of bitopological spaces and to examine which of these separation properties are preserved under pair homeomorphism.

A bitopological space is denoted by a triplet (X, τ_1, τ_2), here τ_1, τ_2 are any two topologies on X. For any subset A of (X, τ_1, τ_2), τ_1–cl(A) and τ_2–cl(A) denote closure of A with respect to τ_1 and τ_2 respectively. Further, τ_1–open (τ_1–closed) and τ_2–open (τ_2–closed) will be used to denote open (closed) set in a bitopological space (X, τ_1, τ_2), with respect to τ_1 and τ_2 respectively. Any subset in a bitopological space (X, τ_1, τ_2) is termed as pairwise open (closed) if and only if the set is open (closed) with respect to τ_1 and τ_2.

Definition 1 [7]. (X, τ_1, τ_2) is said to be weak pairwise T_1 if and only if for each pair of distinct points *x*, *y*, there exists a τ_1-open set U and a τ_2-open set V such that either $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$ or $x\in V$, $y\notin V$ and $y\in U$, $x\notin U$.

Definition 2 [7]. (X, τ_1, τ_2) is said to be pairwise T_1 if and only if for each pair of distinct points *x*, *y*, there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$.

Definition 3 [8]. (X, τ_1, τ_2) is said to be weak pairwise T_2 if and only if for each pair of distinct points *x*, *y*, there exists a τ_1-open set U and a τ_2-open set V such that either $x\in U$, $y\in V$ or $x\in V$, $y\in U$ and $U\cap V=\phi$.

Definition 4 [1]. In a bitopological space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if for each point $x\in X$ and for each τ_1 -closed set A such that $x\notin A$, there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $A\subseteq V$ with $U\cap V=\phi$.

Definition 5 [1]. Bitopological space (X, τ_1, τ_2) is said to be pairwise regular if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Definition 6 [6] Bitopological space (X, τ_1, τ_2) is said to be weak pairwise T_3 if and only if it is pairwise regular and weak pairwise T_1.

Definition 7 [6] A function f from (X, τ_1, τ_2) into (X, τ'_1, τ'_2) is pair continuous if and only if the induced functions f from (X, τ_1) into (Y, τ'_1) and (X, τ_2) into (Y, τ'_2) are pair continuous.

Definition 8 [6]. A bitopological space (X, τ_1, τ_2) is pairwise completely regular if and only if for each for each τ_1 -closed set A and for each point $x\notin A$, there exists a pair continuous function $f : (X, \tau_1, \tau_2)\rightarrow([0, 1], R, L)$ such that $f(x)=1$ and $f(A)=\{0\}$, and for each τ_2 -closed set B and for each point $y\notin B$, there exists a pair continuous function $g : (X, \tau_1, \tau_2)\rightarrow([0, 1], R, L)$ such that $g(y)=0$ and $f(B)=\{1\}$.

Definition 9 [6] Bitopological space (X, τ_1, τ_2) is said to be weak pairwise $T_{3\frac{1}{2}}$ and only if it is pairwise completely regular and weak pairwise T_1.

Definition 10 [1]. Bitopological space (X, τ_1, τ_2) is said to be pairwise normal if and only if for each τ_1-closed set A and τ_2-closed set B disjoint from A, there exists a τ_1-open set U and a τ_2-open set V such that $A\subseteq V$, $B\subseteq U$ and $U\cap V=\phi$.

Definition11 [6] Bitopological space (X, τ_1, τ_2) is said to be weak pairwise T_4 if and only if it is pairwise pairwise normal and weak pairwise T_1.

Definition 12 [9] If $Y\subseteq X$, then the collections $\tau^Y_1 = \{A\cap Y: A\in \tau_1\}$ and $\tau^Y_2 = \{B\cap Y: B\in \tau_2\}$ are the relative topologies on *Y*.

A bitopological space (Y, τ^Y_1, τ^Y_2) is then called a subspace of (X, τ_1, τ_2). Moreover, Y is said to be pairwise closed subspace of X if Y is both τ^Y_1 -closed and τ^Y_2 -closed in X. The pairwise open subspace is defined in the similar way.

SEPARATION PROPERTIES AS HEREDITARY PROPERTIES

Theorem 1. Bitopological space (X, τ_1, τ_2) is pairwise T_1 if and only if every singleton subset is pairwise closed.

Proof. Suppose that every singleton subset of the bitopological space is pairwise closed. Let x and y are any two distinct members of X. Now $x\notin\{y\}\Rightarrow x\notin \tau_1\text{-cl}\{y\}\Rightarrow$ there exists τ_1-open set U containing x such that $\{y\}\cap U=\phi$, i.e., $y\notin U$. Again, $y\notin\{x\}\Rightarrow y\notin \tau_2\text{-cl}\{x\}\Rightarrow$ there exists τ_2-open set V containing y such that $\{x\}\cap V=\phi$, i.e., $x\notin V$. Thus, Bitopological space (X,τ_1,τ_2) is pairwise T_1.

Conversely, assume that Bitopological space (X, τ_1, τ_2) is pairwise T_1. If possible, let $\{x\}$ is not pairwise closed. Let $\tau_2\text{-cl}\{x\}\neq\{x\}$ (say), i.e., $\tau_2\text{-cl}\{x\}\not\subset\{x\}$. Therefore, there exists $y\in \tau_2\text{-cl}\{x\}$ such that $y\notin\{x\}$, i.e., $x\neq y$. Also, there a τ_2-open set V containing y such that $\{x\}\cap V\neq\phi \Rightarrow x\in V$. It is contradiction with given hypothesis. Thus, $\{x\}$ is τ_2-closed. Following similar steps, it can be proved that $\{x\}$ is τ_1-closed. It completes required proof.

Theorem 2. In a bitopological space (X, τ_1, τ_2), the property of being weak pairwise T_1 is hereditary.

Proof. To show that every subspace (Y,τ^Y_1, τ^Y_2) of a weak pairwise T_1 bitopological space

(X, τ_1, τ_2) is also weak pairwise T_1. Let x, y are any two distinct members of Y, as $Y\subseteq X$ and

(X, τ_1, τ_2) is weak pairwise T_1, therefore there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$ or $x\in V$, $y\notin V$ and $y\in U$, $x\notin U$. Since, U is a τ_1-open set and V is a τ_2-open set therefore, $U\cap Y$ is a τ^Y_1-open set and $V\cap Y$ is a τ^Y_2-open set. Further, $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$ or $x\in V$, $y\notin V$ and $y\in U$, $x\notin U \Rightarrow x\in U\cap Y$, $y\notin U\cap Y$ and $y\in V\cap Y$, $x\notin V\cap Y$ or $x\in V\cap Y$, $y\notin V\cap Y$ and $y\in U\cap Y$, $x\notin U\cap Y$. Thus, subspace (X, τ_1, τ_2) is also weak pairwise T_1.

Theorem 3. In a bitopological space (X, τ_1, τ_2), the property of being weak pairwise T_2 is hereditary.

Proof. To show that every subspace (X, τ_1, τ_2) of a weak pairwise T_2 bitopological space

(X, τ_1, τ_2) is also weak pairwise T_2. Let x, y are any two distinct members of Y, as $Y\subseteq X$ and (X, τ_1, τ_2) is weak pairwise T_2, therefore there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $y\in V$ or $x\in V$, $y\in U$ and $U\cap V=\phi$.. Since, U is a τ_1-open set and V is a τ_2-open set therefore, $U\cap Y$ is a τ^Y_1-open set and $V\cap Y$ is a τ^Y_2-open set. Evidently, $x\in U\cap Y$, $y\in V\cap Y$ or $x\in V\cap Y$, $y\in U\cap Y$ and $(U\cap Y)\cap(V\cap Y)=(U\cap V)\cap Y=\phi$. Thus, subspace (Y,τ^Y_1, τ^Y_2) is also weak pairwise T_2.

Theorem 4. In a bitopological space (X, τ_1, τ_2), the property of being pairwise regular is hereditary.

Proof. To show that every subspace (Y,τ^Y_1, τ^Y_2) of a pairwise regular bitopological space

(X, τ_1, τ_2) is also pairwise regular. For any $y\in$ Y and for each τ^Y_1-closed set A' such that $y\notin A'$. Evidently, $A'=A\cap Y$, here A is a τ_1-closed set. Since, A is a τ_1-closed set not containing y and τ_1 is regular with respect to τ_2 , therefore there exists a τ_1-open set U and a τ_2-open set V such that $y\in U$, $A \subseteq V$ with $U\cap V=\phi$. Clearly, $U\cap Y$ is a τ^Y_1-open set containing y and $V\cap Y$ is a τ^Y_2-open set containing $A'=A\cap Y$ such that $(U\cap Y)\cap(V\cap Y) =\phi$. Consequently, τ^Y_1 is regular with respect τ^Y_2 and vice-versa. Hence, required result follows.

Theorem 5. In a bitopological space (X, τ_1, τ_2), the property of being weak pairwise T_3 is hereditary.

Proof. Proof immediately follows from theorem 2 and theorem 4.

Theorem 6. Every pairwise completely regular bitopological space is pairwise regular.

Proof. Let A is any τ_1-closed set not containing arbitrary $x\in X$. As (X,τ_1,τ_2) is pairwise completely regular bitopological, therefore there exists a pair continuous function

$f : (X, \tau_1 , \tau_2)\rightarrow([0, 1], R, L)$ such that $f(x)=1$ and $f(A)=\{0\}$. Since, $(-\infty,\frac{1}{2})$ is L-open in $(\mathbb{R}, R,L)$ and $(\frac{1}{2},\infty)$ is R-open $(\mathbb{R}, R,L)$, so $[0,\frac{1}{2})$ is L-open in $([0, 1], R, L)$ and $(\frac{1}{2},1]$ is R-open $([0,1], R,L)$. Evidently, $U=f^{-1}(\frac{1}{2},1]$ is a τ_1-open set and $V= f^{-1}[0,\frac{1}{2})$ a τ_2-open set. Now, $f(x)=1\in(\frac{1}{2},1]$, i.e., $x\in f^{-1}(\frac{1}{2},1]=U$ and for any $a\in A$ $f(a)= 0\in[0,\frac{1}{2})$, i.e., $a\in f^{-1}[0,\frac{1}{2})= V$ it means $A\subseteq V$. Thus, τ_1 is regular with respect to τ_2. Similarly, it can be proved that τ_2 is regular with respect to τ_1. (X,τ_1,τ_2) This completes required proof.

Remark 1. Above theorem clearly indicates that weak pairwise $T_{3\frac{1}{2}}$ bitopological space is also weak pairwise T_3.

Theorem 7. Every pairwise T_4 bitopological space (X,τ_1,τ_2) is pairwise $T_{3\frac{1}{2}}$ and hence pairwise completely regular.

Proof. Here (X,τ_1,τ_2) is pairwise normal and T_1. It is sufficient to prove that (X,τ_1,τ_2) is pairwise completely regular. Consider any τ_1-closed set A not containing arbitrary $x\in X$. Now A is a τ_1-closed set and $\{x\}$, being singleton set in pairwise T_2, is a τ_2-closed set with $A\cap\{x\}=\phi$. Therefore, by generalization of Urysohn's lemma [1] there exists a pair continuous function $f : (X, \tau_1 , \tau_2)\rightarrow([0, 1], R, L)$ such that f(A)={0} and f({x})={1}. Evidently, f(x)=1 and f(A)={0}. Hence, (X,τ_1 ,τ_2) is pairwise $T_{3\frac{1}{2}}$.

Theorem 8. In a bitopological space (X, τ_1, τ_2), the property of being pairwise completely regular is hereditary.

Proof. It is required to prove that every subspace (Y,$\tau^Y{}_1$, $\tau^Y{}_2$) of a pairwise completely regular bitopological space (X, τ_1, τ_2) is also pairwise completely regular. Let A' is any

$\tau^Y{}_1$-closed set not containing arbitrary $y\in$Y. Then, A'=A∩Y, where A is a τ_1-closed set. Now, A is a τ_1-closed set not containing $y\in$X and (X, τ_1, τ_2) is pairwise completely regular.

Therefore, there exists a pair continuous function f : (X, τ_1 , τ_2)→([0, 1], R, L) such that f(y)=1 and f(A)={0}. Consider a new map h:(Y,$\tau^Y{}_1$, $\tau^Y{}_2$) →([0, 1], R, L) defined by h(x)=f(x), for each $x\in$Y. Evidently, g is pair continuous, also $y\in$Y we have h(y)=f(y)=1. Further, for each a'∈A'= A∩Y, we have h(a')=f(a')=0, therefore h(A')= {0}. Consequently, for each $\tau^Y{}_1$-closed set A' not containing arbitrary $y\in$Y, it is possible to find a pair continuous map h:(Y,$\tau^Y{}_1$, $\tau^Y{}_2$) →([0, 1], R, L) satisfying h(y)=1 and h(A')= {0}. Thus, (Y,$\tau^Y{}_1$, $\tau^Y{}_2$) is pairwise completely regular.

Theorem 9. In a bitopological space (X, τ_1, τ_2), the property of being weak pairwise $T_{3\frac{1}{2}}$ is hereditary.

Proof. Proof immediately follows from theorem 2 and theorem 7.

Theorem 10. Every pairwise closed subspace of a pairwise normal bitopological space

(X,τ_1 ,τ_2) is pairwise normal.

Proof. Let (Y, $\tau^Y{}_1$, $\tau^Y{}_2$*)* be a subspace of a pairwise normal bitopological space (X, τ_1, τ_2), here Y is a pairwise closed subset of (X, τ_1, τ_2). Consider any $\tau^Y{}_1$-closed set A and any $\tau^Y{}_2$-closed set B disjoint from A. Now A=A_1∩Y and B=B_1∩Y, A_1 is τ_1-closed set and B_1 is τ_2-closed set. As Y is pairwise closed in (X, τ_1, τ_2) therefore, A is τ_1-closed set and B is τ_2-closed set. Since, (X, τ_1, τ_2) is pairwise normal therefore, there exists a τ_1-open set U and a τ_2-open set V such that A⊆ V, B⊆ U and U∩V= ϕ. Clearly, A⊆ V∩Y, B⊆ U∩Y and (U∩Y)∩(V∩Y)=(U∩V) ∩Y= ϕ. Evidently, U∩Y is $\tau^Y{}_1$-open and V∩Y is $\tau^Y{}_2$ -open. Hence, subspace (Y, $\tau^Y{}_1$, $\tau^Y{}_2$) is also pairwise normal.

Theorem 11. In a bitopological space (X, τ_1, τ_2), the property of being weak pairwise T_4 is hereditary.

Proof. Let (Y, $\tau^Y{}_1$, $\tau^Y{}_2$) be a subspace of a weak pairwise T_4 bitopological space (X, τ_1, τ_2). Evidently, (Y, $\tau^Y{}_1$, $\tau^Y{}_2$) is weak pairwise T_1, therefore it is left to establish that (Y, $\tau^Y{}_1$, $\tau^Y{}_2$) is pairwise normal. Let A is any $\tau^Y{}_1$-closed set and B is any $\tau^Y{}_2$-closed set with A∩B=ϕ. Let $x\in$A and $y\in$B be arbitrary. As singleton set in a bitopological space are pairwise closed so, choose {x} to be τ_1-closed and {y} to be τ_2-closed. Since, {x}∩{y}=ϕ and (X, τ_1, τ_2) is pairwise normal. Therefore, there exists a τ_1-open set U_x and a τ_2-open set V_y such that {x}⊆V_y , {y}⊆ U_x and U_x∩V_y =ϕ. Set U=∪{ U_x : $x\in$A}, a τ_1-open set and V=∪{ V_y : $y\in$B}, a τ_2-open set with U∩V=ϕ. It can be readily concluded that A⊆ V∩Y, B⊆ U∩Y and (U∩Y)∩(V∩Y)=(U∩V) ∩Y= ϕ. Evidently, U∩Y is $\tau^Y{}_1$-open and V∩Y is $\tau^Y{}_2$ -open. Therefore subspace (Y, $\tau^Y{}_1$, $\tau^Y{}_2$) is also pairwise normal and hence it is weak pairwise T_4.

CONCLUSION

Few results analogues well-known results of the classical topology are generalized for bitopological spaces. Furthermore, these results are used to establish that weak separation properties/separation properties: weak pairwise T_1, weak pairwise T_2, pairwise regular, pairwise completely regular, weak pairwise $T_{3\frac{1}{2}}$, pairwise normal and weak paiwise T_4 are hereditary properties.

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