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On β^*-Convergence Nets and Filters

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Abstract

In this paper we introduce and study some topological properties of β^*-convergence and β^*-cluster points of net and filter by using the concept of β^*-open sets.

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1. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms, compactness, connectedness etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of β^*-open sets was introduced by Mubarki et. al. in 2014. In this paper we introduce and study some topological properties of β^*-convergence and β^*-cluster points of net and filter by using the concept of β^*-open sets.Through out this paper(X,τ) and (Y,σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned.

2. PRELIMINARIES

For a subset A of a topological space (X, τ), Cl(A) and Int(A) denote the closure of A and the interior of A, respectively.

Definition 2.1.[2] The δ-closure of A, denoted by $Cl_\delta(A)$, is defined to be the set of all $x \in X$ such that A∩Int(Cl(U))≠∅ for every open neighbourhood U of X. If A=$Cl_\delta(A)$, then A is called δ-closed. The complement of a δ-closed set is called a δ-open set. The δ-interior of A is defined by the union of all δ-open sets contained in A and is denoted by $Int_\delta(A)$.

Definition 2.2.[1]A subset S of a topological space (X ,τ) is said to be β^*-open if $S \subset Int(Cl(Int(S))) \cup Int(Cl_\delta(S))$. The complement of a β^*-closed set is called a β^*-open set. The family of all β^*-open (β^*-closed) subsets of (X ,τ) is denoted by β^* O(X)(β^* C(X)).

***Definition 2.3.**[1]* The intersection of all β^*-closed sets containing A⊂X is called the β^*-closure of A and is denoted by β^* Cl(A). The union of all β^*- open sets contained in A⊂ X is called the β^*-interior of A and is denoted by β^* Int(A).

Definition2.4.[1]A subset $M(x)$ of a topological space (X ,τ) is called a β^*-neighbourhood of a point $x \in X$ if there exists a β^*-open set S such that $x \in S \subset M(x)$.

Lemma 2.5.[1] Let (X, τ)be a topological space and A a subset of X.

Then

(1) β^* Int(A) is β^*-open;

(2) β^* Cl(A) is β^*-closed;

(3) A is β^*-open if and only if A = β^* Int(A);

(4) A is β^*-closed if and only if A = β^* Cl(A);

(5) β^* Int(X\A)=X\ β^* Cl(A);

(6) β^* Cl(X\A)=X\ β^* Int(A).

Lemma 2.6.[1] Let (X, τ) be a topological space and A⊂X. A point $x \in \beta^* Cl(A)$ if and only if $U \cap A \neq \phi$ for every $U \in \beta^* O(X,x)$.

Definition2.7.[1]A subset B_x of a topological space(X ,τ) is said to be β^*-neighbourhood of a point $x \in X$ if there exists a β^*-open set U such that $x \in U \subset B_x$. The family of all β^*- neighbourhoods of a point $x \in X$ is denoted by $N_{\beta^*}(x)$.

Definition 2.8. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be β^*- irresolute if $f^{-1}(V)$ is β^*-closed (resp. β^*-open) in X for every β^*- closed (resp. β^*-open) subset V of Y .

Definition2.9. A topological space (X ,τ) is said to be β^*-T2[1] if for each pair of distinct points x and y in X , there exist disjoint β^*-open sets U and V in X such that $x\in U$ and $x\in V$.

3. PROPERTIES OF β^*-CONVERGENCE NETS AND FILTERS

Definition 3.1. Let $\{x_d\}_{d\in D}$ be a net in a topological space (X, τ).Then $\{x_d\}_{d\in D}$ β^*-converges to $x\in X$ (written $x_d \xrightarrow{\beta^*} x$) if for each β^*-neighborhood U of x, there is some d0 ∈ D such that d ≥ d0 implies $x_d\in U$ .It is clear that $x_d \xrightarrow{\beta^*} x$ if and only if for each β^*-neighborhood of x contains a tail of $\{x_d\}_{d\in D}$. This is sometimes said $\{x_d\}_{d\in D}$ β^*-converges to x if and only if it is eventually in every β^* - neighborhood of x. The point x is called a β^*-limit point of $\{x_d\}_{d\in D}$.

Definition 3.2. Let $\{x_d\}_{d\in D}$ be a net in a topological space (X ,τ) .Then $\{x_d\}_{d\in D}$ is said to have $x\in X$ as a β^*-cluster point (written $x_d \overset{\beta^*}{\alpha} x$) if for each β^*-neighborhood U of x and for each d ∈ D, there is some d0 ≥ d such that x_{d_0} ∈U. This is sometimes said $\{x_d\}_{d\in D}$ has x as a β^*-cluster point if $\{x_d\}_{d\in D}$ is frequently in every β^*-neighborhood of x .

Theorem 3.3. Let (X, τ)be a topological space, $\{x_d\}_{d\in D}$ a net in X and $x\in X$. Then we have the following

(1) if $x_d \xrightarrow{\beta^*} x$,then $x_d \overset{\beta^*}{\alpha} x$

(2) if $x_d \xrightarrow{\beta^*} x$ ($x_d \overset{\beta^*}{\alpha} x$) ,then $x_d\to x$ ($x_d\,\alpha\, x$), respectively.

The following examples show that the converses of Theorem 3.3 are not true in general.

Example3.4. Let($\Re$, μ) be the usual topological space where $\Re$ be the set of all real numbers, then the net $(s_n)_{n\in N}=(n+(-1)^n n)_{n\in N}$ in $\Re$ has 0 as a β^*-cluster point but not a β^*-limit point. Since if U is a β^*-neighborhood of 0 in $\Re$, then for each $n\in N$,either n is odd or even. If n is odd ,then $n_0=n\Rightarrow S_{n0}=0\in U$ and if n is even, then $n_0=n+1\Rightarrow S_{n0}=0\in U$, thus $s_n \overset{\beta^*}{\alpha} 0$ .But s_n does not β^*- converge to 0, since U=(−1,1)is a β^*-neighborhood of 0 and $s_n\notin(-1,1),\forall n\in N_e$.

Example 3.5. Let (N,I) be the indiscrete topological space where N be the set of all natural numbers and $(s_n)_{n\in N}=(n)_{n\in N}$ be a net in N. Observe that $s_n\to 1(s_n\alpha 1)$ .But s_n does not β^*- converge to 1(does not β^*-cluster to 1), since {1} is a β^*-neighborhood of 1 and $s_n\notin\{1\}$,∀n >1.

Theorem3.6. Let A be a subset of a topological space (X ,τ) and $x\in X$. Then $x\in\beta^* Cl(A)$ if and only if there exists a net $\{x_d\}_{d\in D}$ in A such that $x_d \overset{\beta^*}{\alpha} x$.

Proof. Assume that there exists a net $\{x_d\}_{d\in D}$ in A such that $x_d \overset{\beta^*}{\alpha} x$ .Let U be a β^*-open subset of X containing x . Since $x_d \overset{\beta^*}{\alpha} x$, then by Definition 3.2 for each d ∈ D there exists d0∈ D such that x_{d_0} ∈ U for all d0 ≥ d. But $x_d\in A$ for all d ∈D implies $U\cap A\neq\phi$;hence $x\in\beta^* Cl(A)$ .Conversely, suppose $x\in\beta^* Cl(A)$ and let D be a set which is defined by D={(O, U):O is a β^*-open subset of X, $x\in O$ and U is open set in X, $x\in U$ and O ⊆ U }.Ordered D by(O1, U1) ≤ (O2, U2) ⇔ U2⊆U1. Now we shall prove D is a directed set. Since U ⊆U , (O, U) ≤ (O, U) for every (O, U) ∈D; hence ≤ is reflexive. If (O1, U1) ≤ (O2, U2) and (O2, U2) ≤ (O3, U3), then U2⊆ U1and U3⊆U2. It follows that U3⊆ U1⇒(O1, U1) ≤ (O3, U3); hence ≤ is transitive. Let (O1, U1), (O2, U2) ∈D. Then U1and U2 are open subsets of X and x ∈ U1∩ U2. Since O1⊆U1 and O2⊆U2,O2∩U1⊆U1∩U2,where O2∩U1 is a β^*-open set in X. Since U1∩U2⊆U1,(O1,U1)≤(O2∩U1,U1∩U2) and U1∩U2⊆U2,(O2,U2) ≤(O2∩U1,U1∩U2). Thus D is a directed set. Since $x\in O$ and $x\in U$, x ∈O∩U$\neq\phi$,O∩U is β^*-open in X and $x\in\beta^* Cl(A)$; hence(O∩U)∩A$\neq\phi$ .Define x : D→X by x (O,U)= x_{OU}∈(O ∩ U) ∩ A = O ∩ A ⇒ (x_{OU})OU∈D is a net in A. Now we shall prove that $x_{OU} \overset{\beta^*}{\alpha} x$. Let B be a β^*-open set in X, $x\in B$ and let OU = (O, U) ∈ D ∋ O is β^*-open in X, $x\in O$ and U is open set in X, $x\in U$ ⇒ O ⊆ U ⇒ O ∩ B ⊆ U ∩ B ⊆ U . This implies that there exists (U∩B,U)∈D such that $x_{(U\cap B,U)}$∈B for every(O,U)≤(U∩B,U).Hence $x_{OU} \overset{\beta^*}{\alpha} x$. □

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Theorem3.7. Let A be a subset of a topological space (X ,τ) and $x \in X$. Then $x \in \beta^* Cl(A)$ if and only if there exists a net $\{x_d\}_{d\in D}$ in A such that $x_d \xrightarrow{\beta^*} x$.

Proof. Similar to the proof of Theorem 3.6. □

Theorem3.8. Let $\{x_d\}_{d\in D}$ be a net in a topological space (X,τ) and for each d in D. let A_d be the set of all x_{d_0} for $d_0 \geq d$. Then x is a β^*-cluster point of $\{x_d\}_{d\in D}$ if and only if x belongs to the β^*-closure of A_d for each d in D.

Proof. If x is a β^*-cluster point of $\{x_d\}_{d\in D}$ for each d, A_d intersects each β^*-neighborhood of x as $\{x_d\}_{d\in D}$ is frequently in each β^*-neighborhood. Then x is in the β^*-closure of each A_d .Conversely, if x is not a β^*-cluster point of $\{x_d\}_{d\in D}$, then there is a β^*-neighborhood U of x such that $\{x_d\}_{d\in D}$ is not frequently in U. Hence for some d in D if $d_0 \geq d$, then $x_{d_0} \notin U$, so that U and A_d are disjoint. Consequently x is not in the β^*-closure of A_d. □

Definition3.9.A filter $\mathcal{F}$ on a topological space (X ,τ) is said to be β^*-converges to $x \in X$ (written $\mathcal{F} \xrightarrow{\beta^*} x$) if $N_{\beta^*}(x) \subseteq \mathcal{F}$.

Definition 3.10. Afilter $\mathcal{F}$ on a topological space(X ,τ) has $x \in X$ *as a* β^*-cluster point (written $\mathcal{F} \overset{\beta^*}{\alpha} x$) if for each $F \in \mathcal{F}$ meets each N ∈ $N_{\beta^*}(x)$.

Theorem3.11.A filter $\mathcal{F}$ on a topological space(X ,τ) has $x \in X$ as a β^*-cluster point if and only if $x \in \cap\{\beta^*$ Cl(F):F∈$\mathcal{F}\}$.

Proof. We shall prove $\mathcal{F} \overset{\beta^*}{\alpha} x$ if and only if $x \in \cap\{\beta^*$ Cl(F) : F∈$\mathcal{F}\}$. Then $\mathcal{F} \overset{\beta^*}{\alpha} x$ if and only if for every $N \in N_{\beta^*}(x)$ and for every F∈$\mathcal{F}$,N∩F≠∅.This shows that F∩N≠∅ holds for every N ∈ $N_{\beta^*}(x)$. Hence $x \in \beta^* Cl(F)$ if and only if for every F ∈$\mathcal{F}$; hence $x \in \cap\{\beta^*$ Cl(F):F∈$\mathcal{F}\}$holds for every F∈$\mathcal{F}$. □

Theorem3.12.Let (X,τ) be a topological space and $\mathcal{F}$a filter on X and $x \in X$. Then we have the following

(1). If $\mathcal{F} \xrightarrow{\beta^*} x$, then $\mathcal{F} \overset{\beta^*}{\alpha} x$

(2). If $\mathcal{F} \xrightarrow{\beta^*} x$ ($\mathcal{F} \overset{\beta^*}{\alpha} x$), then $\mathcal{F} \rightarrow x$ ($\mathcal{F} \propto x$), respectively.

(3). If $\mathcal{F} \xrightarrow{\beta^*} x$,then every filter than $\mathcal{F}$ also β^*-converges to x.

Proof. The proof is clear. □

Definition 3.13.A filter base $\mathcal{F}_0$ on a topological space (X, τ) β^*-converges to $x \in X$ ($\mathcal{F}_0 \xrightarrow{\beta^*} x$) if the filter generated by $\mathcal{F}_0$ β^*-converges to x.

Definition 3.14.A filter base $\mathcal{F}_0$ on a topological space (X ,τ) has $x \in X$ as a β^* cluster point ($\mathcal{F}_0 \xrightarrow{\beta^*} x$) if for each $F_0 \in \mathcal{F}_0$ meets each N∈ $N_{\beta^*}(x)$ (if and only if the filter generated by $\mathcal{F}_0$ β^*-clusters at x).

Theorem 3.15.A filter base $\mathcal{F}_0$ on a topological space (X, τ) β^*- converges to $x \in X$ if and only if for each N ∈ $N_{\beta^*}(x)$ there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq$ N.

Proof. If $\mathcal{F} \xrightarrow{\beta^*} x$, then the filter $\mathcal{F}$ generated by $\mathcal{F}_0$ β^*-converges to x, that is , $\mathcal{F} \xrightarrow{\beta^*} x$ .Then $N_{\beta^*}(x) \subseteq \mathcal{F} \Rightarrow$ for every N∈ $N_{\beta^*}(x)$, N∈$\mathcal{F} \Rightarrow$ there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq$N. Conversely, to prove that $\mathcal{F}_0 \xrightarrow{\beta^*} x$,that is, $\mathcal{F}$ generated by $\mathcal{F}_0$ β^*- converges to x. Let N∈ $N_{\beta^*}(x)$, then by hypothesis, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq$ N, since $\mathcal{F}$ is a filter, then N∈$\mathcal{F} \Rightarrow N_{\beta^*}(x) \subset \mathcal{F}$. Hence

$\mathcal{F} \xrightarrow{\beta^*} x \Rightarrow \mathcal{F}_0 \xrightarrow{\beta^*} x$. □

Theorem3.16.A net $\{x_d\}_{d\in D}$ in a topological space (X ,τ) has $x \in X$ as a β^*-cluster point if and only if the filter generated by $\{x_d\}_{d\in D}$ has x as a β^*-cluster point.

Proof. Suppose that $x_d \overset{\beta^*}{\alpha} x$ .Let N∈ $N_{\beta^*}(x)$ and $B_{d_0} \in \mathcal{F}_0$, then for all $d_0 \in D$, there exists d ∈D, d ≥ d_0 such that $x_d \in N$, , but $x_d \in B_{d_0} \Rightarrow N \cap B_{d_0} \neq \phi$. Therefore $\mathcal{F}_0 \overset{\beta^*}{\alpha} x$. hence $\mathcal{F} \overset{\beta^*}{\alpha} x$. Conversely, suppose that a filter generated by $\{x_d\}_{d\in D}$ has x as a β^* - cluster point. Let N∈$N_{\beta^*}(x)$ and $d_0 \in D$, then $N \cap B_{d_0} \neq \phi \Rightarrow \exists x_d \in N$ and $x_d \in B_{d_0}$, d ≥ d_0.Thus $x_d \overset{\beta^*}{\alpha} x$. □

Theorem 3.17. A filter $\mathcal{F}$ on a topological space (X, τ) has $x \in X$ as a β^*-cluster point if and only if the net based on $\mathcal{F}$ has x as a β^*-cluster point.

Proof. Suppose that $\mathcal{F} \overset{\beta^*}{\alpha} x$. Let N∈ $N_{\beta^*}(x)$ and d=(x_0,F_0)∈$D_{\mathcal{F}}$, then N∩F≠∅ for every F ∈ $\mathcal{F}$. Pick $x' \in N \cap F$,let D_0=(x' ,F_0)∈$D_{\mathcal{F}}$. Hence $x_{d_0} = x_{(x',F)} = x' \in N$ for all $d_0 \geq$ d, thus the net based on $\mathcal{F}$ has x as a β^*-cluster point. Conversely, suppose that the net based on $\mathcal{F}$ has x as a β^*-cluster point. Let N ∈ $N_{\beta^*}(x)$ and F ∈$\mathcal{F}$ ⇒for all d = (x_0, F_0) ∈ $D_{\mathcal{F}}$, there exists d_0= (x , F) ∈ $D_{\mathcal{F}}$ ∋ $x_{d_0} = x \in N$ for all $d_0 \geq$ d. But $x \in F$, thus N∩F≠∅.□

Theorem 3.18. A topological space (X, τ) is β^*-T_2 if and only if every β^*-convergent net in X has a unique β^*-limit point.

Proof. Let (X, τ) be a β^*-T_2-space and $(x_d)_{d\in D}$ a net in X such that $x_d \xrightarrow{\beta^*} x$, $x_d \xrightarrow{\beta^*} y$ and $x \neq y$. Since(X ,τ) is a β^*-T_2 –space, there exists U∈ $N_{\beta^*}(x)$ and V∈ $N_{\beta^*}(y)$ such that U∩V=∅.since $x_d \xrightarrow{\beta^*} x$, there exists d_0∈D such that $x_d \in U$, for all d ≥ d_0. Since $x_d \xrightarrow{\beta^*} y$, there exists d_1∈ D such that $x_d \in V$, for all d ≥ d_1. Since D is a directed set and d_0, d_1∈ D, there exists d_2∈D such that $d_2 \geq d_0$ and $d_2 \geq d_1$. Hence $x_d \in U$ for all d ≥ d_2 and $x_d \in V$ for all d ≥ d_2. Then U∩V ≠∅, which is a contradiction. Conversely, suppose that not (X,τ) is not a β^*-T_2-space. Then there exist $x, y \in X$, $x \neq y$ such that for every U∈ $N_{\beta^*}(x)$ and for every $V \in N_{\beta^*}(y)$,U∩V≠∅.Then($N_{\beta^*}(x)$,⊆) and ($N_{\beta^*}(y)$, ⊆) are directed sets by inclusion. Let ρ = $N_{\beta^*}(x) \times N_{\beta^*}(y)$. Define a relation ≥ on ρ as follows: for every (U, V), (W, S) ∈ ρ, we have (U, V) ≥ (W, S) ⇔ U ≥ W and V ≥ S. It is easy to verify that (ρ,≥) is a directed set. Let (U,V)∈ρ ⇒ $x \in U$, $y \in V$ and $U \cap V \neq \phi$ as $U \cap V \neq \phi$. Then there exists $x_{(U,V)}$∈U∩V. Define $x \in \rho \to X$ by $x(U,V) = x_{(U,V)}$ for every (U, V) ∈ ρ. Then ($x_{(U,V)}$)$_{(U,V)\in\rho}$ is a net in X. We will show that ($x_{(U,V)}$)$_{(U,V)\in\rho}$ is β^*-convergent to both x *and* y . For if U ∈ $N_{\beta^*}(x)$ and V ∈ $N_{\beta^*}(y)$, then for each (N,M)∈ ρ such that (N,M)≥(U,V) ,we have $x(N,M) = x_{(N,M)} \in N \cap M \subseteq U \cap V$. Then $x_{(N,M)} \in U$ and $x_{(N,M)} \in V$. Hence $x_{(U,V)} \xrightarrow{\beta^*}$ x and $x_{(U,V)} \xrightarrow{\beta^*}$ y, which is a contradiction. Thus (X, τ) is a β^*-T_2-space. □

Definition3.19. Let (X ,τ) be a topological space and A⊆X. A point $x \in X$ is said to be β^*-limit point of A if and only if every β^*-open set U in X containing x contains a point of A different from x .

Theorem 3.20. Let (X, τ) be a topological space and A ⊆ X. Then

(1) A point $x \in X$ is a β^*-limit point of A if and only if there is a net $(x_d)_{d\in D}$ in A\{x} β^*-converging to x .

(2) A set A is β^*-closed in X if and only if no net in A β^*-converges to a point in $X \setminus A$.

(3) A set A is β^*-open in X if and only if no net in $X \setminus A$ β^*-converges to a point in A.

Proof.(1). Let x be a β^*-limit point of A. We shall prove that there exists a net$(x_d)_{d\in D}$ in A\{x} such that $x_d \xrightarrow{\beta^*} x$. Since x is a β^*- limit point of A, $N \cap A \setminus \{x\} \neq \phi$ holds for every N ∈ $N_{\beta^*}(x)$ .Then $(N_{\beta^*}(x), \subseteq)$ is a directed set by inclusion. Since $N \cap A \setminus \{x\} \neq \phi$ for every N∈ $N_{\beta^*}(x)$ .Then exists $x_N \in N \cap A \setminus \{x\}$. Define $x : N_{\beta^*}(x)$ →A\{x}by $x(N) = x_N$ for every N ∈ $N_{\beta^*}(x)$. Therefore $(x_N)_{N \in N_{\beta^*}(x)}$ is a net in A\{x}. Let N∈ $N_{\beta^*}(x)$ to find d_0∈ D such that $x_d \in N$, for all d ≥ d_0. If d_0= N , then $d = M \in N_{\beta^*}(x)$ for every d ≥ d_0. That is, M ≥N ⇔ M ⊆N . Therefore, x_d= x(d) = x(M) = x_M ∈M ∩ A\{x} ⊆ M ⊆ N ⇒ $x_M \in N$. Hence $x_d \in N$ for all d ≥ d_0. Thus $x_N \xrightarrow{\beta^*} x$. Conversely, suppose that there exists a net$(x_d)_{d\in D}$ in A\{x}such that $x_d \xrightarrow{\beta^*} x$ .Let $U \in N_{\beta^*}(x)$ .Since $x_d \xrightarrow{\beta^*} x$,there exists d_0∈ D such that $x_d \in U$ for all d ≥ d_0.But $x_d \in A\setminus\{x\}$ for all d ∈ D. Then U∩A\{x}≠∅ for every U∈ $N_{\beta^*}(x)$ .Thus x is a β^* - limit point of A.

(2) .Let A be a β^* -closed in X. Now we shall prove that there exists no net in A β^* -converges to a point in $X\setminus A$. Suppose there exists a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{\beta^*} x$ and $x \in X\setminus A$. By Theorem 3.7 $x \in \beta^* Cl(A)$; hence $x \in A$. But $x \in X\setminus A$, then (X\A)∩A≠∅,which is a contradiction. Hence no net in A β^* - converges to a point in X \A. Conversely, suppose that there exists no net in A β^* -converge to a point in $X\setminus A$. Let $x \in \beta^* Cl(A)$. Then by Theorem 3.7 there exists a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{\beta^*} x$. By hypothesis , we get every net in A β^* -converges to a point in A. Then we have for every $x \in A$, we have β^* Cl(A) ⊂ A; hence A is β^* -closed.

(3). By (2), A is β^* -open in X if and only if $X\setminus A$ is β^* - closed in X if and only if no net in $X\setminus A$ β^* -converges to a point in A. □

Remark 3.21. Let $(x_d)_{d\in D}$ be a net in a topological space (X,τ) and $x \in X$. Then

(1) If $x_d \xrightarrow{\beta^*} x$, then every subnet of $(x_d)_{d\in D}$ β^* - converges to x .

(2) If every subnet of $(x_d)_{d\in D}$ has a subnet β^*– convergent to x , then $x_d \xrightarrow{\beta^*} x$.

(3) If $x_d = x$ for all d ∈D, then $x_d \xrightarrow{\beta^*} x$.

Theorem 3.22. A function $f:(X,\tau)\to(Y,\sigma)$ is β^*- irresolute if and only if whenever $(x_d)_{d\in D}$ is a net in X such that $x_d \xrightarrow{\beta^*} x$, then $f(x_d) \xrightarrow{\beta^*} f(x)$.

Proof. Let $V \in N_{\beta^*}(f(x))$ .Since f is β^* -irresolute, there exists U∈ $N_{\beta^*}(x)$ such that $f(U)\subseteq V$. Since $U \in N_{\beta^*}(x)$ and $x_d \xrightarrow{\beta^*} x$, there exists d_0∈ D such that $x_d \in U$ for all d ≥ d_0. Then there exists d_0 ∈ D such that $f(x_d) \in f(U) \subseteq V$ for all d ≥ d_0.Therefore for every $V \in N_{\beta^*}(f(x))$, there exists d_0∈D such that $f(x_d) \in V$ for all d ≥ d_0.Hence $f(x_d) \xrightarrow{\beta^*} f(x)$ .Conversely, suppose that $f:(X,\tau)\to(Y,\sigma)$ is not a β^* -irresolute function. Then there exists $V \in N_{\beta^*}(f(x))$ such that for every $U \in N_{\beta^*}(x)$, $f(U) \not\subseteq V$. Therefore for every $U \in N_{\beta^*}(x)$, there exists $x_U \in U$ such that $f(x_U) \notin V$. Hence $(N_{\beta^*}(x), \subseteq)$ is a directed set by inclusion. Define x : $N_{\beta^*}(x) \to$ X by $x(U) = x_U$ for every $U \in N_{\beta^*}(x)$ .Therefore $(x_U)_{U\in N_{\beta^*}(x)}$ is a net in X. Let $U \in N_{\beta^*}(x)$ to find d_0∈ D such that $x_d \in U$ for all $d \geq d_0$. If d_0= U⇒, then $d = N \in N_{\beta^*}(x)$ holds for all $d \geq d_0$. That is, N ≥ U⇔ N ⊆ U. Therefore x (N) = x_N ∈N ⊆ U⇒ x_N ∈U for all $d \geq d_0$ ⇒ $x_U \xrightarrow{\beta^*} x$ .But $(f(x_U))$ does not β^* -converges to $f(x)$, since $f(x_U) \notin V$ for every $U \in N_{\beta^*}(x)$. This is a contradiction. Thus $f:(X,\tau)\to(Y,\sigma)$ is a β^* - irresolute function. □

Theorem3.23. A filter $\mathcal{F}$ on a topological space(X ,τ) has $x \in X$ as a β^* -cluster point if and only if there is a filter $\mathcal{F}'$ finer than $\mathcal{F}$ which β^* -converges to x .

Proof. If $\mathcal{F} \overset{\beta^*}{\alpha} x$, then by Definition 3.14 each F ∈ $\mathcal{F}$ meets each $N \in N_{\beta^*}(x)$ .Then $\mathcal{F}_0'$ ={N∩F:N∈ $N_{\beta^*}(x)$,F∈$\mathcal{F}$} is a filter base for some filter $\mathcal{F}'$ which is finer than $\mathcal{F}$ and β^* -converges to x. Conversely given $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F}' \xrightarrow{\beta^*} x$ ⇒$\mathcal{F}\subseteq\mathcal{F}'$ and $N_{\beta^*}(x) \subseteq \mathcal{F}'$. Then for every F∈$\mathcal{F}$ and each $N \in N_{\beta^*}(x)$ belong to $\mathcal{F}'$ .Since $\mathcal{F}'$ is a filter , $N \cap F \neq \phi$. Hence $\mathcal{F} \overset{\beta^*}{\alpha} x$. □

Theorem3.24. Let(X ,τ) be a topological space and A⊂X .Then $x \in \beta^* Cl(A)$ if and only if there is a filter $\mathcal{F}$ such that A ∈ $\mathcal{F}$ and $\mathcal{F} \xrightarrow{\beta^*} x$

Proof. If $x \in \beta^* Cl(A)$, then $U \cap A \neq \phi$ for every $U \in N_{\beta^*}(x)$. Hence $\mathcal{F}_0$= {U ∩ A : $U \in N_{\beta^*}(x)$ } is a filter base for some filter $\mathcal{F}$. The resulting filter contains A and $\mathcal{F} \xrightarrow{\beta^*} x$.Conversely, if A∈$\mathcal{F}$ and $\mathcal{F} \xrightarrow{\beta^*} x$, then A ∈ $\mathcal{F}$ and $N_{\beta^*}(x) \subseteq \mathcal{F}$ .Since $\mathcal{F}$ is a filter , $U \cap A \neq \phi$ for every $U \in N_{\beta^*}(x)$. Hence $x \in \beta^* Cl(A)$. □

Definition 3.25. If $f:(X,\tau)\to(Y,\sigma)$ is a function and $\mathcal{F}$ a filter on X, then f ($\mathcal{F}$) is the filter on (Y ,σ) having for a base of the sets { f (F) : F ∈ $\mathcal{F}$ }.

Theorem 3.26. A function $f:(X,\tau)\to(Y,\sigma)$ is β^*-irresolute if and Only if whenever $\mathcal{F} \xrightarrow{\beta^*} x$ in X, then $f(\mathcal{F}) \xrightarrow{\beta^*} f(x)$ in Y.

Proof. Let $V \in N_{\beta^*}(f(x))$. Since f is β^*-irresolute, there exists a β^*-neighborhood U of x such that $f(U) \subseteq V$. Since $\mathcal{F} \xrightarrow{\beta^*} x$,then U ∈ $\mathcal{F}$ ⇒ $f(U) \in f(\mathcal{F})$. But $f(U) \subseteq V$, then V ∈ f ($\mathcal{F}$). Thus $f(\mathcal{F}) \xrightarrow{\beta^*} f(x)$. Conversely, suppose that whenever $\mathcal{F} \xrightarrow{\beta^*} x$ in X ,then $f(\mathcal{F}) \xrightarrow{\beta^*} f(x)$ in Y. Let $\mathcal{F}$={U:U∈ $N_{\beta^*}(x)$ }.Then $\mathcal{F}$ is a filter on X and $\mathcal{F} \xrightarrow{\beta^*} x$ .By hypothesis $f(\mathcal{F}) \xrightarrow{\beta^*} f(x)$ gives for each $V \in N_{\beta^*}(f(x))$ belongs to f ($\mathcal{F}$).Then there exists U∈ $N_{\beta^*}(x)$ such that $f(U) \subseteq V$; hence $f:(X,\tau)\to(Y,\sigma)$ is a β^* -irresolute function. □

Theorem 3.27.Let (X, τ) be a topological space and A ⊆ X. Then a point $x \in X$ is a β^*-limit point of A if and only if $A \setminus \{x\}$ belongs to some filter which β^*-converges to x.

Proof. If x is a β^*-limit point of A, then $U \cap A \setminus \{x\} \neq \phi$ for every $U \in N_{\beta^*}(x)$. Then $\mathcal{F}_0$={U ∩ A\{x} : $U \in N_{\beta^*}(x)$ } is a filter base for some filter $\mathcal{F}$. The resulting filter contains A\{x} and $\mathcal{F} \xrightarrow{\beta^*} x$. Conversely , if $A \setminus \{x\}$ ∈$\mathcal{F}$ and $\mathcal{F} \xrightarrow{\beta^*} x$, $N_{\beta^*}(x)$ ⊂$\mathcal{F}$. Since $\mathcal{F}$ is a filter, $U \cap A \setminus \{x\} \neq \phi$ for every $U \in N_{\beta^*}(x)$ .Thus x is a β^* – limit point of A. □

Definition3.28. If$(x_d)_{d\in D}$ is a net in a topological space (X,τ) ,the filter generated by the filter base $\mathcal{F}_0$ consisting of the sets $B_{d_0} = \{x_d : d \geq d_0\}$,d0∈D is called the filter generated by $(x_d)_{d\in D}$.

Theorem3.29. A net $(x_d)_{d\in D}$ in a topological space (X ,τ) β^*-converges to $x \in X$ if and only if the filter generated by $(x_d)_{d\in D}$ β^*-converges to x.

Proof. The net $(x_d)_{d\in D}$ β^*-converges to x if and only if each β^*-neighborhood of x contains a tail for $(x_d)_{d\in D}$, since the tails of$(x_d)_{d\in D}$ are a base for the filter generated by$(x_d)_{d\in D}$, the result follows. □

Definition 3.30. If $\mathcal{F}$ is a filter on a topological space (X, τ), let $D_{\mathcal{F}}$ = {(x , F) : x ∈ F ∈ $\mathcal{F}$}. Then $D_{\mathcal{F}}$ is directed by the relation $(x_1,F_1) \leq (x_2,F_2)$ if and only if $F_2 \subseteq F_1$, so the function p : $D_{\mathcal{F}}$→X defined by p(x ,F)= x is a net in X. It is called the net based on $\mathcal{F}$.

Theorem3.31. A filter $\mathcal{F}$ on a topological space (X ,τ) β^*-converges to $x \in X$ if and only if the net based on $\mathcal{F}$ β^*-converges to x.

Proof. Suppose that $\mathcal{F} \xrightarrow{\beta^*} x$ .If N∈ $N_{\beta^*}(x)$, then N∈ $\mathcal{F}$. Since N≠∅, there exists p ∈N .Let d_0= (p ,N) ∈ $D_{\mathcal{F}}$. Hence for every d =(q ,F) ≥ d_0=(p,N) we have $x_d = x_{(q,F)} = q \in F \subseteq N$ .Then the net based on $\mathcal{F}$ β^*-converges to x. Conversely, suppose that the net based on $\mathcal{F}$ β^*-converges to x. Let $N \in N_{\beta^*}(x)$, then there exists $d_0 = (p_0,F_0) \in D_{\mathcal{F}}$ such that $x_{(p,F)} = p \in N$ for every $(p,N) \geq (p_0,F_0)$. Then $F_0 \subseteq N$; otherwise, there is some $q \in F_0 \setminus N$, and then $(q, F_0) \geq (p_0,F_0)$, but $x_{(q, F_0)} = q \notin N$. Hence $N \in \mathcal{F}$, so $\mathcal{F} \xrightarrow{\beta^*} x$. □

Theorem3.32. A topological space (X ,τ) is β^*-T_2 if and only if every β^*-convergent filter in X has a unique β^*-limit point.

Proof. Let (X ,τ) be a β^*-T_2– space, $\mathcal{F}$ a filter in X such that $\mathcal{F} \xrightarrow{\beta^*} x$, $\mathcal{F} \xrightarrow{\beta^*} y$ and $x \neq y$ .Since (X ,τ) is a β^*-T_2 –space, there exists $U \in N_{\beta^*}(x)$ and $V \in N_{\beta^*}(y)$ such that $U \cap V = \phi$. Since $\mathcal{F} \xrightarrow{\beta^*} x$ and $\mathcal{F} \xrightarrow{\beta^*} y$, $N_{\beta^*}(x) \subseteq \mathcal{F}$ and $N_{\beta^*}(y) \subseteq \mathcal{F}$. Hence $N_{\beta^*}(y) \subseteq \mathcal{F}$ and $U \in N_{\beta^*}(x) \subseteq \mathcal{F}$ and $V \in N_{\beta^*}(y) \subseteq \mathcal{F}$. This shows that $U,V \in \mathcal{F}$. Since $\mathcal{F}$ is a filter , $U \cap V \neq \phi$.This is a contradiction. Hence $\mathcal{F}$ β^*-converges to a unique β^*- limit point. Conversely, suppose that (X ,τ) is not a β^*-T_2 –space. Then there exists $x, y \in X$, $x \neq y$ such that for every $U \in N_{\beta^*}(x)$ and $V \in N_{\beta^*}(y)$, $U \cap V \neq \phi$.Then $\mathcal{F}_0 = \{U \cap V : U \in N_{\beta^*}(x), V \in N_{\beta^*}(y)\}$ is a filter base for some filter $\mathcal{F}$. The resulting filter β^*-converges to x *and* y. This is a contradiction. Thus (X ,τ) is a β^*-T_2 –space. □

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