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The Accurate Independent Dominating Energy of a Graph

B. Basavanagoud*, **Sujata Timmanaikar** + and **Sunilkumar Hosamani** ++

**Department of Mathematics, Karnatak University, Dharwad, India.*

+Department of Mathematics, Government Engineering College, Haveri, India.

++Department of Mathematics, Rani Channamma University, Belagavi, India.

Abstract

Let $G = (V, E)$ be a graph. An independent dominating set $D_i \subseteq V(G)$ of G is said to be an accurate independent dominating set D_{i_a} such that no $|D_i|$-element subset of $V(G) - D_i$ is an independent dominating set. In this paper, we study a new graph-energy like invariant viz., the accurate independent dominating energy(AIDE) of a graph G. As a chemical application of this new parameter, we show that the predicting power of AIDE is quite better than classical graph energy for enthalpy of vaporization for the set of octane isomers. Next, we obtain bounds for AIDE and finally concluded with some open problems.

Keywords: Accurate independent domination number, Energy, Accurate independent dominating energy.

AMS Subject Classification : 05C69, 05C90.

INTRODUCTION

Let $G = (V, E)$ be a connected, nontrivial, undirected graph without loops and multiple edges. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. By the open neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. By $K_{r,\ s}$ we denote a complete bipartite graph with partite sets of cardinalities r and s. By a star we mean the graph $K_{1,\ n-1}$. For undefined terminologies in this paper, we suggest the interested reader to refer [14].

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \backslash D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set D_i is an independent dominating set if $\langle D_i \rangle$ has no edges. The independent domination number $i(G)$ is the minimum cardinality of an independent dominating set. A dominating set D_a is an accurate dominating set such that no $|D|$-element subset of $V(G) \backslash D_a$ is a dominating set of G. The accurate domination number $\gamma_a(G)$ of G is the cardinality of a smallest accurate dominating set of G [9, 20]. An accurate dominating set is an accurate independent dominating set if $\langle D_a \rangle$ has no edges. The accurate independent domination number $i_a(G)$ is the minimum cardinality of an accurate independent dominating set of G [4]. For a comprehensive survey of domination in graphs, see [15, 16].

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced almost 30 years ago [11] and having a clear connection to chemical problems [13], has in newer times attracted much attention of mathematicians and mathematical chemists [6, 8, 10, 21, 23, 24, 26, 28].

In connection with energy (that is defined in terms of the eigenvalues of the adjacency matrix), energy-like quantities were considered also for the other matrices: Laplacian [13], distance [17], incidence [18], minimum covering energy [1] etc. Recall that a great variety of matrices has so far been associated with graphs [2, 3, 8, 27]. In this paper we study a new matrix, called the accurate independent dominating matrix(AIDM) of a graph, its eigenvalues and energy.

The accurate independent dominating matrix(AIDM) of G is the $n \times n$ matrix defined by $A_{i_a}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & if\ v_i v_j \in E\ ; \\ 1, & if\ \ i = j\ and\ v_i \in D_{i_a}; \\ 0, & otherwise. \end{cases}$$

The characteristic polynomial of $A_{i_a}(G)$ is denoted by $\Phi_n(G, \mu) = det(\mu I - A_{i_a}(G))$.

The accurate independent dominating eigenvalues of a graph G are the eigenvalues of $A_{i_a}(G)$. Since $A_{i_a}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The accurate independent dominating energy of G is then defined as

$$E_{i_a}(G) = \sum_{i=1}^{n} |\mu_i|.$$

ON CHEMICAL APPLICABILITY OF THE $E_{i_a}(G)$

The productivity of $E_{i_a}(G)$ was tested using a data set of octane isomers, found at http://www.moleculardescriptors.eu /dataset.htm. The octane data set consists of the following data: boiling point, melting point, heat capacities, molar refraction, acentric factor, total surface area, octanol-water partition coefficient and entropy. The $E_{i_a}(G)$ was correlated with each of these properties and surprisingly, we can see that the $E_{i_a}(G)$ has a good correlation with the enthalpy of vaporization of octane isomers.

The following structure-property relationship model has been developed for the $E_{i_a}(G)$.

$$HVAP = [23.010(\pm1.973) - 1.640(\pm0.311)]E_{i_a} \quad (1)$$

The correlation coefficient of Accurate independent dominating energy is $r = 0.896$ whereas the correlation coefficient with classical graph energy is $r = 0.287$.

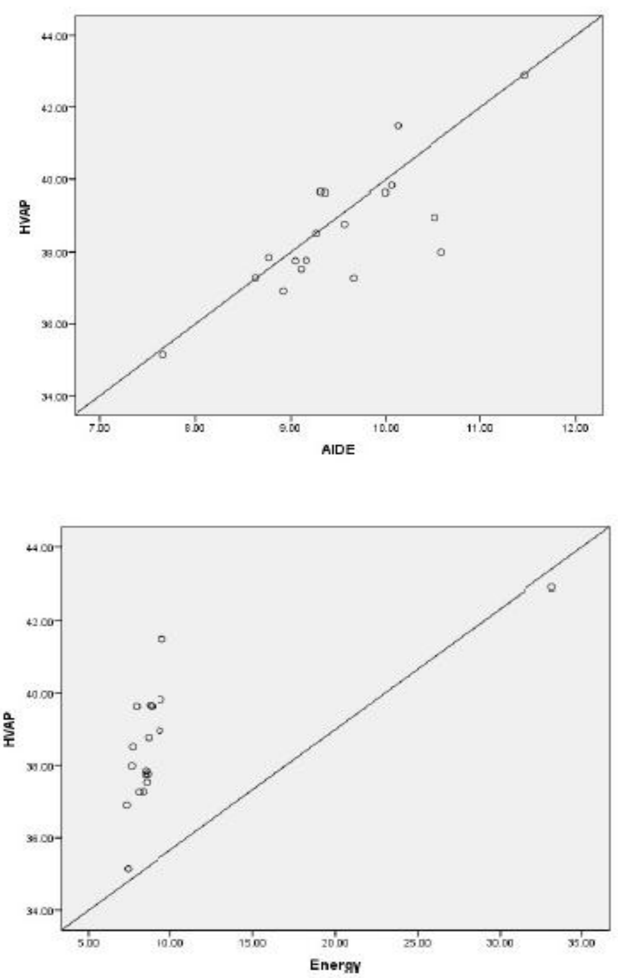

Figure 1. Correlation of AIDE with enthalpy of vaporization of octane isomers and correlation of E(G) with enthalpy of vaporization of octane isomers

RESULTS

We begin with the following straightforward observations.

Observation 1. 1 *Trace of* $A_{i_a}(G) = i_a(G)$.

Observation 2. 2 *Suppose* $a_0\mu^n + a_1\mu^{n-1} + \cdots + a_n$ *be the characteristic polynomial of an AIDM of* G. *Then*

(i) $a_0 = 1$
(ii) $a_1 = -i_a(G)$
(iii) $a_2 = -(m - i_a(G))$
(iv) $a_n = \det(A_{i_a}(G))$

Observation 3. 3 *Suppose* $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots, \geq \mu_n$ *are the accurate independent eigenvalues of AIDM of* G. *Then*

(i) $\sum_{i=1}^{n} \mu_i = i_a(G)$

(ii) $\sum_{i=1}^{n} \mu^2{}_i = 2m + i_a(G)$

In the next successive propositions we obtain the characteristic polynomial of AIDM of the star graph $K_{1,n-1}$ and the complete bipartite graph $K_{r,\ s}$; $2 \leq r < s$.

Proposition 4. 4 *The characteristic polynomial of AIDM of* $K_{1,\ n-1}$ *is* $\mu^{n-2}[\mu^2 - \mu - (n-1)]$.

Proof. Consider the star graph $K_{1,\ n-1}$ with the central vertex v_1. Clearly, the accurate independent dominating set of $K_{1,\ n-1}$ is $D_{i_a} = \{v_1\}$. Hence the AIDM of $K_{1,\ n-1}$ is

$$A_{i_a}(K_{1,\ n-1}) = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The characteristic polynomial of AIDM of $K_{1,\ n-1}$ is

$$|\mu I_n - A_{i_a}(K_{n-1})| \quad = \quad 0$$

$$\begin{vmatrix} \mu-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & \mu & 0 & 0 & \cdots & 0 \\ -1 & 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \mu \end{vmatrix} = 0.$$

From the above we get the following recurrence relation:

$$\phi_n(K_{1,\ n-1},\ \mu) \quad = \quad -\mu^{n-2} + \mu\phi_{n-1}(K_{1,\ n-2},\ \mu) \quad (2)$$

changing n to $n-1$ in (2), we get

$$\phi_n(K_{1,\ n-2},\ \mu) \quad = \quad -\mu^{n-3} + \mu\phi_{n-2}(K_{1,\ n-3},\ \mu) \quad (3)$$

continuing this process, we obtain

$$\phi_n(K_{1,\ n-1},\ \mu) \quad = \quad \mu^{n-2}[\mu^2 - \mu - (n-1)].$$

Proposition 5. 5 *The characteristic polynomial of AIDM of* $K_{r,\ s}$; $2 \le r < s$ *is*

$$(\mu-1)^{r-1}\mu^{s-1}[\mu^2 - \mu - rs]$$

Proof. Consider the complete bipartite graph $K_{r,\ s}$; $2 \le r < s$ with vertex set $V = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, \cdots, v_r\}$ and $V_2 = \{u_1, u_2, \cdots, u_s\}$. Clearly, $D_{i_a}(K_{r,\ s}) = V_2$. Therefore the AIDM of $K_{r,\ s}$ is

$$A_{i_a}(K_{r,\ s}) \quad = \quad \begin{bmatrix} O_r & J_{rs} \\ J_{sr} & I_s \end{bmatrix}$$

where J_{rs} and J_{sr} are the matrices with all entries $1's$ and I_s is an identity matrix of order s.

The characteristic polynomial of AIDM of $K_{r,s}$ is

$$\begin{aligned}\phi_n(K_{r,\ s},\ \mu) \quad &= \quad \begin{vmatrix} \mu I_r & -J_{rs} \\ -J_{sr} & (\mu-1)I_s \end{vmatrix} \\ &= \quad |(\mu-1)I_s||\mu I_r - J_{rs}\tfrac{I_r}{(\mu-1)}J_{sr}| \\ &= \quad (\mu-1)^{s-r}|\mu(\mu-1) - J_{rs}J_{sr}I_r| \\ &= \quad (\mu-1)^{s-r}(\mu(\mu-1) - rs(\mu(\mu-1)^{r-1})).\end{aligned}$$

Now we obtain bounds for $E_{i_a}(G)$ of G similar to McClelland's inequalities [22] for graph energy.

Theorem 6. 6 *Let* G *be a graph of order* n *and size* m *with accurate independent domination number* $i_a(G)$. *Then*

$$E_{i_a}(G) \quad \le \quad \sqrt{n(2m + i_a(G))}. \qquad (4)$$

Proof. Let $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ be the eigenvalues of $A_{i_a}(G)$. Bearing in mind the Cauchy-Schwarz inequality,

$$(\sum_{i=1}^{n} a_i b_i)^2 \le (\sum_{i=1}^{n} a_i)^2(\sum_{i=1}^{n} b_i)^2$$

we choose $a_i = 1$ and $b_i = |\mu_i|$, which implies

$$\begin{aligned}E_{i_a}^2 \quad &\le \quad (\sum_{i=1}^{n} 1)^2(\sum_{i=1}^{n} |\mu_i|)^2 \\ &\le \quad n(\sum_{i=1}^{n} |\mu_i|^2) \\ &= \quad n\sum_{i=1}^{n} \mu_i^2 \\ &= \quad n(2m + i_a(G))\end{aligned}$$

$$E_{i_a}(G) \le \sqrt{n(2m + i_a(G))}.$$

Theorem 7. 7 *Let* G *be a graph of order* n *and size* m *with accurate independent domination number* $i_a(G)$. *Let* $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ *be a non-increasing arrangement of eigenvalues of* $A_{i_a}(G)$. *Then*

$$E_{i_a}(G) \quad \ge \quad \sqrt{2mn + ni_a(G) - \alpha(n)(|\mu_1| - |\mu_n|)^2} \quad (5)$$

where $\alpha(n) = n\left\lfloor\frac{n}{2}\right\rfloor(1 - \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor)$, where $\lfloor x \rfloor$ denotes the integer part of a real number x.

Proof. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers for which there exist real constants a, b, A and B, so that for each i, $i = 1,2,\cdots,n, a \le a_i \le A$ and $b \le b_i \le B$. Then the following inequality is valid

$$|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i| \quad \le \quad \alpha(n)(A-a)(B-b) \quad (6)$$

where $\alpha(n) = n\left\lfloor\frac{n}{2}\right\rfloor(1 - \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor)$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

We choose $a_i = |\mu_i|, b_i = |\mu_i|$, $a = b = |\mu_n|$ and $A = B = |\mu_1|, i = 1,2,\cdots,n$, inequality (6) becomes

$$|n\sum_{i=1}^{n} |\mu_i|^2 - (\sum_{i=1}^{n} |\mu_i|)^2| \quad \le \quad \alpha(n)(|\mu_1| - |\mu_n|)^2 \quad (7)$$

Since $E_{i_a}(G) = \sum_{i=1}^{n} |\mu_i|$, $\sum_{i=1}^{n} |\mu_i|^2 = 2m + i_a(G)$ and $E_{i_a}(G) \le \sqrt{n(2m + i_a(G))}$, the inequality (7) becomes

$$\begin{aligned}n(2m + i_a(G)) - (E_{i_a})^2 \quad &\le \quad \alpha(n)(|\mu_1| - |\mu_n|)^2 \\ (E_{i_a})^2 \quad &\ge \quad 2mn + ni_a(G) - \alpha(n)(|\mu_1| - |\mu_n|)^2.\end{aligned}$$

Hence the equality holds if and only if $\mu_1 = \mu_2 = \cdots = \mu_n$.

Corollary 8. 8 *Let* G *be a graph of order* n *and size* m *with accurate independent domination number* $i_a(G)$. *Let*

$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ *be a non-increasing arrangement of eigenvalues of* $A_{i_a}(G)$*. Then*

$$E_{i_a}(G) \geq \sqrt{2mn + ni_a(G) - \frac{n^2}{4}(|\mu_1| - |\mu_n|)^2} \quad (8)$$

Proof. Since $\alpha(n) = n\left\lfloor\frac{n}{2}\right\rfloor\left(1 - \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) \leq \frac{n^2}{4}$, therefore by (5), result follows.

Theorem 9.9 *Let* G *be a graph of order* n *and size* m *with accurate independent domination number* $i_a(G)$*. Let* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ *be a non-increasing arrangement of eigenvalues of* $A_{i_a}(G)$*. Then*

$$E_{i_a}(G) \geq \frac{|\mu_1||\mu_2|n + 2m + i_a(G)}{|\mu_1| + |\mu_n|} \quad (9)$$

Proof. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers for which there exist real constants r and R so that for each i, $i = 1,2,\cdots,n$ holds $ra_i \leq b_i \leq Ra_i$. Then the following inequality is valid.

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\sum_{i=1}^{n} a_i b_i. \quad (10)$$

Equality of (10) holds if and only if, for at least one i, $1 \leq i \leq n$ holds $ra_i = b_i = Ra_i$.

For $b_i = |\mu_i|$, $a_i = 1$ $r = |\mu_n|$ and $R = |\mu_1|$, $i = 1,2,\cdots,n$ inequality (10) becomes

$$\sum_{i=1}^{n} |\mu_i|^2 + |\mu_1||\mu_n|\sum_{i=1}^{n} 1 \leq (|\mu_1| + |\mu_n|)\sum_{i=1}^{n} |\mu_i|. \quad (11)$$

Since $\sum_{i=1}^{n} |\mu_i|^2 = \sum_{i=1}^{n} \mu_i^2 = 2m + i_a(G)$,

$\sum_{i=1}^{n} |\mu_i| = E_{i_a}(G)$, from inequality (11),

$$2m + i_a(G) + |\mu_1||\mu_n|n \leq (\mu_1 + \mu_n)E_{i_a}(G)$$

Hence the result.

Theorem 10.10 *Let* G *be a graph of order* n *and size* m *with accurate independent domination number* $i_a(G)$*. Let* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ *be a non-increasing arrangement of eigenvalues of* $A_{i_a}(G)$*. If* $\xi = detA_{i_a}(G)$*, then*

$$E_{i_a}(G) \geq \sqrt{2m + i_a(G) + n(n-1)\xi^{\frac{2}{n}}}. \quad (12)$$

Proof.

$$(E_{i_a}(G))^2 = \left(\sum_{i=1}^{n} |\mu_i|\right)^2$$

$$= \sum_{i=1}^{n} |\mu_i|^2 + \sum_{i \neq j} |\mu_i||\mu_j|.$$

Employing the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{n(n-1)}\sum_{i \neq j} |\mu_i||\mu_j| \geq \left(\prod_{i \neq j} |\mu_i||\mu_j|\right)^{\frac{1}{n(n-1)}}.$$

Thus,

$$(E_{i_a}(G))^2 \geq \sum_{i=1}^{n} \mu_i^2 + n(n-1)\left(\prod_{i \neq j} |\mu_i||\mu_j|\right)^{\frac{1}{n(n-1)}}$$

$$\geq \sum_{i=1}^{n} \mu_i^2 + n(n-1)\left(\prod_{i \neq j} |\mu_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= 2m + i_a(G) + n(n-1)\xi^{\frac{2}{n}}.$$

Next, we obtain Koolen and Moulton's [13] type inequality for $E_{i_a}(G)$.

Theorem 11. 11 *If* G *is a graph of order* n *and size* m *and* $2m + i_a(G) \geq n$*, then*

$$E_{i_a}(G) \leq \frac{2m + i_a(G)}{n} + \sqrt{(n-1)\left[(2m + i_a(G)) - \left(\frac{2m + i_a(G)}{n}\right)^2\right]} \quad (13)$$

Proof. Bearing in mind the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2\left(\sum_{i=1}^{n} b_i\right)^2.$$

Put $a_i = 1$ and $b_i = |\mu_i|$ then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq (n-1)\left(\sum_{i=1}^{n} b_i\right)^2$$

$$(E_{i_a}(G) - \mu_1)^2 \leq (n-1)(2m + i_a(G) - \mu_1^2)$$

$$E_{i_a}(G) \leq \mu_1 + \sqrt{(n-1)(2m + i_a(G) - \mu_1^2)}.$$

Let

$$f(x) = x + \sqrt{(n-1)(2m + i_a(G) - x^2)} \quad (14)$$

For decreasing function

$$f'(x) \leq 0$$

$$\Rightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)(2m + i_a(G) - x^2)}} \leq 0$$

$$x \geq \sqrt{\frac{2m + i_a(G)}{n}}.$$

Since $(2m + i_a(G)) \geq n$,

we have $\sqrt{\frac{2m+i_a(G)}{n}} \leq \frac{2m+i_a(G)}{n} \leq \mu_1$.

Also $f(\mu_1) \leq f(\frac{2m+i_a(G)}{n})$.

i.e $E_{i_a}(G) \leq f(\mu_1) \leq f(\frac{2m+i_a(G)}{n})$.

i.e $E_{i_a}(G) \leq f(\frac{2m+i_a(G)}{n})$

Hence, the result (13) follows.

OPEN PROBLEMS

We close with the following list of open problems for further research in spectral theory of graphs:

Problem 1 Construct graphs with $E(G) = E_{i_a}(G)$.

Problem 2 Characterize graphs with equal Laplacian energy and accurate independent dominating energy.

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REFERENCES

[1] C. Adiga, A. Bayad, I. Gutman and S. Srinivas, *The minimum covering energy of a graph,* Kragujevac Journal of Science 34 (2012), 39--56.

[2] A. Balban, *Chemical applications of graph theory,* Academic Press (1976).

[3] R. Bapat, *Graphs and Matrices,* Hindustan Book Agency (2011).

[4] B. Basavanagoud and Sujata Timmanaikar, Accurate independent domination in graphs, Int. J. Math. Combin (In press). Vol.2 (2018), 87-96.

[5] M. Biernacki, H. Pidek and C. Ryll-Nardzewski, *Sur une inégalité entre des intégrales définies,* Univ. Marie Curie-Sktoodowska A4 (1950), 1--4.

[6] C. Coulson and G. Rushbrooke, *Note on the method of molecular orbitals,* Mathematical Proceedings of the Cambridge Philosophical Society 36 (1940), 193--200.

[7] V. Consonni and R. Todeschini, *New spectral index for molecule description,* MATCH Communications in Mathematical and in Computer Chemistry 60 (2008), 3-14.

[8] D. Cvetković, P. Rowlinson and S. Simić, *Eigenspaces of Graphs,* Cambridge University Press (1997).

[9] 9 J. Cyman, M. A. Henning and J. Topp, On accurate domination in graphs, arXiv:1710.03308v1 math. CO 9 Oct 2017.

[10] J. Diaz and F. Metcalf, *Stronger forms of a class of inequalities of G. Pólya-G.Szegö and L. V. Kantorovich,* Bulletin of the American Mathematical Society 69 (1963), 415--418.

[11] I. Gutman, *The energy of a graph,* Ber. Math.-Statist. Sekt. Forschungszentrum Graz 103 (1978), 1--22.

[12] I. Gutman and O. Polansky, *Mathematical Concepts in Organic Chemistry,* Springer--Verlag, Berlin, 1986.

[13] I. Gutman and B. Zhou, *Laplacian energy of a graph,* Linear Algebra and its Applications 414 (2006), 29--37.

[14] F. Harary, *Graph Theory,* Addison--Wesely, Reading, 1969.

[15] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc, New York (1998).

[16] T. Haynes, S. Hedetniemi and P. Slater (eds.), *Domination in Graphs: Advanced Topics,* Marcel Dekker, New York, 1998.

[17] G. Indulal, I. Gutman and A. Vijayakumar, *On distance energy of graphs,* MATCH Communications in Mathematical and in Computer Chemistry 60 (2008), 461--472.

[18] M. Jooyandeh, D. Kiani and M. Mirzakhah, *Incidence energy of a graph,* MATCH Communications in Mathematical and in Computer Chemistry 62 (2009), 561--572.

[19] J. Koolen and V. Moulton, *Maximal energy graphs,* Advances in Applied Mathematics 26 (2001), 47--52.

[20] V. R. Kulli and M. B. Kattimani, The accurate domination number of a graph, In V. R. Kulli, ed., Advances in Domination Theory-I, Vishwa International Publications, Gulbarga, India (2012) 1--8.

[21] J. Liu and B. Liu, *A Laplacian-energy like invariant of a graph,* MATCH Communications in Mathematical and in Computer Chemistry 59 (2008), 355--372.

[22] B. McClelland, *Properties of the latent roots of a matrix: The estimation of π-electron energies,* The Journal of Chemical Physics 54 (1971), 640--643.

[23] I. Milovanović, E. Milovanović and A. Zakić, *A short note on graph energy,* MATCH Communications in Mathematical and in Computer Chemistry 72 (2014), 179--182.

[24] J. Rada, *Energy ordering of catacondensed hexagonal systems,* Discrete Applied Mathematics 145 (2005), 437--443.

[25] H. Sachs, *beziehungen zwischen den in einem graphen enthaltenen kreisen und seinem charakteristischen polynom,* Ibid. 11 (1963), 119--134.

[26] I. Shparlinski, *On the energy of some circulant graphs,* Linear Algebra and its Applications 414 (2006), 378--382.

[27] N. Trinajstić, *Chemical graph theory,* CRC Press (1992).

[28] B. Zhou, *Energy of a graph,* MATCH Communications in Mathematical and in Computer Chemistry 51 (2004), 111--118.