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Conjunctive variety of l-fuzzy languages

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Abstract:
Here we study l-fuzzy languages recognized by finite idempotent semirings. We show that the class of semiring recognizable l-fuzzy languages is closed under quotients and inverse homomorphic images. We introduce the concept of conjunctive variety of l-fuzzy languages. We also obtain an Eilenberg's type variety theorem for semiring recognizable l-fuzzy languages.

Keywords : Generalized fuzzy languages, Conjunctive variety of l-fuzzy languages, Syntactic Semiring.
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1. INTRODUCTION

Theory of formal languages is one of the fundamental concept in theoretical computer science. Zadeh [10] introduced the notion of a fuzzy subset of an ordinary set as a method of representing uncertainty. Later it came as a useful tool for describing real-life problems. Zadeh and Lee [6] generalized the classical notion of languages to the concept of fuzzy languages in 1969. In [7] Petkovic introduced the notion of syntactic monoid of a fuzzy languages. He proved that the class of all recognizable fuzzy languages is a variety and obtained an Eilenberg type variety for fuzzy languages.
Semiring recognizable languages was first studied by Polak [8]. In [9], he introduced the concept of syntactic semiring of a language and studied its properties. Also he established a one-one correspondence between the lattices of all conjunctive variety of languages and pseudovariety of finite idempotent semirings. We generalized the notion of semiring recognizability and syntactic semiring etc to the class of l-fuzzy languages in [2].
In this paper we introduce the notion of variety of semiring recognizable l-fuzzy languages. Also we obtain a one to one correspondence between varieties of semiring recognizable l-fuzzy languages and all pseudovarieties of finite idempotent semirings.

2. PRELIMINARIES

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms are as in [4, 5]. A lattice l is a partially ordered set in which every subset $\{a, b\}$ consisting of two elements has a least upper bound($a \vee b$) and a greatest lower bound($a \wedge b$). We denote it by $(l, \wedge, \vee)$. A lattice l is said to be distributive if for any element a, b and c of l, we have the following distributive properties.

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A lattice l is called complemented if it is bounded and if every element in l has a complement. A lattice l is called a complete lattice if every nonempty subset of l has greatest lower bound and least upper bound in l.
A semiring is a nonempty set S together with two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that

1) $(S, +, 0)$ is a commutative monoid.

2) $(S, \cdot, 1)$ is a monoid.

3) the distributive laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for every $a, b, c \in S$.

4) $0 \cdot a = a \cdot 0 = 0$ for every a.

The semiring $(S', +, \cdot)$ is a subsemiring of the semiring $(S, +, \cdot)$ if and only if $S' \subseteq S$. A morphism of a semiring S into a semiring S' is a mapping from S to S' compatible with the constant elements and operations in S and S'. A semiring $(S, +, \cdot)$ is called additively idempotent if $(S, +)$ is an idempotent semigroup. Here we say idempotent semiring means additively idempotent semiring.

Let $(S,+,\cdot,0,1)$ and $(S',+,\cdot,0',1')$ be semirings. Then $(S\times S',+,\cdot,(0,0'),(1,1'))$ is a semiring, where addition and multiplication are defined by $(a_1,a_1')+(a_2,a_2')=(a_1+a_2,a_1'+a_2')$ and $(a_1,a_1')(a_2,a_2')=(a_1a_2,a_1'a_2')$. The semiring $S\times S'$ is called the direct product of S and S'.
A class of finite idempotent semirings is called a pseudovariety if it is closed with respect to forming finite products, substructures and homomorphic images [8].
A semiring S is called a partially ordered semiring if and only if S is a partially ordered set under the relation $\leq$ and the following conditions hold.

i) $0\leq a$

ii) $a_1\leq a_2$ implies $a_1+a\leq a_2+a$

ii) $a_1\leq a_2$ implies $a_1a\leq a_2a$ and $aa_1\leq aa_2$ for all $a,a_1,a_2\in S$.

Let $(S,+,\cdot)$ be a (partially ordered) idempotent semiring. A subset I of S is called an ordered ideal if

i) $a\in I, b\in S$ and $b\leq a$ implies $b\in I$.

ii) $a,b\in I$ implies $a+b\in I$.

Let A be a nonempty finite set called an alphabet. Elements of A are called letters. A finite sequence of letters of A is called a word. The length of the word w is the number of letters of A occurring in w. A word of length zero is called empty word and is denoted by ε. A^+ denotes the set of all nonempty words over an alphabet A and $A^*=A^+\cup\{\varepsilon\}$ is a monoid under the operation concatenation, called free monoid over A. A subset L of A^* is called a language over an alphabet A.
Let $F(A^*)$ denote the set of all finite subsets of A^*. This set equipped with the operations usual union and multiplication $U\cdot V=\{uv\mid u\in U, v\in V\}$ form the free idempotent semiring over the alphabet A.
Let X be a nonempty set. A l-fuzzy set (or more precisely l-fuzzy subset) λ of X is a function from X into the complete complemented distributive lattice l. A l-fuzzy language λ over an alphabet A is a l-fuzzy subset of $A^*(resp. A^+)$.
For l-fuzzy languages λ_1,λ_2 over an alphabet A, their meet, left quotient and right quotient are defined as follows:
For $u\in A^*$,

$$(\lambda_1\wedge\lambda_2)(u)=\lambda_1(u)\wedge\lambda_2(u),$$

$$(\lambda_1^{-1}\lambda_2)(u)=\bigvee_{v\in A^*}(\lambda_2(vu)\wedge\lambda_1(v)),$$

$$(\lambda_2\lambda_1^{-1})(u)=\bigvee_{v\in A^*}(\lambda_2(uv)\wedge\lambda_1(v)).$$

Let A and B be finite alphabets and $\phi: A^*\to B^*$ be a homomorphism. Let λ be a l-fuzzy language over B. The inverse of λ under ϕ is a l-fuzzy language $\lambda\phi^{-1}$ over A defined by

$$(\lambda\phi^{-1})(u)=\lambda(\phi(u)), u\in A^*.$$

Let λ be a l-fuzzy language over an alphabet A. We say that λ is recognizable if there exist a finite monoid M, a homomorphism $\phi: A^*\to M$ and a l-fuzzy subset $\pi: M\to l$ such that $\lambda=\pi\phi^{-1}$ where $\pi\phi^{-1}(u)=\pi(\phi(u))$, $u\in A^*$. We also say that the monoid M recognizes λ by a morphism ϕ.
The class of all recognizable l-fuzzy languages over A is denoted by $lF(A^*)$. $lF(A^*)$ is closed under join, meet and complementation [1].

Definition 2.1. *[3] A fuzzy subset λ of a semiring $(S,+,\cdot)$ is called a fuzzy left (right) ideal of S if*

1) $\lambda(x+y)\geq\lambda(x)\wedge\lambda(y)$

2) $\lambda(x\cdot y)\geq\lambda(y)$ ($\lambda(x\cdot y)\geq\lambda(x)$) *for all* $x,y\in S$.

A fuzzy subset λ of a semiring S is called a fuzzy two-sided ideal or simply a fuzzy ideal of S if it is both a fuzzy left ideal or a fuzzy right ideal of S. A fuzzy ideal of a semiring S is called fuzzy ordered ideal if it satisfies the condition, $x\leq y$ implies $\lambda(x)\geq\lambda(y)$ for all $x,y\in S$.
The following theorem gives a characterization for fuzzy ideals.

Theorem 2.2. *[3] A fuzzy subset λ of a semiring S is a fuzzy ideal of S if and only if each nonempty level subset $\lambda_t=\{x\in S\mid\lambda(x)\geq t\}$ of λ is an ideal of S for all $t\in[0,1]$.*

3. GENERALIZED FUZZY LANGUAGES

Let λ be a l-fuzzy language over A. The function $\lambda_{min}: F(A^*)\to l$ defined by $\lambda_{min}(U)=\bigwedge_{u\in U}\lambda(u), U\in F(A^*)$ is called the generalized fuzzy language determined by λ. If $\mid U\mid=1$ then $\lambda_{min}(u)=\lambda(u)$. So we can view λ_{min} as a generalization of λ.
The following properties of generalized fuzzy languages follows immediately from the definition.

Theorem 3.1. *[2] Let $\lambda_1 and \lambda_2$ be l-fuzzy languages over A and let $L_1,L_2\in F(A^*)$. Then*

(1) $(\lambda_1\wedge\lambda_2)_{min}=\lambda_{1min}\wedge\lambda_{2min}$.

(2) $\lambda_{min}(L_1\cup L_2)=\lambda_{min}(L_1)\wedge\lambda_{min}(L_2)$.

4. SEMIRING RECOGNIZABLE FUZZY LANGUAGES

Definition 4.1. *[2] Let λ be a l-fuzzy language over A. We say that λ is recognized by an idempotent semiring S, if there exists a semiring homomorphism $\beta: F(A^*)\to S$, a l-fuzzy ordered ideal γ of S such that $\lambda_{min}=\gamma\beta$.*

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Let λ be a l-fuzzy language. Then λ is said to be semiring recognizable if it is recognized by a finite idempotent semiring.

Theorem 4.2. *[2] Let λ_1 and λ_2 be l-fuzzy languages over A. If λ_1 and λ_2 are semiring recognizable, then $\lambda_1 \wedge \lambda_2$ is semiring recognizable.*

Let λ be a l-fuzzy language over A. Define a relation $\sim_{min}$ on $F(A^*)$ as follows:
For $U, V \in F(A^*)$, $U \sim_{min} V \Leftrightarrow \lambda_{min}(\bigcup_{u\in U} puq) = \lambda_{min}(\bigcup_{v\in V} pvq)$ for all $p, q \in A^*$. The relation $\sim_{min}$ is a congruence on $F(A^*)$. Thus the quotient structure $F(A^*)/\sim_{min}$ is a semiring under the binary operations

$$[U]_{\sim_{min}} \cup [V]_{\sim_{min}} = [U \cup V]_{\sim_{min}} \quad \text{and}$$

$$[U]_{\sim_{min}} \cdot [V]_{\sim_{min}} = [U \cdot V]_{\sim_{min}}.$$

$F(A^*)/\sim_{min}$ is an idempotent semiring, called the syntactic semiring of λ or λ_{min}. It is denoted by Syn(λ_{min}).

5. ORDERED SYNTACTIC MONOID

Let λ be a l-fuzzy language over A. Define a relation $(\sim_\lambda)$ on A^* as follows:
For $u, v \in A^*, u \sim_\lambda v$ if and only if $\lambda(puq) = \lambda(pvq)$, for all $p, q \in A^*$. Then the relation $\sim_\lambda$ is a congruence on A^* called syntactic congruence of λ. The quotient monoid $A^*/\sim_\lambda=$ Syn(λ) is called syntactic monoid of λ.
Now we consider a relation $\leq$ on syntactic monoid Syn(λ), defined by $[u]_{\sim_\lambda} \leq [v]_{\sim_\lambda} \Leftrightarrow \lambda(pvq) \leq \lambda(puq)$ for all $p, q \in A^*$. The relation $\leq$ is a partial order on Syn(λ). (Syn $(\lambda), \leq$) is called the ordered syntactic monoid and is denoted by $O(\lambda)$.

Theorem 5.1. *Let λ be a l-fuzzy language over an alphabet A and λ_{min} be the generalized fuzzy language determined by λ. Then the function θ defined by $\theta([u]_{\sim_\lambda}) = [\{u\}]_{\sim_{min}}, u \in A^*$ is an injective semigroup homomorphism of ordered syntactic monoid $(O(\lambda), \cdot, \leq)$ into the syntactic semiring $(Syn(\lambda_{min}), \cup, \cdot)$.*

Proof. Let $u, v \in A^*$ and $[u]_{\sim_\lambda} = [v]_{\sim_\lambda}$ then $u \sim_\lambda v$. Thus for all $p, q \in A^*$, $\lambda(puq) = \lambda(pvq)$. That is, $\lambda_{min}(p\{u\}q) = \lambda_{min}(p\{v\}q)$ for all $p, q \in A^*$. Thus $\{u\} \sim_{min} \{v\}$. Hence $[\{u\}]_{\sim_{min}} = [\{v\}]_{\sim_{min}}$. Thus θ is well defined. Also we have

$$\begin{aligned}\theta([u]_{\sim_\lambda} \cdot [v]_{\sim_\lambda}) &= \theta([u \cdot v]_{\sim_\lambda})\\ &= [\{u \cdot v\}]_{\sim_{min}}\\ &= [\{u\}]_{\sim_{min}} \cdot [\{v\}]_{\sim_{min}}\\ &= \theta([u]_{\sim_\lambda}) \cdot \theta([v]_{\sim_\lambda}).\end{aligned}$$

So θ is a homomorphism.

Let $\theta([u]_{\sim_\lambda}) = \theta([v]_{\sim_\lambda})$. Then $[\{u\}]_{\sim_{min}} = [\{v\}]_{\sim_{min}}$. Thus we get, $\{u\} \sim_{min} \{v\}$. Hence $\lambda_{min}(p\{u\}q) = \lambda_{min}(p\{v\}q)$ for all $p, q \in A^*$. That is, $\lambda(puq) = \lambda(pvq)$, for all $p, q \in A^*$. Thus $u \sim_\lambda v$. Hence $[u]_{\sim_\lambda} = [v]_{\sim_\lambda}$. So θ is one to one.
Let $u, v \in A^*$ and let $[u]_{\sim_\lambda} \leq [v]_{\sim_\lambda}$. Then

$$\lambda(pvq) \leq \lambda(puq), \text{ for all } p, q \in A^*. \tag{1}$$

For proving $\theta([u]_{\sim_\lambda}) \leq \theta([v]_{\sim_\lambda})$, we first prove that for all $p, q \in A^*$, $\lambda_{min}(p\{u, v\}q) = \lambda_{min}(p\{v\}q)$. We have

$$\begin{aligned}\lambda_{min}(p\{u, v\}q) &= \lambda_{min}(\{puq, pvq\})\\ &= \lambda_{min}(\{puq\} \cup \{pvq\}).\end{aligned}$$

By Theorem 3.1, we get $\lambda_{min}(p\{u, v\}q) = \lambda_{min}(p\{u\}q) \wedge \lambda_{min}(p\{v\}q)$. From the definition of λ_{min}, $\lambda_{min}(p\{u, v\}q) = \lambda(puq) \wedge \lambda(pvq)$. So by (1), $\lambda_{min}(p\{u, v\}q) = \lambda(pvq) = \lambda_{min}(p\{v\}q)$. Thus $\{u, v\} \sim_{min} \{v\}$. Hence $[\{u, v\}]_{\sim_{min}} = [\{v\}]_{\sim_{min}}$. That is, $[\{u\} \cup \{v\}]_{\sim_{min}} = [\{v\}]_{\sim_{min}}$. Thus $[\{u\}]_{\sim_{min}} \cup [\{v\}]_{\sim_{min}} = [\{v\}]_{\sim_{min}}$. Since Syn($\lambda_{min}$) is an idempotent semiring, we get $[\{u\}]_{\sim_{min}} \leq [\{v\}]_{\sim_{min}}$. That is, $\theta([u]_{\sim_\lambda}) \leq \theta([v]_{\sim_\lambda})$. Thus θ is an injective semigroup homomorphism from $(O(\lambda), \cdot, \leq)$ into (Syn$(\lambda_{min}), \cup, \cdot)$. □

Theorem 5.2. *The syntactic semiring $(Syn(\lambda_{min}), \cup, \cdot)$ of a l-fuzzy language λ over A is isomorphic to the semiring $(F(O(\lambda)), \cup, \cdot)$.*

Proof. Let $U, V \in F(A^*)$. Define a function ρ from (Syn$(\lambda_{min}), \cup, \cdot)$ to $(F(O(\lambda)), \cup, \cdot)$ by

$$\rho([U]_{\sim_{min}}) = \bigcup_{u\in U} ([u]_{\sim_\lambda}).$$

Let $U, V \in F(A^*)$ and let $[U]_{\sim_{min}} = [V]_{\sim_{min}}$. This can be written as $\left[\bigcup_{u\in U}\{u\}\right]_{\sim_{min}} = \left[\bigcup_{v\in V}\{v\}\right]_{\sim_{min}}$. Hence $\bigcup_{u\in U}[\{u\}]_{\sim_{min}} = \bigcup_{v\in V}[\{v\}]_{\sim_{min}}$. That is, $\bigcup_{u\in U}[u]_{\sim_\lambda} = \bigcup_{v\in V}[v]_{\sim_\lambda}$. Thus $\rho([U]_{\sim_{min}}) = \rho([V]_{\sim_{min}})$. Hence ρ is well defined.
For $[U]_{\sim_{min}}, [V]_{\sim_{min}} \in$ Syn(λ_{min}), we have

$$\begin{aligned}\rho([U]_{\sim_{min}} \cup [V]_{\sim_{min}}) &= \rho([U \cup V]_{\sim_{min}})\\ &= \bigcup_{u\in U\cup V} ([u]_{\sim_\lambda})\\ &= \bigcup_{u\in U}([u]_{\sim_\lambda}) \cup \bigcup_{v\in V}([v]_{\sim_\lambda})\\ &= \rho([U]_{\sim_{min}}) \cup \rho([V]_{\sim_{min}}).\end{aligned}$$

Also we have

$$\begin{aligned}\rho([U]_{\sim_{min}} \cdot [V]_{\sim_{min}}) &= \rho([U \cdot V]_{\sim_{min}}) \\ &= \bigcup_{u \in U \cdot V} ([u]_{\sim_\lambda}) \\ &= \bigcup_{u \in U} ([u]_{\sim_\lambda}) \cdot \bigcup_{v \in V} ([v]_{\sim_\lambda}) \\ &= \rho([U]_{\sim_{min}}) \cdot \rho([V]_{\sim_{min}}).\end{aligned}$$

Hence ρ is a homomorphism.
For $[U]_{\sim_{min}}, [V]_{\sim_{min}} \in$ Syn(λ_{min}), we have $\rho([U]_{\sim_{min}}) = \rho([V]_{\sim_{min}})$. That is, $\bigcup_{u \in U} ([u]_{\sim_\lambda}) = \bigcup_{v \in V} ([v]_{\sim_\lambda})$. Thus $\bigcup_{u \in U} [\{u\}]_{\sim_{min}} = \bigcup_{v \in V} [\{v\}]_{\sim_{min}}$. That is, $[\bigcup_{u \in U} \{u\}]_{\sim_{min}} = [\bigcup_{v \in V} \{v\}]_{\sim_{min}}$. Thus $[U]_{\sim_{min}} = [V]_{\sim_{min}}$. Hence ρ is one to one. Therefore ρ is an isomorphism from $(\text{Syn}(\lambda_{min}), \cup, \cdot)$ to $(F(O(\lambda)), \cup, \cdot)$. □

We have already proved that any monoid is a syntactic monoid of a l-fuzzy language (cf.[1], Theorem 4.3). Theorem 5.1 shows that there exists an injective semigroup homomorphism from an ordered syntactic monoid $(O(\lambda), \cdot, \leq)$ into the syntactic semiring $(\text{Syn}(\lambda_{min}), \cup, \cdot)$. The syntactic semiring $(\text{Syn}\lambda_{min}), \cup, \cdot)$ of a l-fuzzy language λ is isomorphic to the semiring $(F(O(\lambda)), \cup, \cdot)$, by Theorem 5.2. Polak proved that every finite idempotent semiring is isomorphic to the syntactic semiring of a recognizable language (cf.[9], Proposition 6). From the above discussion we have the following result.

Theorem 5.3. *Every finite idempotent semiring is a syntactic semiring of a l-fuzzy language.*

6. MEET, QUOTIENT AND INVERSE HOMOMORPHIC IMAGE OF GENERALIZED FUZZY LANGUAGES

Let λ_{1min} and λ_{2min} be generalized fuzzy languages determined by λ_1 and λ_2 respectively. Then their meet, left quotient and right quotient are defined as follows :
For $U \in F(A^*)$,

$$(\lambda_{1min} \wedge \lambda_{2min})(U) = \lambda_{1min}(U) \wedge \lambda_{2min}(U).$$

$$(\lambda_{1min}^{-1}\lambda_{2min})(U) = \bigvee_{v \in A^*} (\lambda_{2min}(vU) \wedge \lambda_{1min}(v)),$$

$$(\lambda_{2min}\lambda_{1min}^{-1})(U) = \bigvee_{v \in A^*} (\lambda_{2min}(Uv) \wedge \lambda_{1min}(v)).$$

Let A and B be finite alphabets and φ from $F(A^*)$ to $F(B^*)$ be a semiring homomorphism and λ_{min} be a generalized fuzzy language determined by a l-fuzzy language λ over B. Then the inverse homomorphic image of λ_{min} is a l-fuzzy subset $\lambda_{min}\varphi^{-1}$ of $F(A^*)$ defined by

$$(\lambda_{min}\varphi^{-1})(U) = \lambda_{min}(\varphi(U)), \;\; U \in F(A^*).$$

The following properties of generalized fuzzy languages follows immediately from the definition.

Theorem 6.1. *Let λ_1 and λ_2 be l-fuzzy languages over A. Then*

(i) $(\lambda_1^{-1}\lambda_2)_{min} = \lambda_{1min}^{-1}\lambda_{2min}$.

(ii) $(\lambda_2\lambda_1^{-1})_{min} = \lambda_{2min}\lambda_{1min}^{-1}$.

Theorem 6.2. *Let A and B be finite alphabets and φ from $F(A^*)$ to $F(B^*)$ be a semiring homomorphism. If λ is a l-fuzzy language over B, then $(\lambda\varphi^{-1})_{min} = \lambda_{min}\varphi^{-1}$.*

Theorem 6.3. *Let A and B be finite alphabets and φ from $F(A^*)$ to $F(B^*)$ be a semiring homomorphism. Let λ be a l-fuzzy language over B. If λ is semiring recognizable, then $\lambda\varphi^{-1}$ is semiring recognizable.*

Proof. Assume that λ is semiring recognizable. Then there exist an idempotent semiring S, a semiring homomorphism $\beta' : F(B^*) \to S$ and a l-fuzzy ordered ideal γ of S such that $\lambda_{min} = \gamma\beta'$. Define a map $\beta : F(A^*) \to S$ by

$$\beta(U) = \beta'(\varphi(U)), \; U \in F(A^*).$$

Since β' and φ are well defined, β is well defined.
For $U, V \in F(A^*)$, we have

$$\begin{aligned}\beta(U \cdot V) = \beta'(\varphi(U \cdot V)) &= \beta'(\varphi(U) \cdot \varphi(V)) \\ &= \beta'(\varphi(U)) \cdot \beta'(\varphi(V)) \\ &= \beta(U) \cdot \beta(V),\end{aligned}$$

and

$$\begin{aligned}\beta(U \cup V) = \beta'(\varphi(U \cup V)) &= \beta'(\varphi(U) \cup \varphi(V)) \\ &= \beta'(\varphi(U)) + \beta'(\varphi(V)) \\ &= \beta(U) + \beta(V)\end{aligned}$$

where $+$ denotes the addition (commutative binary operation) on S and $\cdot$ denote the product on S. Thus β is a semiring homomorphism from $F(A^*)$ to S. By Theorem 6.2, we have

$$\begin{aligned}(\gamma\beta)(U) &= \gamma(\beta(U)) \\ &= \gamma(\beta'(\varphi(U)) = (\gamma\beta')(\varphi(U)) \\ &= \lambda_{min}(\varphi(U)) = (\lambda_{min}\varphi^{-1})(U) \\ &= (\lambda\varphi^{-1})_{min}(U).\end{aligned}$$

Thus $(\lambda\varphi^{-1})_{min} = \gamma\beta$. Hence the semiring S recognizes $\lambda\varphi^{-1}$. □

Theorem 6.4. *Let λ_1 and λ_2 be l-fuzzy languages over A. If λ_1 and λ_2 are semiring recognizable, then $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}$ are semiring recognizable.*

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Proof. Assume that λ_1 and λ_2 are semiring recognizable. Since λ_2 is semiring recognizable, there exist a finite idempotent semiring S, a semiring homomorphism $\beta : F(A^*) \to S$ and a l-fuzzy ordered ideal γ of S such that $\lambda_{2min} = \gamma\beta$. Define a map $\gamma' : S \to l$ by

$$\gamma'(s) = \bigvee_{v \in A^*} (\gamma(\beta(vU)) \wedge \lambda_1(v))$$

where $s = \beta(U), U \in F(A^*)$. Since β and γ are well defined, γ' is well defined. Next we prove that γ' is a l-fuzzy ordered ideal of S. By Theorem 2.2, it suffices to prove that the level subset γ'_t is an ordered ideal of S for all $t \in l$. Let $s_1, s_2 \in \gamma'_t$ and let $U_1, U_2 \in F(A^*)$ such that $\beta(U_1) = s_1$ and $\beta(U_2) = s_2$. Then $\gamma'(s_1) \geq t$ and $\gamma'(s_2) \geq t$. Since γ is a l-fuzzy ordered ideal of S, we have

$$\begin{aligned}
\gamma'(s_1 + s_2) &= \gamma'(\beta(U_1) + \beta(U_2)) \\
&= \gamma'(\beta(U_1 \cup U_2)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(v(U_1 \cup U_2))) \wedge \lambda_1(v)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(vU_1 \cup vU_2)) \wedge \lambda_1(v)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(vU_1) + \beta(vU_2)) \wedge \lambda_1(v)) \\
&\geq \bigvee_{v \in A^*} (\gamma(\beta(vU_1)) \wedge \gamma(\beta(vU_2)) \wedge \lambda_1(v)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(vU_1)) \wedge \gamma(\beta(vU_2)) \wedge \lambda_1(v) \\
&\wedge \lambda_1(v)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(vU_1)) \wedge \lambda_1(v) \wedge \\
&\gamma(\beta(vU_2)) \wedge \lambda_1(v)) \\
&= (\bigvee_{v \in A^*} (\gamma(\beta(vU_1)) \wedge \lambda_1(v))) \\
&\wedge (\bigvee_{v \in A^*} (\gamma(\beta(vU_2)) \wedge \lambda_1(v))) \\
&= \gamma'(\beta(U_1)) \wedge \gamma'(\beta(U_2)) \\
&= \gamma'(s_1) \wedge \gamma'(s_2).
\end{aligned}$$

That is, $\gamma'(s_1 + s_2) \geq \gamma'(s_1) \wedge \gamma'(s_2) \geq t$. Thus $s_1 + s_2 \in \gamma'_t$. Let $s_1 \in S, s_2 \in \gamma'_t$ and $s_1 \leq s_2$. Then $\gamma'(s_2) \geq t$ and $\beta(U_1) \leq \beta(U_2)$. By the definition of ordered semiring, $\beta(v)\beta(U_1) \leq \beta(v)\beta(U_2)$ for $v \in A^*$. That is, $\beta(vU_1) \leq \beta(vU_2)$. Since γ is a l-fuzzy ordered ideal of S, $\gamma(\beta(vU_1)) \geq \gamma(\beta(vU_2))$. Thus $\gamma(\beta(vU_1)) \wedge \lambda_1(v) \geq \gamma(\beta(vU_2)) \wedge \lambda_1(v)$. So $\bigvee_{v \in A^*} (\gamma(\beta(vU_1)) \wedge \lambda_1(v)) \geq \bigvee_{v \in A^*} (\gamma(\beta(vU_2)) \wedge \lambda_1(v))$. That is, $\gamma'(s_1) \geq \gamma'(s_2)$. Since $s_2 \in \gamma'_t$, we get $\gamma'(s_1) \geq t$. Thus $s_1 \in \gamma'_t$. Hence γ'_t is an ordered ideal of S, for all $t \in l$. Thus γ' is a l-fuzzy ordered ideal of S.

For all $U \in F(A^*)$, we have

$$\begin{aligned}
(\gamma'\beta)(U) &= \gamma'(\beta(U)) \\
&= \bigvee_{v \in A^*} (\gamma(\beta(vU)) \wedge \lambda_1(v)) \\
&= \bigvee_{v \in A^*} (\lambda_{2min}(vU) \wedge \lambda_{1min}(v)) \\
&= (\lambda_{1min}^{-1}\lambda_{2min})(U) \\
&= (\lambda_1^{-1}\lambda_2)_{min}(U),
\end{aligned}$$

by Theorem 6.1. Thus $(\lambda_1^{-1}\lambda_2)_{min} = \gamma'\beta$. Hence the semiring S recognizes $\lambda_1^{-1}\lambda_2$. Similarly we can prove that $\lambda_2\lambda_1^{-1}$ is recognized by S. ☐

7. CONJUNCTIVE VARIETY OF L-FUZZY LANGUAGES

Here we define the conjunctive variety of l-fuzzy languages, a tool for classifying semiring recognizable l-fuzzy languages.

Definition 7.1. *Let $l\mathscr{C}$ be a family of l-fuzzy languages and $l\mathscr{C}_{min}$ be the family of associated generalized l-fuzzy languages. We say that $l\mathscr{C}$ is a conjunctive variety if $l\mathscr{C}_{min}$ is closed under finite meet, quotients and inverse homomorphic images.*

Theorem 7.2. *The class $lF(A^*)$ of l-fuzzy languages is a conjunctive variety.*

Proof. Follows from Theorems 4.2, 6.3 and 6.4. ☐

Let $lF(A^*)$ be a conjunctive variety of l-fuzzy languages. Then we define

$$lF^s = \{\text{Syn}\ (\lambda_{min}) : \lambda \in lF(A^*)\}.$$

Theorem 7.3. *lF^s is a pseudovariety of finite idempotent semirings.*

Proof. Let $S_1, S_2 \in lF^s$. Then there exist $\lambda_1, \lambda_2 \in lF(A^*)$ such that $S_1 = \text{Syn}(\lambda_{1min})$ and $S_2 = \text{Syn}\ (\lambda_{2min})$. Since $lF(A^*)$ is a conjunctive variety of l-fuzzy languages, $\lambda_{1min} \wedge \lambda_{2min}$ belongs to $lF_{min}(A^*)$. That is, $(\lambda_1 \wedge \lambda_2)_{min}$ belongs to $lF_{min}(A^*)$. Thus $\lambda_1 \wedge \lambda_2$ belongs to $lF(A^*)$. By Theorem 4.2, the l-fuzzy language $\lambda_1 \wedge \lambda_2$ is recognized by the idempotent semiring $S_1 \times S_2$. Thus $S_1 \times S_2$ belongs to lF^s and hence lF^s is closed under finite meet. By Theorem 5.3, lF^s is closed under substructures and homomorphic images. Thus lF^s is a pseudo variety of finite idempotent semirings. ☐

lF^s has the following property.

Theorem 7.4. *Let $lF_1(A^*)$ and $lF_2(A^*)$ be conjunctive varieties of l-fuzzy languages. Then if $lF_1(A^*) \subseteq lF_2(A^*)$ (subvarieties), then $lF_1^s(A^*) \subseteq lF_2^s(A^*)$.*

If $\mathscr{S}$ is a pseudo variety of idempotent semirings, then we define $\mathscr{S}^f = \{\lambda \mid \lambda \in lF(A^*)$ for some A and $\text{Syn}(\lambda_{min}) \in \mathscr{S}\}$.

Theorem 7.5. *$\mathscr{S}^f$ is a conjunctive variety of l-fuzzy languages.*

Proof. Let $\lambda_1, \lambda_2 \in \mathscr{S}^f$. Then there exist S_1, S_2 in $\mathscr{S}$ such that $S_1 = \text{Syn}(\lambda_{1min})$ and $S_2 = \text{Syn}(\lambda_{2min})$. Since $\mathscr{S}$ is a pseudovariety of idempotent semirings, $S_1 \times S_2$ belongs to $\mathscr{S}$. By Theorem 4.2, the idempotent semiring $S_1 \times S_2$ recognizes the l-fuzzy language $\lambda_1 \wedge \lambda_2$. Thus $\lambda_1 \wedge \lambda_2$ belongs to $\mathscr{S}^f$. Hence $(\lambda_1 \wedge \lambda_2)_{min} \in \mathscr{S}^f_{min}$. That is, $(\lambda_{1min} \wedge \lambda_{2min}) \in \mathscr{S}^f_{min}$. Thus $\mathscr{S}^f_{min}$ is closed under finite meet. By Theorem 6.4, S_2 recognizes the l-fuzzy languages $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}$. Thus $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}$ belong to $\mathscr{S}^f$. Hence $(\lambda_1^{-1}\lambda_2)_{min}$ and $(\lambda_2\lambda_1^{-1})_{min}$ belong to $\mathscr{S}^f_{min}$. That is, $\lambda_{1min}^{-1}\lambda_{2min},\ \lambda_{2min}\lambda_{1min}^{-1} \in \mathscr{S}^f_{min}$. Thus $\mathscr{S}^f_{min}$ is closed under quotients. Also by Theorem 6.3, $\mathscr{S}^f_{min}$ is closed under inverse homomorphic images. Thus $\mathscr{S}^f$ is a conjunctive variety of l-fuzzy languages. □

$\mathscr{S}^f$ has the following property.

Theorem 7.6. *If $\mathscr{S}_1$ is a pseudo subvariety of $\mathscr{S}_2$, then $\mathscr{S}_1^f \subseteq \mathscr{S}_2^f$.*

From Theorems 7.4 and 7.6, we have the following result.

Theorem 7.7. *Let $lF(A^*)$ be a conjunctive variety of l-fuzzy languages and $\mathscr{S}$ be a pseudo variety of idempotent semirings. Then*

(i) $lF^{sf} = lF$.

(ii) $\mathscr{S}^{fs} = \mathscr{S}$.

Proof. **(i)** By the definition of lF^{sf}, we have

$$\begin{aligned} \lambda \in lF^{sf} &\Leftrightarrow \text{ Syn } (\lambda_{min}) \in lF^s \\ &\Leftrightarrow \lambda \in lF. \end{aligned}$$

Thus $lF^{sf} = lF$.

(ii) By the definition of $\mathscr{S}^f$ and lF^s we have $\mathscr{S} \subseteq \mathscr{S}^{fs}$. Let S belongs to $\mathscr{S}^{fs}$, then there exists a l-fuzzy language λ in $\mathscr{S}^f$ such that $S = \text{Syn}(\lambda_{min})$. Since $\lambda \in \mathscr{S}^f$, $\text{Syn}(\lambda_{min}) \in \mathscr{S}$. Thus we get $\mathscr{S}^{fs} \subseteq \mathscr{S}$. Hence $\mathscr{S}^{fs} = \mathscr{S}$. □

8. VARIETY THEOREM

The following result gives an Eilenberg type variety theorem for semiring recognizable l-fuzzy languages.

Theorem 8.1. *The mappings $lF \longrightarrow lF^s$ and $\mathscr{S} \longrightarrow \mathscr{S}^f$ are mutually inverse lattice isomorphisms between the lattices of all conjunctive varieties of recognizable l-fuzzy languages and all pseudovarieties of finite idempotent semirings.*

Proof. By Theorems 7.3 and 7.5, lF^s is a pseudo variety of idempotent semirings and $\mathscr{S}^f$ is a conjunctive variety of l-fuzzy languages. If $lF_1 \subseteq lF_2$, then by Theorem 7.4, $lF_1^s \subseteq lF_2^s$. Also if $\mathscr{S}_1 \subseteq \mathscr{S}_2$, then by Theorem 7.6, $\mathscr{S}_1^f \subseteq \mathscr{S}_2^f$. By Theorem 7.7, we have $lF^{sf} = lF$and $\mathscr{S}^{fs} = \mathscr{S}$. Hence the theorem. □

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