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Haar Wavelet Based Numerical Method for the Solution of Multidimensional Stochastic Integral Equations

S. C. Shiralashetti*, Lata Lamani[1]

P. G. Department of Studies in Mathematics,
Karnatak University, Dharwad-580003, Karnataka, India.

Abstract

In this paper, we develop an accurate and efficient Haar wavelet based numerical method for the solution of Multi-dimensional Stochastic Integral equations. Initially, we study the properties of stochastic integral equations and Haar wavelets. Then, Haar wavelets operational matrix of integration and Haar wavelets stochastic operational matrix of integration are developed. Convergence and error analysis of the Stochastic Haar wavelet method is presented for the solution of Multi-dimensional Stochastic Integral equations. Efficiency of the proposed method is justified through the illustrative examples.

Keywords: Haar wavelets, Multi-dimensional Stochastic Integral equations, stochastic operational matrix of integration.

AMS Subject Classification: 65T60, 60H05.

1. INTRODUCTION

Wavelet is newly emerging area in the field of mathematics. Wavelets are extensively used for signal processing in communications and physics, and it is one of the best mathematical tools [4]. Integral equations are important tools in applied mathematics for describing knowledge models. Since in many cases, the exact solution of integral equations does not exist, the numerical approximation of integral equations become necessary, since in many cases the exact solution to these equations does not exist. There are various methods for approximating these equations and different basis functions [10] are used.

Stochastic integrals are used in modeling various phenomena in science, engineering and physics [12]. Many authors have studied the numerical approximation of stochastic integral equations. Some of them are Platen [1], Oksendal [3], Maleknejad et al [7], [13], Cortes et al [9], Douglas et al [12] and Zhang [8].

Haar wavelet is one of the most important families of Wavelets. It has become an important tool for solving differential equations, integral equations, integro-differential equations and much other class of equations. Lepik applied Haar wavelet method for solving integral equations [10, 11]. Shiralashetti et al. applied Haar wavelet collocation method for numerical solution of multi-term fractional differential equations [5] and numerical solution of initial value problems [6]. Solution of nonlinear oscillator equations, Stiff systems, regular Sturm-Liouville problems, etc. using Haar wavelets were studied by Bujurke et al[14, 15]. In 2012, Maleknejad et al applied block-pulse functions for the solution of multidimensional stochastic integral equations [13]. In this paper, we developed the numerical method for the solution of stochastic integral equations using Haar wavelets. Consider the stochastic Ito-volterra integral equation,

$$\mathrm{U(t) = g(t)} + \int_0^t k_0(s,t)\,\mathrm{U(s)ds} + \sum_{i=1}^{n}\int_0^t k_i(s,t)\,\mathrm{U(s)dB_i(s)}, \quad t \in [0,T) \tag{1.1}$$

where $U(t), g(t), k_0(s,t)$ and $k_i(s,t), i=1,2,...,n$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) and $U(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_i(s,t)U(s)dB_i(s),\ i=1,2,...,n$ ares the Ito integrals [3].

This paper is presented as follows. In section 2, we study some basic definitions arising in Stochastic calculus, properties of wavelets, Haar wavelets, Haar wavelets operational matrix of Integration and Haar wavelets stochastic operational matrix of integration. In section 3, Proposed method of solution for multi-dimensional stochastic integral equations is given. In section 4, we study the convergence and error analysis of the proposed method. In section 5, some examples are presented to show the efficiency of the presented method. Finally, in section 6, conclusion is drawn.

2. PROPERTIES OF STOCHASTIC CALCULUS AND WAVELETS

In this section, we discuss some basic definitions arising in Stochastic calculus, properties of Wavelets, Haar wavelets, Haar wavelets operational matrix of Integration and Haar wavelets stochastic operational matrix of integration. Finally, some results which will be used in further sections are mentioned.

* *Corresponding author*
e-mail id: shiralashettisc@gmail.com;

2.1. Stochastic calculus

Definition 2.1: *A standard Brownian motion defined on the interval* $[0,T)$ *is a random variable* $B(t)$ *which depends on* $t \in [0,T)$ *and satisfies the following conditions:*

- $B(0)=0$ with probability 1.
- For $0 \le s < t \le T$, the random variable given by increment $B(t)-B(s)$ is distributed normally with mean zero and variance $t-s$, equivalently, $B(t)-B(0)=\sqrt{t-s}N(0,1)$, where $N(0,1)$ is a random variable distributed normally with mean zero and variance 1.
- The increments $B(t)-B(s)$ and $B(v)-B(u)$ are independent for $0 \le s < t < u < v \le T$

  Let $p \ge 2$. Let us consider a random variable U with distribution f_U, so that

$$E[U^p] = \int_{-\infty}^{\infty} U^p f_U dY < \infty$$

Let $L^p(\Omega,H)$ be the collection of all strongly measurable, p- integrable and H-valued random variables. It is a routine to check that L^p -space is a Banach space with

$$\|V\|_{L^p(\Omega,H)} = \left[E\|V\|^p\right]^{\frac{1}{p}}$$

for each $V \in L^p(\Omega,H)$. Here we consider $p=2$ i.e, $L^2(\Omega, H)$.

Definition 2.2.1[2] *The sequence* U_n *converge to* U *in* L^2 *if for each* n, $E(|U_n|^2) < \infty$.

Let us assume that $0 \le s \le T$, let $v = v(s,T)$ be the class of functions that $g(t,w):[0,\infty]\times\Omega \to R^n$, satisfy ,

- the function $(t,w)\to g(t,w)$ is $\beta \times G$ measurable, where β is Borel algebra.
- g is adapted to G_t.
- $E\left[\int_s^T g(t,w)^2 dt\right] < \infty$.

Definition 2.3. *(The Ito-integral [3]) Let* $g \in v(s,T)$, *then the Ito-integral of* g *is defined by*

$$\int_s^T g(t,w)dB(t)(w) = \lim_{n\to\infty}\int_s^T \varphi_{t,w} dB(t)(w),$$

where, $\{\varphi\}$ is the sequence of elementary functions such that

$$E\left[\int_s^T (g-\varphi_n)^2 dt\right] \to 0 \text{ a.s}, \quad n\to\infty$$

2.2. Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right) \quad (2.1)$$

If we restrict the parameters a and b to discrete values as,

$$a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0$$

we have the following family of discrete wavelets,

$$\psi_{n,k}(t) = |a_0|^{-\frac{1}{2}}\psi(a_0^k t - nb_0)$$

where $\psi_{n,k}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{n,k}(x)$ forms an orthonormal basis.

2.3. Haar wavelets, Haar Wavelets Operational matrix of Integration and Stochastic operational matrix of Haar wavelets

2.3.1. Haar Wavelets

For the family of Haar wavelet the scaling function $h_0(t)$ is defined as

$$h_0(t) = \begin{cases} 1, & t \in (0,1] \\ 0, & \text{otherwise} \end{cases} \quad (2.2)$$

The Haar Wavelet family for $t \in [0,1)$ is defined as,

$$h_i(t) = \begin{cases} 1, & t \in (\alpha,\beta] \\ -1, & t \in (\beta,\gamma] \\ 0, & \text{elsewhere} \end{cases} \quad (2.3)$$

where $\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m}, \text{and } \gamma = \frac{k+1}{m}$

$m = 2^j$, $j = 0,1,...,J$, J is the level of resolution and $k = 0,1,...,m-1$ is the translation parameter. Maximum level of resolution is J. The index i in equation (2.3) is calculated using $i = m+k+1$. In case of minimal values, $m=1$, $k=0$ then $i=2$. The maximal value of i is $N = 2^{J+1}$.

Let us define the collocation points $t_l = \frac{l-0.5}{N}$, $l = 1,2,...,N$, then, the Haar coefficient matrix

$H(i,l) = h_i(t_l)$ has dimension $N \times N$. For instance, if $J = 2 \Rightarrow N = 8$, we have

$$H(8,8) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

2.3.2. Haar Wavelets Operational Matrix of Integration

Now, we write the operational matrix of integration for Haar wavelets.

By integrating equation (2.3), we get the operational matrix of integration as,

$$\int_0^t H(s)ds = PH(t) \tag{2.4}$$

where $H(t)$ is the Haar wavelet vector given by $[h_0(t), h_1(t), h_2(t), \ldots, h_{N-1}(t)]$

By using equation (2.2) P can be evaluated as,

$$P = \begin{cases} t - \dfrac{k}{m}, & t \in (\alpha, \beta] \\ \dfrac{k+1}{m} - t, & t \in (\beta, \gamma] \\ 0, & \text{elsewhere} \end{cases} \tag{2.5}$$

where $\alpha = \dfrac{k}{m}, \beta = \dfrac{k+0.5}{m}, \text{and } \gamma = \dfrac{k+1}{m}$.

For instance, if $J = 2 \Rightarrow N = 8$, we have

$$Ph(8,8) = (1/16)\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.3.3. Haar Wavelets Stochastic Operational Matrix of Integration

The Ito-Integral of Haar wavelets can be computed as follows:

$$\int_0^t H(s)dB(s) = P_s H(s) \tag{2.6}$$

where P_s is called Haar Wavelets Stochastic Operational Matrix of integration and is given by,

$$P_S = \begin{cases} B\left(t - \dfrac{k}{m}\right), & t \in (\alpha, \beta] \\ B\left(\dfrac{k+1}{m} - t\right), & t \in (\beta, \gamma] \\ 0, & \text{elsewhere} \end{cases} \tag{2.7}$$

For instance, for $N = 8$, using (2.7), we have

$$P_s(8,8) = \begin{bmatrix} B\left(\frac{1}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{5}{16}\right) & B\left(\frac{7}{16}\right) & B\left(\frac{9}{16}\right) & B\left(\frac{11}{16}\right) & B\left(\frac{13}{16}\right) & B\left(\frac{15}{16}\right) \\ B\left(\frac{1}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{5}{16}\right) & B\left(\frac{7}{16}\right) & B\left(\frac{7}{16}\right) & B\left(\frac{5}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{1}{16}\right) \\ B\left(\frac{1}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{1}{16}\right) & B(0) & B(0) & B(0) & B(0) \\ B(0) & B(0) & B(0) & B(0) & B\left(\frac{1}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{3}{16}\right) & B\left(\frac{1}{16}\right) \\ B\left(\frac{1}{16}\right) & B\left(\frac{1}{16}\right) & B(0) & B(0) & B(0) & B(0) & B(0) & B(0) \\ B(0) & B(0) & B\left(\frac{1}{16}\right) & B\left(\frac{1}{16}\right) & B(0) & B(0) & B(0) & B(0) \\ B(0) & B(0) & B(0) & B(0) & B\left(\frac{1}{16}\right) & B\left(\frac{1}{16}\right) & B(0) & B(0) \\ B(0) & B(0) & B(0) & B(0) & B(0) & B(0) & B\left(\frac{1}{16}\right) & B\left(\frac{1}{16}\right) \end{bmatrix}$$

where $B(0) = 0$ using definition (2.1).

Remark 2.1: *Using equation (2.3) for a N-vector G, we have*

$$H(t)H^T(t)G = \tilde{G}H(t) \tag{2.8}$$

where, $H(t)$ is the Haar wavelet coefficient matrix and $\tilde{G}$ is an $N \times N$ matrix given by

$$\tilde{G} = H\overline{G}H^{-1} \tag{2.9}$$

where $\overline{G} = diag(H^{-1}G)$. Also, for a $N \times N$ matrix X, we have

$$H^t XH(t) = \hat{X}^T H(t) \tag{2.10}$$

where, $\hat{X}^T = VH^{-1}$ and $V = diag(H^T XH)$ is a N-vector.

3. METHOD OF SOLUTION

Consider the Stochastic integral equation,

$$\begin{aligned} U(t) &= g(t) + \int_0^t k_0(s,t)\, U(s)ds \\ &\quad + \sum_{i=1}^{n} \int_0^t k_i(s,t)\, U(s)dB_i(s), \quad t \in [0,T) \end{aligned} \tag{3.1}$$

where $U(t), g(t), k_0(s,t)$ and $k_i(s,t), i = 1,2,...,n$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) and $U(t)$ is unknown. Also $B(t) = (B_1(s), B_2(s),..., B_n(s))$ is a multidimensional Brownian motion process and $\int_0^t k_i(s,t)\, U(s)\, dB_i(s),\ i = 1,2,...,n$ are the Ito integrals. We approximate $U(t),\ g(t),\ k_i(s,t), i = 1,2,...,n$ as follows:

$$U(t) \simeq U^T H(t) = UH^T(t) \tag{3.2}$$

$$g(t) \simeq G^T H(t) = GH^T(t) \tag{3.3}$$

$$\begin{aligned} k_i(s,t) &\simeq H^T(s) K_i H(t) \\ &= H^T(t) K_i^T H(s), i = 0,1.2....,n \end{aligned} \tag{3.4}$$

where U and G are Haar wavelet coefficient vectors and $K_i, i = 1,2,...,n$ are Haar wavelet matrices obtained on approximating $k_i(s,t), i = 1,2,...,n$ by using Haar wavelets. Substituting (3.2), (3.3), and (3.4) in (3.1), we have

$$\begin{aligned} U^T H(t) &\simeq G^T H(t) + H^T(t) K_0 \left(\int_0^t H(s) H^T(s) U ds \right) \\ &\quad + \sum_{i=1}^{n} H^T(t) K_i \left(\int_0^t H(s) H^T(s) U dB_i(s) \right) \end{aligned} \tag{3.5}$$

Using Haar Wavelets Operational matrix of Integration, Stochastic operational matrix of Haar wavelets and remark (2.1), we have

$$\begin{aligned} U^T H(t) &\simeq G^T H(t) + H^T(t) K_0 \tilde{U} P H(t) \\ &\quad + \sum_{i=1}^{n} H^T(t) K_i \tilde{U} P_s H(t) \end{aligned} \tag{3.6}$$

Using $V_0 = K_0 \tilde{U} P$ and $Y_1 = K_i \tilde{U} P_s, i = 1,2,...,n$ and using (2.1), we get

$$U^T H(t) \simeq G^T H(t) + \hat{V}_0^T H(t) + \sum_{i=1}^{n} \hat{V}_i^T H(t) \tag{3.7}$$

in $\hat{V}_i$, $i = 1,2,...,n$ and this gives,

$$U^T - \hat{V}_0^T - \sum_{i=1}^{n} \hat{V}_i^T \simeq G^T \tag{3.8}$$

where $\hat{V}_0$ and $\sum_{i=1}^{n} \hat{V}_i^T$, $i = 1,2,3,...,n$ are linear functions of the unknown vector U and (3.8) is a linear system of equations. Solving this linear system, we get the unknown vector U. Substituting this in (3.2), we get the solution of the stochastic integral equation (3.1).

4. CONVERGENCE AND ERROR ANALYSIS

In this section, we study the convergence and error analysis of the proposed method.

Theorem 2: Let $u(t) \in L^2[0,1)$ having the bounded first derivative on $[0,1)$ i.e, $\exists\ M > 0\ \forall t \in [0,1) \ni |u'(t)| \le M$. Then

$$\|E_N(t)\| = \|u(t) - u_i(t)\|_E \le \frac{M}{\sqrt{3}} \frac{1}{k} = O\left(\frac{1}{k}\right).$$

Proof: Let $u_i(t) = \sum_{i=0}^{N-1} u_i h_i(t)$,

where, $i = m + k + 1,\ N = 2^{J+1},\ j = 0,1,..., J$.

Now,

$$\begin{aligned} \|u(t) - u_i(t)\|_E^2 &= \int_0^1 (u(t) - u_i(t))^2 \\ &= \sum_{i=N}^{\infty} \sum_{i'=N}^{\infty} u_i u_{i'} \int_0^1 h_i(t) h_{i'}(t) dt \\ &= \sum_{i=N}^{\infty} u_i^2 \end{aligned} \tag{4.1}$$

We have $h_i(t) = 2^{\frac{j}{2}} h(2^j t - k),\ k = 0,1,..., m-1,\ m = 2^j$.

Therefore,

$$u_i = \int_0^1 2^{\frac{j}{2}} u(t) h(2^j t - k) dt$$

From (2.3),

$$h(2^j t - k) = \begin{cases} 1, & \frac{k}{2^j} \le t < \frac{k+0.5}{2^j} \\ -1, & \frac{k+0.5}{2^j} \le t < \frac{k+1}{2^j} \\ 0, & \text{elsewhere} \end{cases}$$

And therefore,

$$u_i = 2^{\frac{j}{2}}\left(\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} u(t)dt + \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} u(t)dt \right)$$

By mean value theorem, we have

$$\exists\, t_1,\, t_2 \,:\, \frac{k}{2^j} \le t_1 < \frac{k+0.5}{2^j},\ \frac{k+0.5}{2^j} \le t_2 < \frac{k+1}{2^j}$$

such that

$$u_i = 2^{\frac{j}{2}}\left\{\left(\left(\frac{k+0.5}{2^j}\right)-\left(\frac{k}{2^j}\right)\right)u(t_1)-\left(\left(\frac{k+1}{2^j}\right)-\left(\frac{k+0.5}{2^j}\right)\right)u(t_2)\right\}$$
$$= 2^{\frac{-j}{2}-1}\left(u(t_1)-u(t_2)\right)$$

Applying mean value theorem, we have

$$u_i^2 = 2^{-j-2}\left(t_2-t_1\right)^2 u'^2\left(t_0\right) \qquad (t_1 < t_0 < t_2)$$
$$\le 2^{-j-2}.2^{-2j}.M^2$$
$$= 2^{-3j-2}.M^2$$

From (4.1), we have

$$\|u(t)-u_i(t)\|_E^2 = \sum_{i=N}^{\infty} u_i^{\ 2}$$
$$= \sum_{j=J+1}^{\infty}\left(\sum_{i=m}^{2^{j+1}-1} u_i^2\right)$$
$$\le M^2 \sum_{j=J+1}^{\infty} 2^{-3j-2}\left(2^{j+1}-1-2^j+1\right)$$
$$= \frac{M^2}{3}\frac{1}{k^2}$$

Therefore,

$$\|E_N(t)\| = \|u(t)-u_i(t)\|_E \le \frac{M}{\sqrt{3}}\frac{1}{k} = O\left(\frac{1}{k}\right)$$

This completes the proof.

5. NUMERICAL EXPERIMENTS

In this section we consider some examples to show the efficiency of the proposed method.

Test Problem 1.[13] Let us consider the following three-dimnsional Stochastic integral equation,

$$U(t) = \frac{1}{12} + \int_0^t r(s)U(s)ds + \sum_{i=1}^{3}\int_0^t \alpha_i(s)U(s)dB_i(s),\ s,t \in [0,1) \qquad (5.1)$$

where, $r(s) = s^2, \alpha_1(s) = \sin(s), \alpha_2(s) = \cos(s)$ and $\alpha_3(s) = s$. Exact Solution of this Stochastic integral equation is

$$U(t) = \frac{1}{12}\exp\left(\int_0^t\left(r(s)-\frac{1}{2}\sum_{i=1}^{3}\alpha_i^2(s)\right)ds + \sum_{i=1}^{3}\int_0^t \alpha_i(s)dB_i(s)\right) \qquad (5.2)$$

where $U(t)$ is the unknown three-dimensional stochastic process defined on the probability space (Ω, F, P), and $(B_1(t), B_2(t), B_3(t))$ is the three-dimensional Brownian motion process.

Method of implementation for N=16.

Firstly, we approximate

$$g(t) = GH^T(t) \qquad (5.3)$$

$$U(t) = UH^T(t) \qquad (5.4)$$

$$k_i(s,t) \simeq H^T(s)K_iH(t)$$
$$= H^T(t)K_i^TH(s), i = 0,1.2....,n \qquad (5.5)$$

where K_i's, $i = 0,1.2....,n$ are $N \times N$ matrices. Substituting these $g(t)$, $U(t)$, $k_i(s,t)$, $i = 0,1.2....,n$ in (5.1) and using the collocation points $t_i = \frac{i-0.5}{N}$, we get N algebraic equations with N unknowns. Solving these N algebraic equations we get the N unknowns.

For instance $N = 16$, we get,

$$G = \begin{bmatrix} 0.8333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.6)$$

$$K_0 = \begin{bmatrix}
0.020813 & -0.020813 & 0 & -0.041626 & 0 & 0 & 0 & -0.083252 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1665 \\
-0.015625 & 0.015625 & 0 & 0.03125 & 0 & 0 & 0 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.125 \\
-0.0039062 & 0.0039062 & 0 & 0.0078125 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03125 \\
-0.011719 & 0.011719 & 0 & 0.023438 & 0 & 0 & 0 & 0.046875 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.09375 \\
-0.00097656 & 0.00097656 & 0 & 0.0019531 & 0 & 0 & 0 & 0.0039062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0078125 \\
-0.00097656 & 0.00097656 & 0 & 0.0019531 & 0 & 0 & 0 & 0.0039062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0078125 \\
-0.0029297 & 0.0029297 & 0 & 0.0058594 & 0 & 0 & 0 & 0.011719 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.023438 \\
-0.0048828 & 0.0048828 & 0 & 0.0097656 & 0 & 0 & 0 & 0.019531 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.039062 \\
-0.0068359 & 0.0068359 & 0 & 0.013672 & 0 & 0 & 0 & 0.027344 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.054688 \\
-0.00024414 & 0.00024414 & 0 & 0.00048828 & 0 & 0 & 0 & 0.00097656 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0019531 \\
-0.00073242 & 0.00073242 & 0 & 0.0014648 & 0 & 0 & 0 & 0.0029297 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0058594 \\
-0.0012207 & 0.0012207 & 0 & 0.0024414 & 0 & 0 & 0 & 0.0048828 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0097656 \\
-0.001709 & 0.001709 & 0 & 0.003418 & 0 & 0 & 0 & 0.0068359 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.013672 \\
-0.0021973 & 0.0021973 & 0 & 0.0043945 & 0 & 0 & 0 & 0.0087891 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.017578 \\
-0.0026855 & 0.0026855 & 0 & 0.0053711 & 0 & 0 & 0 & 0.010742 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.021484 \\
-0.0031738 & 0.0031738 & 0 & 0.0063477 & 0 & 0 & 0 & 0.012695 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.025391 \\
-0.0036621 & 0.0036621 & 0 & 0.0073242 & 0 & 0 & 0 & 0.014648 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.029297
\end{bmatrix} \quad 5.7)$$

$$K_1 = \begin{bmatrix}
0.028736 & -0.028736 & 0 & -0.057472 & 0 & 0 & 0 & -0.11494 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.22989 \\
-0.013431 & 0.013431 & 0 & 0.026862 & 0 & 0 & 0 & 0.053724 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10745 \\
-0.0075315 & 0.0075315 & 0 & 0.015063 & 0 & 0 & 0 & 0.030126 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.060252 \\
-0.0056875 & 0.0056875 & 0 & 0.011375 & 0 & 0 & 0 & 0.02275 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0455 \\
-0.0038714 & 0.0038714 & 0 & 0.0077427 & 0 & 0 & 0 & 0.015485 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.030971 \\
-0.0036307 & 0.0036307 & 0 & 0.0072613 & 0 & 0 & 0 & 0.014523 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.029045 \\
-0.0031642 & 0.0031642 & 0 & 0.0063284 & 0 & 0 & 0 & 0.012657 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.025314 \\
-0.002501 & 0.002501 & 0 & 0.0050021 & 0 & 0 & 0 & 0.010004 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.020008 \\
-0.001949 & 0.001949 & 0 & 0.003898 & 0 & 0 & 0 & 0.007796 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015592 \\
-0.0019186 & 0.0019186 & 0 & 0.0038372 & 0 & 0 & 0 & 0.0076743 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015349 \\
-0.0018582 & 0.0018582 & 0 & 0.0037165 & 0 & 0 & 0 & 0.0074329 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.014866 \\
-0.0017689 & 0.0017689 & 0 & 0.0035378 & 0 & 0 & 0 & 0.0070755 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.014151 \\
-0.0016519 & 0.0016519 & 0 & 0.0033039 & 0 & 0 & 0 & 0.0066077 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.013215 \\
-0.0015092 & 0.0015092 & 0 & 0.0030184 & 0 & 0 & 0 & 0.0060368 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.012074 \\
-0.0013429 & 0.0013429 & 0 & 0.0026858 & 0 & 0 & 0 & 0.0053717 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.010743 \\
-0.0011557 & 0.0011557 & 0 & 0.0023114 & 0 & 0 & 0 & 0.0046227 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0092454
\end{bmatrix} \quad (5.8)$$

$$K_1 = \begin{bmatrix}
0.0526 & -0.0526 & 0 & -0.1052 & 0 & 0 & 0 & -0.2104 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4208 \\
0.0073374 & -0.0073374 & 0 & -0.014675 & 0 & 0 & 0 & -0.02935 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0587 \\
0.0019231 & -0.0019231 & 0 & -0.0038462 & 0 & 0 & 0 & -0.0076924 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.015385 \\
0.0052985 & -0.0052985 & 0 & -0.010597 & 0 & 0 & 0 & -0.021194 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.042388 \\
0.00048646 & -0.00048646 & 0 & -0.00097291 & 0 & 0 & 0 & -0.0019458 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0038916 \\
0.0014291 & -0.0014291 & 0 & -0.0028582 & 0 & 0 & 0 & -0.0057165 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.011433 \\
0.0022829 & -0.0022829 & 0 & -0.0045659 & 0 & 0 & 0 & -0.0091317 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.018263 \\
0.0029948 & -0.0029948 & 0 & -0.0059896 & 0 & 0 & 0 & -0.011979 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.023958 \\
0.00012197 & -0.00012197 & 0 & -0.00024394 & 0 & 0 & 0 & -0.00048788 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00097577 \\
0.00036401 & -0.00036401 & 0 & -0.00072802 & 0 & 0 & 0 & -0.001456 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0029121 \\
0.00060037 & -0.00060037 & 0 & -0.0012007 & 0 & 0 & 0 & -0.0024015 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0048029 \\
0.00082736 & -0.00082736 & 0 & -0.0016547 & 0 & 0 & 0 & -0.0033094 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0066189 \\
0.0010414 & -0.0010414 & 0 & -0.0020829 & 0 & 0 & 0 & -0.0041657 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0083315 \\
0.0012393 & -0.0012393 & 0 & -0.0024785 & 0 & 0 & 0 & -0.0049571 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0099141 \\
0.0014178 & -0.0014178 & 0 & -0.0028355 & 0 & 0 & 0 & -0.005671 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.011342 \\
0.0015741 & -0.0015741 & 0 & -0.0031482 & 0 & 0 & 0 & -0.0062965 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.012593
\end{bmatrix} \quad (5.9)$$

$$K_3 = \begin{bmatrix} 0.03125 & -0.03125 & 0 & -0.0625 & 0 & 0 & 0 & -0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.25 \\ -0.015625 & 0.015625 & 0 & 0.03125 & 0 & 0 & 0 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.125 \\ -0.0078125 & 0.0078125 & 0 & 0.015625 & 0 & 0 & 0 & 0.03125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0625 \\ -0.0078125 & 0.0078125 & 0 & 0.015625 & 0 & 0 & 0 & 0.03125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0625 \\ -0.0039062 & 0.0039062 & 0 & 0.0078125 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03125 \\ -0.0039062 & 0.0039062 & 0 & 0.0078125 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03125 \\ -0.0039062 & 0.0039062 & 0 & 0.0078125 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03125 \\ -0.0039062 & 0.0039062 & 0 & 0.0078125 & 0 & 0 & 0 & 0.015625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03125 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \\ -0.0019531 & 0.0019531 & 0 & 0.0039062 & 0 & 0 & 0 & 0.0078125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015625 \end{bmatrix} \quad (5.10)$$

Substituting (5.6), (5.7), (5.8), (5.9) and (5.10) in (5.1) and solving, we get,

$$1.0001\mathrm{u}_1 - 0.000069972\mathrm{u}_2 - 0.00013994\mathrm{u}_4 - 0.00027989\mathrm{u}_8 - 0.00055978\mathrm{u}_{16} = 0.0833$$

$$1.0\mathrm{u}_2 - 0.000034675\mathrm{u}_1 + 0.00006935\mathrm{u}_4 + 0.0001387\mathrm{u}_8 + 0.0002774\mathrm{u}_{16} = 0$$

$$0.0000041449\mathrm{u}_2 - 0.0000041449\mathrm{u}_1 + \mathrm{u}_3 + 0.0000082897_{\mathrm{u}}4 + 0.000016579\mathrm{u}_8 + 0.000033159\mathrm{u}_{16} = 0$$

$$0.000034032\mathrm{u}_2 - 0.000034032\mathrm{u}_1 + 1.0001\mathrm{u}_4 + 0.00013613\mathrm{u}_8 + 0.00027226\mathrm{u}_{16} = 0$$

$$0.000005591\mathrm{u}_2 - 0.000005591\mathrm{u}_1 + 0.000011182\mathrm{u}_4 + \mathrm{u}_5 + 0.000022364\mathrm{u}_8 + 0.000044728\mathrm{u}_{16} = 0$$

$$0.0000003406\mathrm{u}_1 - 0.0000003406\mathrm{u}_2 - 0.0000006812\mathrm{u}_4 + \mathrm{u}_6 - 0.0000013624\mathrm{u}_8 - 0.0000027248\mathrm{u}_{16} = 0$$

$$0.000001891\mathrm{u}_2 - 0.000001891\mathrm{u}_1 + 0.000003782\mathrm{u}_4 + \mathrm{u}_7 + 0.0000075641\mathrm{u}_8 + 0.000015128\mathrm{u}_{16} = 0$$

$$0.000031699\mathrm{u}_2 - 0.000031699\mathrm{u}_1 + 0.000063398\mathrm{u}_4 + 1.0001\mathrm{u}_8 + 0.00025359\mathrm{u}_{16} = 0$$

$$0.00047439\mathrm{u}_2 - 0.00047439\mathrm{u}_1 + 0.00094878\mathrm{u}_4 + 0.0018976\mathrm{u}_8 + \mathrm{u}_9 + 0.0037951\mathrm{u}_{16} = 0$$

$$0.00088763\mathrm{u}_2 - 0.00088763\mathrm{u}_1 + 0.0017753\mathrm{u}_4 + 0.0035505\mathrm{u}_8 + \mathrm{u}_{10} + 0.0071011\mathrm{u}_{16} = 0$$

$$0.0012164\mathrm{u}_2 - 0.0012164\mathrm{u}_1 + 0.0024328\mathrm{u}_4 + 0.0048657\mathrm{u}_8 + \mathrm{u}_{11} + 0.0097313\mathrm{u}_{16} = 0$$

$$0.0014913\mathrm{u}_2 - 0.0014913\mathrm{u}_1 + 0.0029825\mathrm{u}_4 + 0.005965\mathrm{u}_8 + \mathrm{u}_{12} + 0.01193\mathrm{u}_{16} = 0$$

$$0.0032522\mathrm{u}_2 - 0.0032522\mathrm{u}_1 + 0.0065043\mathrm{u}_4 + 0.013009\mathrm{u}_8 + \mathrm{u}_{13} + 0.026017\mathrm{u}_{16} = 0$$

$$0.0042691\mathrm{u}_2 - 0.0042691\mathrm{u}_1 + 0.0085381\mathrm{u}_4 + 0.017076\mathrm{u}_8 + \mathrm{u}_{14} + 0.034152\mathrm{u}_{16} = 0$$

$$0.0079222\mathrm{u}_2 - 0.0079222\mathrm{u}_1 + 0.015844\mathrm{u}_4 + 0.031689\mathrm{u}_8 + \mathrm{u}_{15} + 0.063378\mathrm{u}_{16} = 0$$

$$0.0070371\mathrm{u}_2 - 0.0070371\mathrm{u}_1 + 0.014074\mathrm{u}_4 + 0.028148\mathrm{u}_8 + 1.0563\mathrm{u}_{16} = 0$$

since $U = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8 \quad u_9 \quad u_{10} \quad u_{11} \quad u_{12} \quad u_{13} \quad u_{14} \quad u_{15} \quad u_{16}]$

where, $u_1, u_2, u_3, u_4, u_5, u_6, u_7\ u_8, u_{9,}\ u_{10,}\ , u_{11}, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ and u_{16} are the unknowns to determined. Solving the above linear system of equations, we get

$$\begin{aligned} \mathrm{U} = [&0.083328 \quad 7.0007\mathrm{e}-05 \quad 9.5938\mathrm{e}-05 \quad 0.00011761 \quad 0.0002565 \\ &0.0003367 \quad 0.00062482 \quad 0.00055501 \quad 2.7348\mathrm{e}-06 \quad 3.269\mathrm{e}-07 \\ &2.6841\mathrm{e}-06 \quad 4.4096\mathrm{e}-07 \quad -2.6863\mathrm{e}-08 \quad 1.4914\mathrm{e}-07 \quad 2.5001\mathrm{e}-06 \quad 3.7415\mathrm{e}-05] \end{aligned}$$

Substituting this in (5.4), we get

$$U = [0.085431 \quad 0.085338 \quad 0.081838 \quad -3.3851e-05 \quad 0.083184 \quad -0.00058664 \quad -6.3365e-08 \quad -3.9793e-05 \quad 0.083258 \quad -2.1677e-05 \quad -8.0202e-05 \quad 6.9814e-05 \quad 2.4079e-06 \quad 2.2431e-06 \quad -1.7601e-07 \quad -3.4915e-05]$$

Using (5.2), we get the exact solution for the collocation points $t_i = \dfrac{i-0.5}{N}$, $i = 1, 2, ..., N$ is as follows

$$U = [0.054617 \quad 0.01829 \quad 0.007051 \quad 0.0023266 \quad 0.000771 \quad 0.00023155 \quad 5.9224e-05 \quad 2.5059e-05 \quad 4.3426e-06 \quad 5.9009e-07 \quad 8.5827e-08 \quad 2.5351e-08 \quad 6.818e-09 \quad 1.0125e-09 \quad 4.7998e-10 \quad 2.7111e-10]$$

Table 1 shows the exact, approxmate and absolute error for test problem 1 for $N = 64$ and table 2 presents the maximum absolute value for different values of N .Figure 1 shows the graph exact and approximate values for $N = 128$ for test problem 1.

Table 1: Comparison of Exact solution, Approximate solution and Absolute error for Test Problem 1 for $N = 64$.

t	Exact	Approximate	Absolute Error
0	0.083333333333333	0.009085463738134	0.074247869595199
0.1	0.003057195000000	-0.000170233570000	0.003227428570000
0.2	0.000181340800000	-0.000048727070000	0.000230067870000
0.3	0.000014440500000	-0.000000040361842	0.000014480861842
0.4	0.000000953032400	-0.000007352625000	0.000008305657400
0.5	0.000000067329150	0.041819729000000	0.041819661670850
0.6	0.000000017442560	-0.000000015274910	0.000000032717470
0.7	0.000000001869459	-0.000000952224200	0.000000954093659
0.8	0.000000000186807	-0.000001423197000	0.000001423383807
0.9	0.000000000150200	-0.000004805537000	0.000004805687200

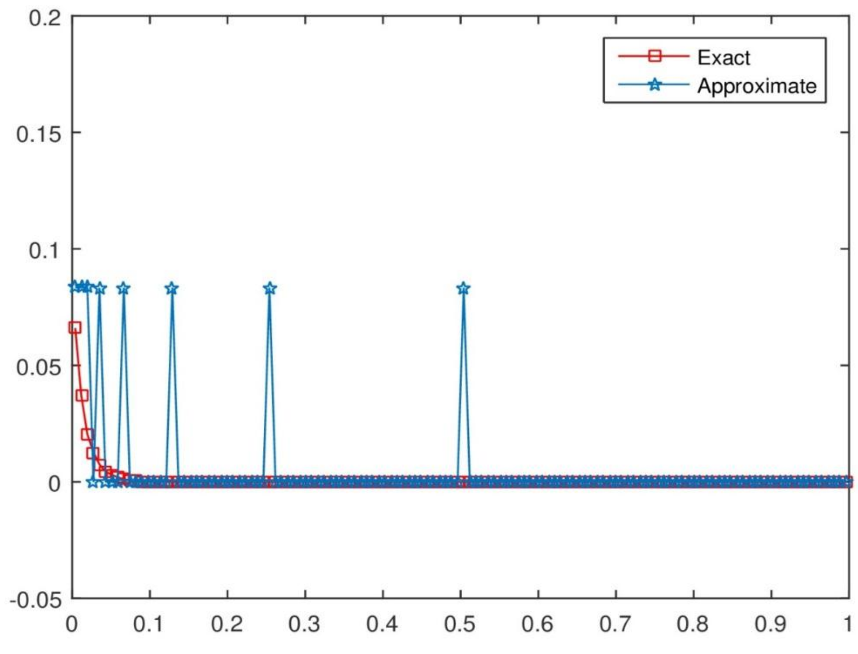

Fig 1: Graphical comparison of exact and approximate solution of test problem 1 for N=128

Table 2: Absolute Errors for different values of N for Test Problem 1.

N	Maximum Absolute Error (E_{max})
16	0.083253
32	0.083323
64	0.08333
128	0.08333

Test Problem 2. Let us consider the following three-dimnsional Stochastic integral equation,

$$U(t) = \frac{1}{12} + \int_0^t r(s,t)U(s)ds + \sum_{i=1}^{3}\int_0^t \alpha_i(s,t)U(s)dB_i(s), \quad s,t \in [0,1) \tag{5.11}$$

where, $r(s,t) = (s+t)^2, \alpha_1(s,t) = \sin(s+t), \alpha_2(s,t) = \cos(s+t)$ and $\alpha_3(s,t) = s+t$. Exact Solution of this Stochastic integral equation is

$$U(t) = \frac{1}{12} exp\left(\int_0^t \left(r(s,t) - \frac{1}{2}\sum_{i=1}^{3}\alpha_i^2(s,t)\right)ds + \sum_{i=1}^{3}\int_0^t \alpha_i(s,t)dB_i(s)\right) \tag{5.12}$$

where $U(t)$ is the unknown three-dimensional stochastic process defined on the probability space (Ω, F, P), and $(B_1(t), B_2(t), B_3(t))$ is the three-dimensional Brownian motion process.

Proceeding as above, we get

$$U = [0.082622 \quad -0.0014984 \quad -0.001327 \quad -0.00069811 \quad 0.000060018 \quad 0.0016352 \quad 0.0033605 \quad 0.0061549$$
$$-0.00030558 \quad -0.00067661 \quad -0.00068702 \quad -0.0010006 \quad -0.0011959 \quad -0.00090803 \quad 0.00033781 \quad -0.0015068]$$

and finally we get solution as,

$$U = [0.0843668 \quad 0.0962524 \quad 0.0678882 \quad 0.000603125 \quad 0.0831491 \quad -0.00782021 \quad 0.000705472 \quad -0.00093492$$
$$0.0841207 \quad -0.000628867 \quad -0.00157521 \quad -0.00279437 \quad 0.000371023 \quad 0.00031362 \quad -0.000287895 \quad 0.00184464]$$

Using (5.16), we get the exact solution for the collocation points $t_i = \frac{i-0.5}{N}$, $i = 1, 2, ..., N$ as

$$U = [0.05432 \quad 0.017856 \quad 0.0067464 \quad 0.002048 \quad 0.00059906 \quad 0.00015118 \quad 0.000026721 \quad 9.6772e-6$$
$$9.792e-7 \quad 6.6263e-8 \quad 4.0992e-9 \quad 5.8171e-10 \quad 7.3917e-11 \quad 3.1719e-12 \quad 4.3435e-13 \quad 8.1027e-14]$$

Table 3 shows the exact, approximate and absolute error for test problem 2 for $N = 64$ and table 4 presents the maximum absolute value for different values of N. Figure 2 shows the graph exact and approximate values for $N = 128$ for test problem 2.

Table 3: Comparison of Exact solution, Approximate solution and Absolute error for Test Problem 4 for $N = 64$.

t	Exact	Approximate	Absolute Error
0	0.08333	0.00924612	0.074088
0.1	0.000118	-0.0034307403	0.003549
0.2	2.5873484e-07	-0.00089326092	0.000893
0.3	1.799083e-10	-0.000136978344	0.000137
0.4	9.130491e-14	-0.0002785063	0.000278
0.5	1.8894825e-17	0.043216155	0.043216
0.6	1.0567269e-21	-2.0356538e-05	2.035653e-05
0.7	1.2635861e-26	-0.00012155036	0.0001215
0.8	4.124415e-32	-6.387428e-05	6.387428e-05
0.9	1.77009273e-39	-0.0001444809	0.000144

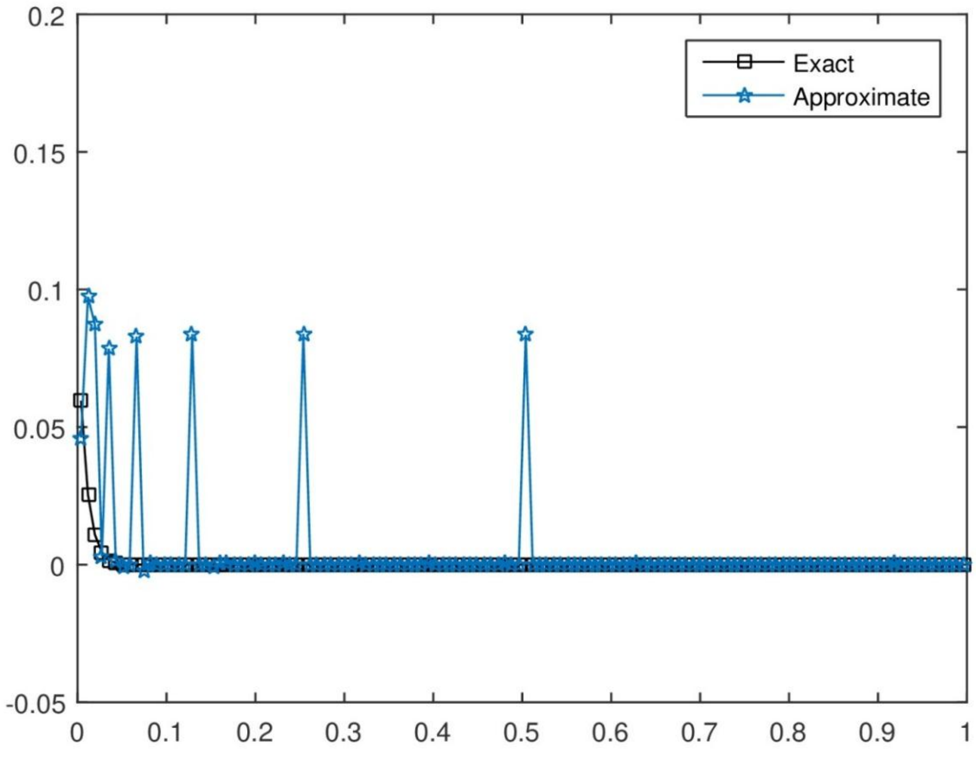

Fig 2: Graphical comparison of exact and approximate solution of test problem 2 for N=128

Table 4: Absolute Errors for different values of N for Test Problem 2.

N	Maximum Absolute Error (E_{max})
16	0.08412
32	0.08410
64	0.084885
128	0.083978

Test Problem 3. We consider another stochastic integral equation

$$U(t) = g + \int_0^t rU(s)ds + \sum_{i=1}^{4}\int_0^t \alpha_i U(s)dB_i(s), \quad s,t \in [0,1) \tag{5.13}$$

where $U(t)$ is the unknown four-dimensional stochastic process defined on the probability space (Ω, F, P), and $(B_1(t), B_2(t), B_3(t), B_4(t))$ is the four-dimensional Brownian motion process.

Case I: Let $g = \frac{1}{200}$, $r = \frac{1}{20}$, $\alpha_1 = \frac{1}{50}, \alpha_2 = \frac{2}{50}, \alpha_3 = \frac{4}{50}$ and $\alpha_4 = \frac{9}{50}$. Exact Solution to this Stochastic integral equation is

$$U(t) = \frac{1}{200} exp\left(\left(\frac{1}{20} - \frac{1}{2}\sum_{i=1}^{4}\alpha_i^2\right)t + \sum_{i=1}^{4}\alpha_i B_i(t)\right) \tag{5.14}$$

Proceeding as above, we get

$U =$[0.005 0.0000069367 0.0000077935 0.0000086539 0.0000095205 0.000010393
0.000011268 0.000012143 0.00000053502 0.0000012411 0.0000019966
0.0000027828 0.0000035905 0.0000044115 0.0000052424 0.0000060845]

and finally we get solution as,

$U =$[0.00502474 0.00502367 0.000064 0.00500392 0.00501148 0.00500749
0.00499148 0.00498592 0.00501653 0.00500934 0.00499481
0.00498599 0.00500175 0.00499126 0.0049783 0.00496613].

Using (5.14), we get the exact solution for the collocation points $t_i = \frac{i - 0.5}{N}$, $i = 1, 2, ..., N$ as

$U =$[0.0048142 0.0044668 0.0041587 0.0043082 0.0048269 0.0052459
0.0055221 0.0063425 0.0074367 0.0081671 0.0079893
0.0076211 0.0074787 0.0068641 0.006319 0.0060238]

Table 5 shows the exact, approximate and absolute error for test problem 3 case I for $N = 64$ and table 6 presents the maximum absolute value for different values of *N*. Figure 3 shows the graph exact and approximate values for $N = 128$

Table 5: Comparison of Exact solution, Approximate solution and Absolute error for Test Problem 2 for $N = 64$.

t	Exact	Approximate	Absolute Error
0	0.005	0.005001675	0.000001675
0.1	0.004714704	0.005018245	0.000303541
0.2	0.004779895	0.005017277	0.000237382
0.3	0.004441919	0.005000602	0.000558683
0.4	0.004503195	0.00499582	0.000492625
0.5	0.00429019	0.005006055	0.000715865
0.6	0.003650508	0.005005107	0.001354599
0.7	0.003657457	0.005004682	0.001347225
0.8	0.003348398	0.004984532	0.001636134
0.9	0.003208872	0.004982866	0.001773994

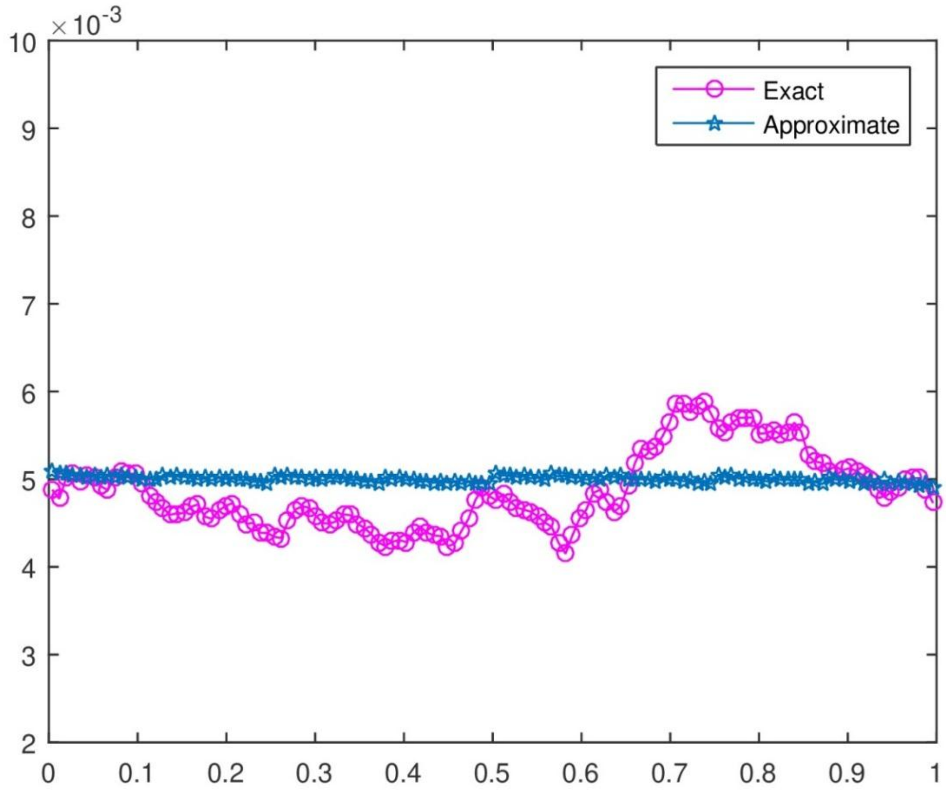

Fig 3: Graphical comparison of exact and approximate solution of test problem 3(Case I) for N=128

Table 6: Absolute Errors for different values of N for Case 1.

N	Maximum Absolute Error (E_{max})
16	0.0032
32	0.0024
64	0.0018
128	9.2206e-4

Case II: Let $g = \frac{1}{400}$, $r = \frac{1}{40}$, $\alpha_1 = \frac{1}{40}, \alpha_2 = \frac{2}{40}, \alpha_3 = \frac{4}{40}$ and $\alpha_4 = \frac{9}{40}$. Exact Solution to this Stochastic integral equation is

$$U(t) = \frac{1}{400} exp\left(\left(\frac{1}{40} - \frac{1}{2}\sum_{i=1}^{4}\alpha_i^2\right)t + \sum_{i=1}^{4}\alpha_i B_i(t)\right) \quad (5.15)$$

where $U(t)$ is the unknown four-dimensional stochastic process defined on the probability space (Ω, F, P), and $(B_1(t), B_2(t), B_3(t), B_4(t))$ is the four-dimensional Brownian motion process.

Proceeding as above, we get

$$U = [0.0024998 \quad 8.5405e-7 \quad 1.0229e-6 \quad 1.1954e-6 \quad 1.3708e-6 \quad 1.5491e-6 \quad 1.7296e-6 \quad 1.9124e-6$$
$$-2.1601e-7 \quad -1.413e-7 \quad -3.4799e-8 \quad 9.0168e-8 \quad 2.2794e-7 \quad 3.749e-7 \quad 5.2919e-7 \quad 6.8911e-7]$$

and finally we get solution as,

$$U = [0.00250282 \quad 0.00250325 \quad 0.00250015 \quad 0.00250043 \quad 0.00250113 \quad 0.0025012 \quad 0.00249816 \quad 0.00249798$$
$$0.00250208 \quad 0.00250163 \quad 0.00249877 \quad 0.00249802 \quad 0.00250018 \quad 0.00249912 \quad 0.00249651 \quad 0.00249513]$$

Using (5.15), we get the exact solution for the collocation points $t_i = \frac{i-0.5}{N}$, $i = 1, 2, ..., N$ as

$$U = [0.002828 \quad 0.003378 \quad 0.0036944 \quad 0.0039131 \quad 0.0037388 \quad 0.0034489 \quad 0.0037546 \quad 0.0040631$$
$$0.0042555 \quad 0.0044919 \quad 0.0042637 \quad 0.0039142 \quad 0.0040318 \quad 0.0040473 \quad 0.003886 \quad 0.0040557]$$

Table 7 shows the exact, approximate and absolute error for test problem 3 case II for $N = 64$ and table 6 presents the maximum absolute value for different values of *N*. Figure 3 shows the graph exact and approximate values for $N = 128$

Table 7: Comparison of Exact solution, Approximate solution and Absolute error for Test Problem 3 for $N = 64$

t	Exact	Approximate	Absolute Error
0	0.005	0.002501054	0.00000105
0.1	0.00251811	0.002510808	0.00000731
0.2	0.00250846	0.002510071	0.00000161
0.3	0.00214671	0.002500746	0.00035403
0.4	0.00221171	0.002497624	0.00028591
0.5	0.00193488	0.002503535	0.00056865
0.6	0.00210954	0.002502825	0.00039328
0.7	0.00201032	0.002502372	0.00049205
0.8	0.00185816	0.002491277	0.00063311
0.9	0.00181232	0.002490202	0.00067787

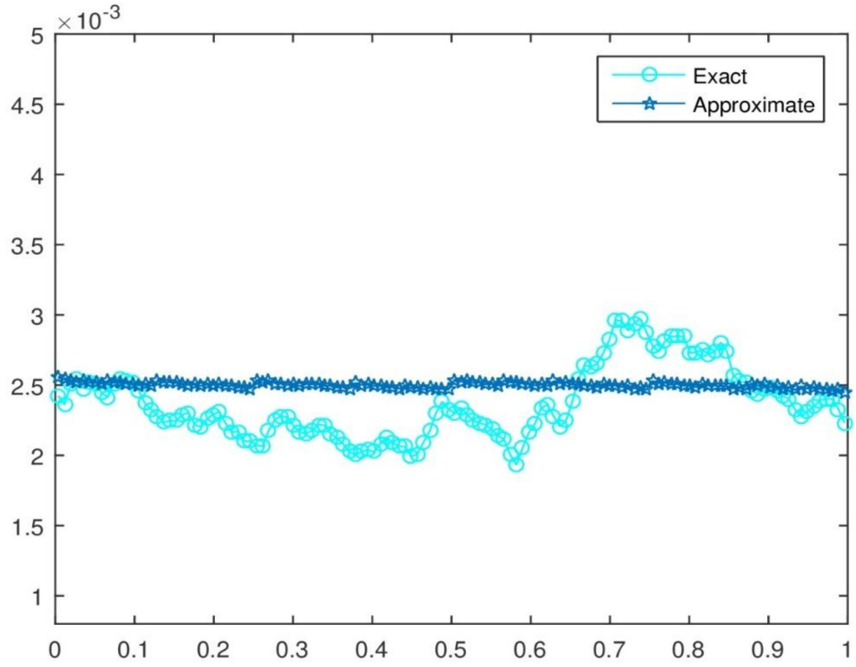

Fig 4: Graphical comparison of exact and approximate solution of test problem 3(Case II) for N=128

Table 8: Absolute Errors for different values of N for Case 2.

N	Maximum Absolute Error (E_{max})
16	0.0019903
32	7.8120e-4
64	7.0239e-4
128	5.9261e-4

6. CONCLUSION

In this paper, we derive a new Haar wavelets stochastic operational matrix of integration to solve multi-dimensional stochastic integral equations. The tables show that the numerical solution obtained by proposed method agrees the exact solution. Also, we find the absolute error to show the efficiency of the proposed method. Hence, the Haar wavelet method is effective for solving multi-dimensional stochastic integral equations.

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REFERENCES

[1]. Kloeden, P. E. and Platen, E., 2013, "Numerical solution of stochastic differential equations" , Springer Science & Business,23.

[2]. Klebaner, F. C., 2012, "Introduction to stochastic calculus with applications", World Scientific Publishing Company.

[3]. Oksendal, B., 2013, "Stochastic differential equations: an introduction with applications", Springer Science & Business Media.

[4]. Strang, G., 1989, "Wavelets and dilation equations: A brief introduction", SIAM review, 31(4), pp.614-627.

[5]. Shiralashetti, S. C. and Deshi, A. B., 2016, "An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations", Nonlinear dynamics, 83(1-2), pp.293-303.

[6]. Shiralashetti, S. C., Deshi, A.B. and Desai, P.M., 2016, "Haar wavelet collocation method for the numerical solution of singular initial value problems", Ain Shams Engineering Journal, 7(2), pp.663-670.

[7]. Maleknejad, K., Khodabin, M. and Rostami, M., 2012, "Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions", Mathematical and Computer Modelling, 55(3-4), pp.791-800.

[8]. Zhang, X., 2010, “Stochastic Volterra equations in Banach spaces and stochastic partial differential equation”, Journal of Functional Analysis, 258(4), pp.1361-1425.

[9]. Cortes, J. C., Jodar, L. and Villafuerte, L., 2007, “Mean square numerical solution of random differential equations: facts and possibilities”, Computers & Mathematics with Applications, 53(7), pp.1098-1106.

[10]. Lepik, U. and Tamme, E., 2004, “Application of the Haar wavelets for solution of linear integral equations”, Proceedings, pp, 494, p.507.

[11]. Lepik, U., 2006, “Haar wavelet method for nonlinear integro-differential equations”, Applied mathematics and Computation, 176(1), pp.324-333.

[12]. Douglas, H. and Peter, P., 2006, “Stochastic Differential Equations In Science And Engineering (With Cd-rom)”, World Scientific.

[13]. Maleknejad, K., Khodabin, M. and Rostami, M., 2012. “A numerical method for solving m-dimensional stochastic Ito-Volterra integral equations by stochastic operational matrix”. Computers & Mathematics with Applications, 63(1), pp.133-143.

[14]. Bujurke, N. M., Salimath, C. S. and Shiralashetti, S. C., 2008. “Numerical solution of stiff systems from nonlinear dynamics using single-term Haar wavelet series”. Nonlinear Dynamics, 51(4), pp.595-605.

[15]. Bujurke, N. M., Salimath, C. S. and Shiralashetti, S. C., 2008. “Computation of eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets”. Journal of Computational and Applied Mathematics, 219(1), pp.90-101.

[16]. M. H. Heydari, M. R. Hooshmandasl, A. Shakiba, and C. Cattani, 2016. “Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations”. Nonlinear Dynamics, *85*(2), pp.1185-1202.