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Some Applications of Multivalent Functions Defined by Extended Fractional Differintegral Operator

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Abstract

In the present paper an extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$ $(-\infty<\lambda<p+1: p\in\mathbb{N})$, suitable for the study of multivalent functions is introduced. The various results obtained here for each of these function classes include coefficient bound, inclusion relation for $(k,\theta)-$ neighborhood of subclass of analytic and multivalent functions with negative coefficient, Hadamard products, Integral means. Further, results based on partial sums of functions belonging to the class are derived.

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1. INTRODUCTION

Let S_p denotes a class of functions of the form:

$$f(z)=z^p+\sum_{n=k}^{\infty}a_n z^n\,(p<k;p,k\in N=\{1,2,.......\}),\tag{1}$$

which are analytic and p - valent in the open unit disk $U=\{z:|z|<1\}$. A function f belong to the class S_p is said to be p-valent starlike of order α in U if and only if

$$\Re\left\{\frac{zf^{'}(z)}{f(z)}\right\}>\alpha,(0\le\alpha<p;z\in U).\tag{2}$$

Also a function f belonging to the class S_p is said to be p-valent convex of order α in U if and only if

$$\Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\}>\alpha,(0\le\alpha<p;z\in U).\tag{3}$$

We denote by $S_p^*(\alpha)$ the class of all functions in S_p which are p-valent starlike of order α in U and by $K_p(\alpha)$ the class of all functions in S_p which are p-valent convex of order α in U. We denote that $S_p^*(0)=S_p^*, S_1^*(\alpha)=S^*(\alpha), K_p(0)=K_{p,}K_1(\alpha)=K(\alpha)$, and

$$f(z)\in K_p(\alpha)\Leftrightarrow\frac{zf^{'}(z)}{p}\in S_p^*(\alpha).\tag{4}$$

The classes $S_p^*(\alpha)$ and $K_p(\alpha)$ were studied by Patil and Thakare [24], Aouf [1] and Owa [20] for $f\in S_p$ given by (1) and $g\in S_p$ given by

$$g(z)=z^p+\sum_{n=k}^{\infty}b_n z^n,(b_n\ge 0).\tag{5}$$

The Hadmard product (or convolution) of f and g is given by

$$(f*g)(z)=z^p+\sum_{n=k}^{\infty}a_n b_n z^n=(g*f)(z).\tag{6}$$

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as

$f\prec g$ in U or $f(z)\prec g(z)$ $(z\in U)$,

If there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z)=g(w(z)), z\in U.$ it is known that

$f(z)\prec g(z)$ $(z\in U)\Rightarrow$ $f(0)=g(0)$ and $f(U)\subset g(U)$.

In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [17], [18])

$f(z)\prec g(z)$ $(z\in U)$ $\Leftrightarrow$ $f(0)=g(0)$ and $f(U)\subset g(U)$.

Furthermore, $f(z)$ is said to be subordinate to $g(z)$ in the disk if the Schwaz lemma that if $f(z)\prec g(z)$ in U, then $f\prec g$ in U_r for every $r(0<r<1)$.

Recently, Patel & .Mishra [23] (see also Aouf et al. [4], Liu [14], Liu and Patel [15], Sharma et al. [33], Srivastava et al. [30], Supramaniam et al. [34], Zhi-Gang Wang and Lei Shi [35]) introduced and investigated an extended fractional differintegral operator $\Omega_z^{(\lambda,p)}f(z):S_p\to S_p$ for a function

$f(z)$ of the form (1) and for a real number $\lambda(-\infty<\lambda<p+1)$ by

$$\Omega_z^{(\lambda,p)}f(z)=z^p+\sum_{n=k}^{\infty}\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}a_n z^n$$

$$=z^p+\sum_{n=k}^{\infty}C_{n,p}^{\lambda}a_n z^n \qquad (7)$$

where $$C_{n,p}^{\lambda}=\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)} \qquad (8)$$

$$=z^p\,{}_2F_1(1,p+1;p+1-\lambda;z)*f(z)$$
$$(-\infty<\lambda<p+1;z\in U).$$

It is easily seen from (7) that

$$z\left(\Omega_z^{(\lambda,p)}f(z)\right)'=(p-\lambda)\Omega_z^{(1+\lambda,p)}f(z)+\lambda\Omega_z^{(\lambda,p)}f(z)$$
$$(-\infty<\lambda<p;z\in U). \qquad (9)$$

We also note that

$$\Omega_z^{(0,p)}f(z)=f(z),\quad \Omega_z^{(1,p)}f(z)=\frac{zf'(z)}{p},$$

and in general

$$\Omega_z^{(\lambda,p)}f(z)=\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z)$$
$$(-\infty<\lambda<p+1;z\in U). \qquad (10)$$

Where $D_z^{\lambda}f(z)$ is, respectively, the fractional integral of $f(z)$ of order $-\lambda$ when $-\infty<\lambda<0$ and the fractional derivative of $f(z)$ of order λ when $0\le\lambda<p+1.$ for integral value of λ, (9) further simplifies to

$$\Omega_z^{(k,p)}f(z)=\frac{(p-k)!z^k f^{(k)}(z)}{p!}\quad (k\in N;k<p+1)$$

and

$$\Omega_z^{(-m,p)}f(z)=\frac{p+m}{z^m}\int_0^z t^{m-1}\Omega_z^{(-m+1,p)}f(t)dt,(m\in N)$$

$$=F_{1,p}oF_{2,p}o....oF_{m,p}(f)(z)$$

$$=F_{1,p}\left(\frac{z^p}{1-z}\right)*F_{2,p}\left(\frac{z^p}{1-z}\right)*..........*F_{m,p}\left(\frac{z^p}{1-z}\right)*f(z).$$

Where $F_{\mu,p}$ is generalized Bernadi-Libra-Livingston integral operator [6] and o stands for the usual composition of functions.

Now, by using the extended fractional differintegral operator $\Omega_z^{(,\lambda p)}$ $(-\infty\le\lambda<p+1),$ we introduce the following sub class of function in S_p.

For fixed parameters β,γ and ξ $(0<\beta\le1,\frac{1}{2}\le\xi\le1,0\le\gamma<\frac{p}{2})$ and $0\le\lambda<1$, we say that a function $f(z)\in S_p$ is in the class $S_p^{\lambda}(\beta,\gamma,\xi)$ if it satisfies the following condition:

$$\left|\frac{\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)}-p}{2\xi\left[\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)}-\gamma\right]-\left[\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)}-p\right]}\right|<\beta \qquad (11)$$

For $\lambda=0$, $k=p+1, p\in\mathbb{N}$ in (11) the class $S_p^{\lambda}(\beta,\gamma,\xi)$ reduces to the class $S_p^0(\lambda,l,\gamma,\beta,\xi)=S_p^0(\gamma,\beta,\xi)$ see Kulkarni et al. [9]).

Let T_p denote the subclass of S_p consisting of functions of the form

$$f(z)=z^p-\sum_{n=k}^{\infty}a_n z^n, a_n\ge0;z\in U. \qquad (12)$$

Further, we define the class $TS_p^{\lambda}(\beta,\gamma,\xi)$ by

$$TS_p^{\lambda}(\beta,\gamma,\xi)=S_p^{\lambda}(\beta,\gamma,\xi)\cap T_p.$$

We note that:

For $\lambda=0$, in (11), the class $TS_p^{\lambda}(\beta,\gamma,\xi)$ reduces to the class $T_p^o(\lambda,l,\gamma,\beta,\xi)=T_p(\gamma,\beta,\xi),$ which for $p=1$ reduces to $T(\gamma,\beta,\xi)$ studies by Kulkarni [10].

In this paper, we aim at proving coefficient inequality, neighborhood, partial sums, integral means, and modified Hadamard product involving the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$.

2. COEFFICIENT INEQUALITY

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0<\beta\le1,\frac{1}{2}\le\xi\le1,0\le\gamma<\frac{p}{2},n\ge k,p<k,$ and $C_{n,p}^{\lambda}$ is given by (8) with $-\infty<\lambda<p+1$ and $z\in U$.

Theorem 2.1 *Let the function* f *be defined by (12). Then* f *is in the class* $TS_p^{\lambda}(\beta,\gamma,\xi)$ *if and only if*

$$\sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\xi\beta(n-\gamma)\right]C_{n,p}^{\lambda}a_n \le 2\beta\xi(p-\gamma) \quad . \tag{13}$$

Proof. Assume that inequality (13) holds true. We find from (12) that

$$\left|z\left[\Omega_z^{(\lambda,p)}f(z)\right]' - p\Omega_z^{(\lambda,p)}f(z)\right| - \beta\left|2\xi\left\{z\left[\Omega_z^{(\lambda,p)}f(z)\right]' - \gamma\Omega_z^{(\lambda,p)}f(z)\right\} - \left\{z\left[\Omega_z^{(\lambda,p)}f(z)\right]' - p\Omega_z^{(\lambda,p)}f(z)\right\}\right|$$

$$= \left|\sum_{n=k}^{\infty} -(n-p)C_{n,p}^{\lambda}a_n z^n\right| - \beta\left|2\xi\left[(p-\gamma)z^p - \sum_{n=k}^{\infty}(n-\gamma)C_{n,p}^{\lambda}a_n z^n\right] + \sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n z^n\right|$$

$$\le \sum_{n=k}^{\infty}\left[(n-p)+2\xi\beta(n-\gamma)-\beta(n-p)\right]C_{n,p}^{\lambda}a_n - 2\beta\xi(p-\gamma)$$

$$= \sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\xi\beta(n-\gamma)\right]C_{n,p}^{\lambda}a_n - 2\beta\xi(p-\gamma) \le 0.$$

Hence by the maximum modulus theorem, we have $f \in TS_p^{\lambda}(\beta,\gamma,\xi)$ conversely, let $f \in TS_p^{\lambda}(\beta,\gamma,\xi)$. Then

$$\left|\frac{\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)} - p}{2\xi\left[\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)} - \gamma\right] - \left[\dfrac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{\Omega_z^{(\lambda,p)}f(z)} - p\right]}\right| < \beta$$

that is, that

$$\frac{\left|\sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n z^n\right|}{\left|2\xi\left[(p-\gamma)z^p - \sum_{n=k}^{\infty}(n-\gamma)C_{n,p}^{\lambda}a_n z^n\right] + \sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n z^n\right|} < \beta \tag{14}$$

Now $\Re e\{f(z)\} \le |f(z)|$ for all z, we have

$$\Re e\left\{\frac{\sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n z^n}{2\xi\left[(p-\gamma)z^p - \sum_{n=k}^{\infty}(n-\gamma)C_{n,p}^{\lambda}a_n z^n\right] + \sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n z^n}\right\} < \beta \tag{15}$$

Choose value of z on the real axis so that $\dfrac{z\left[\Omega_z^{(\lambda,p)}f(z)\right]'}{\Omega_z^{(\lambda,p)}f(z)}$ is real. Then upon clearing the denominator in (15) and letting $z \to 1^-$ through real values, we have

$$\frac{\sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n}{2\xi\left[(p-\gamma) - \sum_{n=k}^{\infty}(n-\gamma)C_{n,p}^{\lambda}a_n\right] + \sum_{n=k}^{\infty}(n-p)C_{n,p}^{\lambda}a_n} \le \beta$$

That is

$$\sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\xi\beta(n-\gamma)\right]C_{n,p}^{\lambda}a_n \le 2\beta\xi(p-\gamma)$$

This is the required condition, which completes the proof of theorem 2.1.

Corollary 2.2 Let the function f be defined by (12). Then f is in the class $TS_p^{\lambda}(\beta,\gamma,\xi)$ if and only if

$$\sum_{n=k}^{\infty}\Psi_{(p,n)}^{\lambda}(\beta,\gamma,\xi)a_n \le 1, \tag{16}$$

Where, $\Psi_{(p,n)}^{\lambda}(\beta,\gamma,\xi)=\dfrac{\left[(n-p)(1-\beta)+2\xi\beta(n-\gamma)\right]C_{n,p}^{\lambda}}{2\beta\xi(p-\gamma)}$, (17)

$0<\beta\le 1, \frac{1}{2}\le\xi\le 1, 0\le\gamma<\frac{p}{2}, n\ge k, p<k, -\infty<\lambda<p+1.$

Corollary 2.3 Let the function f defined by (12) is in the class $TS_p^{\lambda}(\beta,\gamma,\xi)$ then we have

$$a_n \le \frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]C_{n,p}^{\lambda}}, (n\ge k) \tag{18}$$

The result is sharp for the function f given by

$$f(z)=z^p-\frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]C_{n,p}^{\lambda}}z^n, n\ge k. \tag{19}$$

3. NEIGHBORHOOD FOR THE CLASS $TS_p^{\lambda}(\beta,\gamma,\xi)$

Next, following the earlier investigations by Goodman [8], Rucheweyh [26], and others including Srivastava et al. [29], Orhan ([21] and [20]), Altinas et al. [2] (see also [11], [16], [31], [3]), we define the (k,δ)-neighborhood of functions in the family $TS_p^{\lambda}(\beta,\gamma,\xi)$.

Definition 3.1 For $f\in T_p$ of the form (12) and $\delta\ge 0$ we define a (k,δ) - neighborhood of a function $f(z)$ by

$$N_{k,\delta}(f)=\left\{g: g\in T_p, g(z)=z^p-\sum_{n=k}^{\infty}c_n z^n \ \& \sum_{n=k}^{\infty}n\left|a_n-c_n\right|\le\delta\right\}.$$

In particular, for the function, $h(z)=z^p$

We immediately have

$$N_{k,\delta}(h)=\left\{g: g\in T_p, g(z)=z^p-\sum_{n=k}^{\infty}c_n z^n \ \& \sum_{n=k}^{\infty}n\left|c_n\right|\le\delta\right\}.$$

Theorem 3.2 The class $TS_p^{\lambda}(\beta,\gamma,\xi)\subset N_{k,\delta}(h)$, *where* $\delta=\dfrac{(k+1-2p)}{\Psi_{(k,p)}^{\lambda}(\beta,\gamma,\xi)}$.

Proof For the function $f(z)\in TS_p^{\lambda}(\beta,\gamma,\xi)$ of the form (12), corollary 1 immediately yields

$$\left[(k-p)(1-\beta)+2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}\sum_{n=k}^{\infty}a_n \le 2\beta\xi(p-\gamma),$$

$$\sum_{n=k}^{\infty} a_n \le \frac{2\beta\xi(p-\gamma)}{\left[(k-p)(1-\beta)+2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}} = \frac{1}{\Psi_{(k,p)}^{\lambda}(\beta,\gamma,\xi)} \tag{20}$$

On other hand, we also find from (16) and (20) that

$$C_{k,p}^{\lambda}\sum_{n=k}^{\infty} a_n \le 2\beta\xi(p-\gamma)+\left[(1-p)(1-\beta)-2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}\sum_{n=k}^{\infty} a_n$$

$$\le 2\beta\xi(p-\gamma)+\left[(1-p)(1-\beta)-2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}\frac{2\beta\xi(p-\gamma)}{\left[(k-p)(1-\beta)+2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}}$$

$$\le \frac{2\beta\xi(p-\gamma)(k+1-2p)}{\left[(k-p)(1-\beta)+2\xi\beta(k-\gamma)\right]C_{k,p}^{\lambda}} = \frac{(k+1-2p)}{\Psi_{(k,p)}^{\lambda}(\beta,\gamma,\xi)} = \delta,$$

Which in view of definition 3.1, proves Theorem 3.

4. PARTIAL SUMS

Following the earlier works by Silverman [27], N.C .Cho et al. [5] and others (see also [25], [13],), in this section we investigate the ratio of real parts of functions involving (12) and their sequence of partial sums defined by

$$f_1(z)=z^p;\ f_n(z)=z^p-\sum_{n=k}^{r} a_n z^n, r\in N \tag{21}$$

And determine sharp lower bounds for

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\},\Re\left\{\frac{f_n(z)}{f(z)}\right\},\Re\left\{\frac{f'(z)}{f_n'(z)}\right\}\&\Re\left\{\frac{f_n'(z)}{f'(z)}\right\}.$$

Theorem 4.1 *If f of the form (12) satisfies condition (13), then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\}\ge\frac{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)-1}{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)} \tag{22}$$

and

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\}\ge\frac{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)}{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)+1} \tag{23}$$

Where $\Psi_{(p,n)}^{\lambda}(\beta,\gamma,\xi)$ *is given by (17).*

Proof. In order to prove (22) , it is sufficient to show that

$$\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)\left[\frac{f(z)}{f_n(z)}-\left(\frac{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)-1}{\Psi_{(p,k+r)}^{\lambda}(\beta,\gamma,\xi)}\right)\right]\prec\frac{1+z}{1-z} \quad (z\in U).$$

We can write

$$\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\left[\frac{f(z)}{f_n(z)}-\left(\frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)-1}{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)}\right)\right]$$

$$=\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\left[\frac{1-\sum_{n=k}^{\infty}a_n z^{n-p}}{1-\sum_{n=k}^{r}a_n z^{n-p}}-\left(\frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)-1}{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)}\right)\right]$$

$$=\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\left[\frac{1-\sum_{n=k}^{r}a_n z^{n-p}-\sum_{n=k+r}^{\infty}a_n z^{n-p}}{1-\sum_{n=k}^{r}a_n z^{n-p}}-\left(\frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)-1}{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)}\right)\right]=\frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z)=\frac{-\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n z^{n-p}}{2-2\sum_{n=k}^{r}a_n z^{n-p}-\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n z^{n-p}}$$

Obviously $w(0)=0$ and

$$|w(z)|\le\frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n}{2-2\sum_{n=k}^{r}a_n-\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n}$$

Now, $|w(z)|\le 1$ if and only if

$$2\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n\le 2-2\sum_{n=k}^{r}a_n,$$

which is equivalent to

$$\sum_{n=k}^{r}a_n+\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\sum_{n=k+r}^{\infty}a_n\le 1.$$

In view of (13), this is equivalent to showing that

$$\sum_{n=k}^{r}\left[\Psi^{\lambda}_{(p,n)}(\beta,\gamma,\xi)-1\right]a_n+\sum_{n=k+r}^{\infty}\left[\Psi^{\lambda}_{(p,n)}(\beta,\gamma,\xi)-\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)\right]a_n\ge 0.$$

Thus we have completed the proof of (22), the proof of (23) is similar to (22) and will be omitted.

Theorem 4.2 *If $f(z)$ of the form (12) satisfies (13), then*

$$\Re\left\{\frac{f'(z)}{f_n'(z)}\right\}\ge\frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)-k-1}{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)} \qquad (24)$$

and

$$\Re\left\{\frac{f_n^{'}(z)}{f^{'}(z)}\right\} \geq \frac{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)}{\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)+k+1} \tag{25}$$

Where $\Psi^{\lambda}_{(p,k+r)}(\beta,\gamma,\xi)$ *is given by (17).*

5 . INTEGRAL MEANS

The following subordination result due to Littewood [12] will be required in our investigation. The integral means of analytic functions was studied in [25], [19].

Lemma 5.1 if $f(z)$ and $g(z)$ are analytic in U with, $f(z) \prec g(z)$, then $\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\mu} d\theta \leq \sum_0^{2\pi}\left|g(re^{i\theta})\right|^{\mu} d\theta,$ where $\mu > 0, z = re^{i\theta}$ & $0 < r < 1.$

Application of Lemma 5.1 to function $f(z)$ in the class $TS_p^{\lambda}(\beta,\gamma,\xi)$ gives the following result using known procedures.

Theorem 5.2 *Let* $f(z) \in TS_p^{\lambda}(\beta,\gamma,\xi)$ *and* $f_2(z) = z^p - \frac{1}{\Psi^{\lambda}_{(p,n)}(\beta,\gamma,\xi)} z^n$ *where* $\Psi^{\lambda}_{(p,n)}(\beta,\gamma,\xi)$ *is given by (17), if* $f(z)$ *satisfies*

$$\sum_{n=k}^{\infty}|a_n| \leq \left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}\right| \tag{26}$$

Then for $\mu > 0$ *and* $z = re^{i\theta}, (0 < r < 1),$

$$\int_0^{2\pi}|f(z)|^{\mu} d\theta \leq \int_0^{2\pi}|f_2(z)|^{\mu} d\theta. \tag{27}$$

Proof. By putting $z = re^{i\theta}, (0 < r < 1),$ we see that

$$\int_0^{2\pi}|f(z)|^{\mu} d\theta = r^{\mu p}\int_0^{2\pi}\left|1-\sum_{n=k}^{\infty}a_n z^{n-p}\right|^{\mu} d\theta.$$

And

$$\int_0^{2\pi}|f_2(z)|^{\mu} d\theta = r^{\mu p}\int_0^{2\pi}\left|1-\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}z^{n-p}\right|^{\mu} d\theta.$$

Applying lemma (5.1), we have to show that

$$1-\sum_{n=k}^{\infty}a_n z^{n-p} \prec 1-\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}z^{n-p},$$

Let us define the function $w(z)$ by

$$1-\sum_{n=k}^{\infty}a_n z^{n-p} = 1-\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{n-p} \tag{28}$$

or by

$$\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{n-p} = \sum_{n=k}^{\infty} a_n z^{n-p} \tag{29}$$

Since, for $z=0$, $\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}(w(0))^{n-p} = 0,$

there exists an analytic function $w(z)$ in U such that $w(0)=0$.

Next, we prove the analytic function $w(z)$ satisfies $|w(z)|<1 \ (z\in U)$ for

$$\sum_{n=k}^{\infty}|a_n| \le \left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}\right|$$

By the equality (27), we know that

$$\left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{n-p}\right| \le \left|\sum_{n=k}^{\infty} a_n z^{n-p}\right| < \left|\sum_{n=k}^{\infty} a_n\right|,$$

For $z\in U$, hence,

$$\left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{n-p}\right| - \left|\sum_{n=k}^{\infty} a_n\right| < 0. \tag{30}$$

Letting $t=|w(z)| \ (t\ge 0)$ in (30), we define the function $G(t)$ by

$$G(t) = \left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}\right|(t)^{n-p} - \sum_{n=k}^{\infty}|a_n|. \quad (t\ge 0).$$

If $G(1)\ge 0,$ then we have $t<1$ for $G(t)<0.$ therefore, for $|w(z)|<1 \ (z\in U),$ we need

$$G(1) = \left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}\right| - \sum_{n=k}^{\infty}|a_n| \ge 0,$$

That is,

$$\sum_{n=k}^{\infty}|a_n| \le \left|\frac{1}{\Psi^{\lambda}_{(n,p)}(\beta,\gamma,\xi)}\right|.$$

Consequently, if the inequality (26) holds true, there exists an analytic function $w(z)$ with $w(0)=0,$ $|w(z)|<1 \ (z\in U)$, such that $f(z)=f_2(w(z))$. This completes the proof of Theorem (5).

6. MODIFIED HADAMARD PRODUCT

For the functions $f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n \, (a_{n,j}\ge 0; j=1,2; p,k\in N),$ (31)

We denote by $(f_1 * f_2)$ the modified Hadamard product of functions f_1 and f_2, that is,

$$(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n. \tag{32}$$

Theorem 6.1 *Let the functions* $f_j\,(j=1,2)$ *, defined by (31) be in the class* $TS_p^\lambda(\beta,\gamma,\xi)$ *then* $(f_1 * f_2) \in TS_p^\lambda(\beta,\mu,\xi)$ *where*

$$\mu = p - \frac{2\beta\xi(p-\gamma)^2(k-p)\left[(1-\beta)+2\beta\xi\right]}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]^2 C_{n,p}^\lambda - 4\beta^2\xi^2(p-\gamma)^2} \tag{33}$$

The result is sharp.

Theorem 6.2 *Let the function* $f_j\,(j=1,2)$ *defined by (31),* $f_1 \in TS_{n,p}^\lambda(\beta,\mu_1,\xi)$ *and* $f_2 \in TS_{n,p}^\lambda(\beta,\mu_2,\xi)$.

Then $(f_1 * f_2) \in TS_p^\lambda(\beta,\mu,\xi)$, *where*

$$\mu = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(k-p)\left[(1-\beta)+2\beta\xi\right]}{A_1(\mu_1,p,\beta,\xi,k).A_2(\mu_2,p,\beta,\xi,k)C_{n,p}^\lambda - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)} \tag{34}$$

And

$$A_1(\mu_1,p,\beta,\xi,k) = \left[(k-p)(1-\beta)+2\beta\xi(k-\mu_1)\right] \tag{35}$$

$$A_2(\mu_2,p,\beta,\xi,k) = \left[(k-p)(1-\beta)+2\beta\xi(k-\mu_2)\right] \tag{36}$$

Theorem 6.3 *Let the functions* $f_j\,(j=1,2)$ *defined by (31) are in the class* $TS_p^\lambda(\beta,\gamma,\xi)$ *. Then the function*

$$h(z) = z^p - \sum_{n=k}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right) z^n \tag{37}$$

Belongs to the class $TS_p^\lambda(\beta,\tau,\xi)$, *where*

$$\tau = p - \frac{4\beta\xi(p-\gamma)^2(n-p)\left[(1-\beta)+2\beta\xi\right]}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]^2 C_{n,p}^\lambda - 8\beta^2\xi^2(p-\gamma)^2} \tag{38}$$

The result is sharp for the functions $f_j\,(j=1,2)$ *defined by (31).*

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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