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Composite Non Smooth Mathematical Programming Problems with Equilibrium Constraints under Generalized Univexity

L.Venkateswara Reddy
Professor
Dept. of IT
SVEC, Tirupati, India.

Anju Singh
Research Scholar
Dept. of Maths, Mewar University
Gangrar, Chittorgarh (Rajasthan), India.

Dr. S.P. Pandey
Professor
Dept. of Maths, Maharshi University of Information Technology,
IIM Road, Lucknow, India.

Abstract

Here, we will state and prove sufficient optimality conditions for a composite non smooth mathematical programming problem with respect of equilibrium constrains under generalized univexity condition. Further, we deduced Wolfe and Mond-Weir type dual models for the chosen problem using convexiticators. Also, we state and prove we are and strong dudity theorems.

Keywords: Composite program, duality, generalized Univexity, Convexifications

1. INTRODUCTION:

Convexificators concept was introduced by Demyanov [6]. This concept is used to generalize the concepts in optimization and the theory of non smooth analysis (Refer [16], [1], [32], [19], [20]). Further, the concept of Clarke sub differentials, Michel – Pinot sub – differentials and Treiman sub differentials of a Locally Lpschitz real valued functions for convexiticators were introduced by Luc, DT. [20] and the latest development one can refer [16], [17], [18], [19].

Also, the concept of mathematical programming program with respect to equilibrium constraints (MPEC1) is usually studied for optimization problem in which the required constraints functions were defined by using complementary system or by using auxiliary parametric variational inequality. In the literature, various equilibrium phenomena were introduced to study applications on economics and engineering which was characterized either by a variational inequality or an optimization problem. This justifies the name mathematical programming problem with equilibrium constraints. This was studied for both smooth case [11], [36] and for the non smooth case ([29], [30] [37]). In another development, Luc et al. [20] introduced and studied a comprehensive study on mathematical programming with Equilibrium constraints. Consequently, Flegal and Kanzow [8,9] obtained the optimality conditions for MPEC1 by using FJ – conditions. Also in [9], Flegal and Kanzow introduced a new, constraint called "Slater type constraints qualifications and a new Abadie type constraint qualification for the MPEC1.

Further on, the concept of convexity and generalized play an important role in the field of optimization, control theory, Economics, Game Theory and so on. The most important generalization of convexity is invexity of function, which was introduced by Hanson [11] and the name coined by Craven [5]. Since last three and half decades, optimality and duality condition in convexity and generalized convexity (invexity) were introduced many researchers (see [4, 23]; [24], [26]. Duality results have many applications in Numerical Algorithm in the field of nonlinear programming problem for solving certain class of optimization programs. Consequently, the concept of duality helped the society to develop stopping rules and to solve primal and dual problems both in Linear and Nonlinear optimization problem.

In this context, Wolfe [35] and Mond-Weir [27] dual models were very popular in the field of nonlinear programming problem very recently, B.C. Joshi, et.al. [16] derived sufficient optimality conditions for global optimality for the chosen mathematical programming problem with equilibrium constraints under generalized univexity.

By make use of the above arguments in this paper, we introduce composite mathematical programming problem for equilibrium constraints by using generalized univexity assumptions. Also, we state and prove duality results of Wolfe and Mond – Weir types to the MPEC1.

This papers is organized is as follows. Section 2 gives elementary basic definitions and notations. Section 3 contains sufficient optimality condition for the chosen MPEC1 using generalized univexity. Finally, section 4 gives Weak and Strong duality results in the frame work of generalized convexitications with respect to generalized univexity condition.

2. NOTATIONS AND DEFINITIONS:

Definition 2.1: Let $F:R^n \rightarrow RU\{+\infty\}$ be a generalized real-valued function, which admit convexifications at $\tilde{x} \in R^n$ and $\eta: R^n \times R^n \rightarrow R^n$ be a Kernal function, then, f is said to be:

a) $\partial^* - V - \rho -$ univexity, with respect to η, θ if for the very $x \in R^n$, we have

$$\theta\left(\left(x,\tilde{x}\right)\left(F\left(x\right)-F\left(\tilde{x}\right)\right)\right) \geq \left(\xi\eta^T\left(x,\tilde{x}\right)\right), \forall \in \partial^* F\left(F^1\left(\tilde{x}\right)\right)$$

(b) $\partial^* - V - \rho -$ pseudo univexity with respect to η, θ if for every $x \in R^n$, we have

$$\exists \in \partial^* F\left(F^1(\tilde{x})\right), \left(\xi \eta^T(x,\tilde{x})\right) \geq 0$$

$$\Rightarrow F\left(F^1(x)\right) \geq F\left(F^1(\tilde{x})\right).$$

(iii) $\partial^* - V - \rho -$ quasi univexity at $\tilde{x}$ with respect to η and θ if for every $x \in R^n$,

we have

$$\theta(x,\tilde{x})\left(F\left(F^1(x)\right)\right) - F\left(F^1(\tilde{x})\right)$$

$$\leq \left(\xi \eta^T(x,\tilde{x})\right) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0$$

$$\forall \in \partial^* F\left(F^1(\tilde{x})\right).$$

3. OPTIMALITY:

In the sequel, we need the following notations from [16, 32].

They are

$$I_\upsilon^\theta = \left\{ i \in I : \gamma_i^\psi = 0, \psi > 0 \right\}$$

$$I_\gamma^\psi = \left\{ i \in I : \gamma_i^\psi = 0, \gamma_i^\theta = 0 \right\}$$

$$\delta_\gamma^+ = \left\{ i \in \delta : \gamma_i^\theta > 0 \right\}$$

$$K_\gamma^+ = \left\{ i \in K : \gamma_i^\psi > 0 \right\}$$

We will state and prove that the following optimality conditions with respect to $\partial^* - V - \rho -$ univexity.

Theorem 3.1: Suppose $\tilde{x}$ is a feasible – GA – Stationary point of MPEC1. Also, assume that F (F^1) is $\partial^* - V - \rho -$ univexity at $\tilde{x}$ w.r.t to the kernals η, θ and $g_j(G_j), (j \in I_g), \pm h_m(H_m), (m = 1, 2, ..., p)$ $\theta_i(i \in \delta U I^1), -\psi_i(i \in I^1 U K)$ are $\partial^* - V - \rho -$ quasi univexity at $\tilde{x}$ with respect to the some common Kernals η and θ. If $I_\gamma^\theta \cup I_\gamma^\psi \cup v I_\gamma^+ \cup K_\gamma^+ = \phi$, then $\tilde{x}$ is said to be global optimal solution of problem MPEC1.

Proof:

Suppose x is any arbitrary feasible point $\Rightarrow g_j\left(G_j(x)\right) \leq 0 = g_j\left(g_j(\tilde{x})\right)$, by definition of $\partial * - V - \rho$ univexity, we have

$$\left(\xi_i^{g_j}, \eta^T(x,\tilde{x})\right) + \rho \| \theta(x,\tilde{x}) \|^2$$
$$\leq 0, \forall \xi_i^{g_j} \in \partial\, g_j\left(G_j(\tilde{x})\right) \forall j \in I_g \tag{3.7}$$

Similarly, we have

$$\left(T_m, \eta^T(x,\tilde{x})\right) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0$$

$$\forall T_m \in \partial^* h_m\left(H_m(\tilde{x})\right), (\tilde{x}), \forall m = \{1, 2, ..., q\} \tag{3.8}$$

$$\left(\lambda_m, \eta^t\right)(x,\tilde{x}) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0,$$

$$\forall \lambda_m \in \partial^*\left(-h_m(H_m)(\tilde{x})\right), \forall m = \{1, 2, ...q\} \tag{3.9}$$

$$\xi_i^\theta, \eta^T(x,\tilde{x}) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0,$$

$$\forall \xi_i^\theta \in \partial^*.(-\theta'_i)(\tilde{x}), \forall i \in S \cup I \tag{3.10}$$

$$\xi_i^\psi, \eta(x,\tilde{x}) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0$$

$$\forall \in_i^\psi \in \partial^*.(-\psi_i)(\tilde{x}), \forall i \in I \cup K \tag{3.11}$$

Suppose $I_\gamma^\theta U\, I_\gamma^\psi \cup \delta_\gamma^+ \cup K_\gamma^+ = \phi$, also multiply equation (3.7) to (3.11), by $T_i^{g_j} \geq 0$

$(i \in I_g), T_m^h > 0, m = 1, 2, ..., q,$

$T_m^h > 0$, $T_i^\theta > 0$, $i \in \delta \cup I$, $T_i^\psi > 0$ $i \in I \cup K$, respectively and finally by sum rule,

we have

$$\left(\sum_{i \in I_g} T_i^g \xi_i^g + \sum_{m=1}^{q} T_m^h\ T_m^l + \gamma_m^h\ \lambda_m \right.$$

$$\left. + \sum_{i=1}^{I} T_i^\theta\ \xi_i^\theta + \sum_{i=1}^{l} T_i^\psi\ \xi_i^\psi \right),$$

$$\eta^T(x,\tilde{x}) + \rho \| \theta(x,\tilde{x}) \|^2 \leq 0,$$

for all $\xi_i^{g_j} \in$ con $\partial^* g_j(\tilde{x}), T_m \in \text{con}\, \partial^* h_m(\tilde{x}),$

$\lambda_m \in \text{con}\, \partial^*(-h_m)(\tilde{x}), \xi_i^\theta \in \text{con}\, \partial^*(-\theta_i)(\tilde{x})$ and $\xi_i^\psi \in \text{con}\, \partial^*(-\psi_i)(\tilde{x})$.

which implies by using generalized GA-stationary of $\tilde{x}$, and also select & $\xi \in \text{con}\, \partial^* F\left(F^1(\tilde{x})\right)$, so that

$$< \xi, \eta^T(x,\tilde{x}) + \rho \| \theta(x,\tilde{x}) \|^2 \geq 0$$

Also, by definition of $\partial^* - V - \rho -$ univexity of $F\left(F^1\right)$ at $\tilde{x}$ with respect to the some common kernals of η and θ, we have

$b(x,\tilde{x})\left(F\left(F^1(x)\right)-F\left(F^1(\tilde{x})\right)\geq 0,\right.$ for all feasible points of x.

Thus, by def, $\tilde{x}$ is a global optimal point of MPEC1.

4. DUALITY RESULTS

Here, we will discuss Wolfe and Mond-Weir type dual problems under generalized $\partial^* - V - \rho -$univexity with convoxificators.

The Wolfe type dual programming problem for MPEC1 is :

(CWD) $$\max(u,\mu)\left\{F\left(F^1(\mu)+\sum_{j\in I_g}\mu_j^g g_j\left(G_j(\mu)\right)\right)\right\}$$

$$+\sum_{m=1}^{q}\lambda_m^h\ h_m\left(H_m(u)\right)$$

$$-\sum_{j=1}^{q}\left[\mu_j^\theta\,\theta_j\left(\theta^1(u)\right)+\mu_j^\psi\left(\psi_j\left(\psi_j^1(u)\right)\right)\right]$$

subject to conditions:

$$0\in \operatorname{con}\partial^* F\left(F^1(u)\right),$$

$$+\sum_{j\in I_g}\mu_j^g \operatorname{con}\partial^* g_j\left(G_j(u)\right)$$

$$+\sum_{m=1}^{q}\left[\mu_m^h \operatorname{con}\partial^* h_m\left(H_m(u)\right)\right]$$

$$+\gamma_m^h \operatorname{con}\partial^*\left(-h_m\left(H_m\right)(u)\right)$$

$$+\sum_{j=1}^{q}\left[\mu_j^\theta \operatorname{con}\partial^*\left(-\theta_j\right)(u)\right]$$

$$+\mu_j^\psi \operatorname{cond}\partial^*\left(-\psi_g(u)\right)$$

$$\mu_{i_g}^g\geq 0,\ \mu_m^h,\ \gamma_m^h\geq 0,\ m=h$$

$$\mu_j^\partial,\mu_j^\psi,\gamma_i^\psi\geq 0,\ i=1,..........p$$

$$\mu_K^\theta=\mu_8^\psi=\mu_k^\theta=0,\ \forall i\in I,\ \nu_i^\theta=0,\ \gamma_i^\psi=0 \tag{4.1}$$

Theorem 4.1 (Wear Duality)

Let $\tilde{x}$ be a feasible solution for the problem (MPEC1), (u,τ) be feasible for the dual problem (CWD) and the corresponding index sets I_{g_j},δ,I^1,K are defined accordingly.

Suppose that $F\left(F^1\right),g_j\left(G_j\right),\left(J\in I_g\right),\ \pm h_m\left(H_m\right)$ $(m=1,2,......,q),-\theta_j\left(\theta_j^1\right),$ $\left(j\in\delta\cup I^1\right),-\psi_j\left(\psi_j^1\right)(g\in I\cup K)$ admit bounded upper semi – regular convexificators and are $\partial^* - V - \rho -$ univex functions at u, with respect to the some common Kernals η and θ.

If $I_\gamma^\theta VI_\gamma^\psi\cup\delta_\gamma^+\cup K_\gamma^+=\phi$ then any w feasible for the problem MPEC1, we have

$$b(u,w)\left[F\left(F^1(u)\right)-F\left(F^1(w)\right)\right]\geq\sum_{j\in I_0}\lambda_j^g g_j\left(G_j(u)\right)$$

$$+\sum_{m=1}^{q}\ \lambda_m^h\ h_m\left(H_m(u)\right)$$

$$-\sum_{k=1}^{l}\left[\lambda_k^\theta\theta_k\left(\theta_k^1(u)\right)+\lambda_k^\psi\psi_k\left(\psi_k^1(u)\right)\right]$$

Proof:

Suppose w is any feasible point for the composite problem MPEC1.

By def, we have

$$g_j\left(G_j(W)\right)\leq 0$$

and $h_m\left(H_m(w)\right)=0,\quad \forall m=1,2,.......,q.$

Since F (F[1]) is $\partial^* - V - \rho -$univex at u with respect to the some common Kernals η and θ, then, we get

$$\begin{aligned} & b(u,w)\left[F\left(F^1(u)-F\left(F^1(w)\right)\right)\right] \\ & \geq\left(\xi,\eta^T(u,w)\right)+\rho\,||\,\theta(u,w)\,||^2 \\ & \forall\in\partial^* F\left(F^1(u)\right)\end{aligned} \tag{4.2}$$

We can write

$$\begin{aligned} & b(u,w)\left[g_j\left(G_j(u)\right)-g_j\left(G_j(w)\right)\right] \\ & \geq\left(\xi_j^g,\eta^T(u,w)+\rho\,||\,\partial(u,w)\right)||^2, \\ & \forall\xi_j^g\in\partial^* g_j\left(G_j(u)\right),j\in I_g \end{aligned} \tag{4.3}$$

$$b(u,w)\left[h_m\left(H_m(u)-h_m\right)\left(H_m(w)\right)\right]\left(\lambda_m \in D^* h_m\left(H_m(u)\right)\right),$$

$$\geq\left(\lambda_m, \eta^T(u,w)+\rho \| \theta(u,w)\|^2\right) \forall m=1,2,\ldots\ldots.,q \tag{4.4}$$

$$b(u,w)\left\lfloor -h_m\left(H_m(u)-h_m\left(H_m(w)\right)\right)\right\rfloor \forall \mu_m \in \partial^*\left(-h_m\left(H_m(u)\right)\right) \forall\, m=1,2,\ldots\ldots,q \tag{4.5}$$

$$\geq\left(\xi_j^\theta, \eta^\tau(u,w)\right)+\rho \| \theta(u,w)\|^2$$

$$\forall m=1,2,\ldots\ldots\ldots,q \tag{4.6}$$

$$b(u,w)\left[-\theta_j\left(\theta_j^1(u)+\theta_j\left(\theta_j^1\right)(w)\right)\right] \geq\left(\xi_j^\theta, \eta(u,w)+\rho \| \theta(u,w)\|^2\right) \forall \in_j^\theta \partial^*\left(-\partial_j\left(\theta_j^1(w)\right)\right),$$

$$\forall\, j \in \delta \cup I, \tag{4.7}$$

$$b(u,w)\left[-\psi_j\left(\psi_j^1(u)\right)+\psi_j\left(\psi_j^1(w)\right)\right] \geq\left(\xi_j^\psi, \eta^\tau(u,w)\right)+\rho \| \theta(u,w)\|^2\ \forall_j \in I \cup K. \tag{4.8}$$

If $I_\gamma^\theta \cup I_\gamma^\psi \cup S_\gamma^+ \cup K_\gamma^+=\rho$ and then multiply the equations

(4.4), (4.8) with $\lambda_j^g \geq 0\ \left(i \in I_g\right)$, $\lambda_m^h>0$

$\left(j \in \delta \cup I\right), \lambda_j^\psi>0\ \ \left(j \in I \cup K\right)$, respectively and then finally adding the equations
(4.3) – (4.8),
we have

$$b(u,w)\left\lfloor F\left(F^1(u)\right)-F\left(F^1\left(F^1(w)\right)\right)\right\rfloor \geq b(u,w)\left\lfloor \sum_{j \in I_g} \lambda_j^g g_j\left(G_j(u)\right)-\sum_{j \in I_g} \lambda_j^g g_j\left(G_j(w)\right)\right\rfloor$$

$$+\sum_{m=1}^q \lambda_m^h\ h_m\left(H_m(u)\right)-\sum_{m=1}^q \lambda_m^h\ h_m\left(H_m(w)\right)-\sum_{m=1}^q \gamma_m^h\left(h_m\left(H_m(u)\right)+\sum_{m=1}^q \gamma_m^h\ h_m\left(H_m(w)\right)\right)$$

$$-\sum_{k=1}^l T_k^\theta\left(\theta_k\left(\theta_k^1(w)\right)\right)+\sum_{k=1}^l T_k^\theta\left(\theta_k\left(\theta_k^1(w)\right)\right)-\sum_{k=1}^l T_k^\psi \psi_k\left(\psi_k^1(u)\right)+\sum_{k=1}^l T_k^\psi\ \psi_1\left(\psi_1^1(w)\right)$$

$$\geq\left(\xi+\sum_{j \in I_g} \lambda_j^g \xi_j^g+\sum_{m=1}^g\left[\lambda_m^h\ \mu_m+\lambda_m^h\ \mu_m'\right]\right) \sum_{K=1}^l\left[T_k^\theta \xi_j^\theta+T_k^u\ \xi_k^\psi\right], \eta^T(u,w)+\rho \| \theta(u,w)\|^2$$

From equation (2.2), there exists $\bar{\xi}\ \in \operatorname{con} \partial^*\left(F\left(F^1(u)\right)\right)$

$$\xi_j^g \in \operatorname{con} \partial^* g_j\left(G_j(u)\right), \bar{\lambda}_m \in \operatorname{con} \partial^* h_m\left(H_m(u)\right),\ \bar{\mu}_m \in \operatorname{con} \partial^*\left(-h_m\left(H_m\right)(w), \xi_j^\theta \in \operatorname{con} \partial^*\left(-\theta_j\left(\theta_j^1(w)\right)\right)\right)$$

and $\bar{\xi}_j^\psi\ \in \operatorname{con} \partial^*\left(-\psi_\kappa\left(\psi_k^1\right)\right)(w)$ s.t

$$\bar{\xi}+\sum_{j \in I_g} \lambda_j^g \bar{\xi}_j^g+\sum_{n=1}^q\left[\lambda_m^h \bar{\mu}_m+\lambda_m^h \bar{\mu}_m+\sum_{K=1}^l T_i^\theta\ \bar{\xi}_k^\theta+T_l^\psi \bar{\xi}_i^{\ \psi}=0\right]$$

Thus, we have

$$b(u,v)\left(F\left(F^{l}(n)\right)\right)-F\left(F^{l}(w)\right)+\sum_{j\in I_g}\lambda_j^g g_j\left(G_j(u)\right)-\sum_{j\in I_g}\lambda_j^g g_j\left(G_j(w)\right)+\sum_{m=1}^{q}\lambda_m^h\ h_m\left(H_m(u)\right)-\sum_{m=1}^{q}\lambda_m^h\ h_m\ (w)$$

$$-\sum_{m=1}^{q}\mu_m^h\ h_m\left(H_m(u)\right)+\sum_{m=1}^{q}\mu_m^h\ h_m\ \left(H_m(w)\right)+\sum_{k=1}^{l}T_L^{\theta}\ \theta_l\left(\theta_l^l(u)\right)+\sum_{k=1}^{l}T_l^{\theta}\ \left(\theta_l^{\theta}(w)\right)-\sum_{k=1}^{l}T_l^{\psi}\ \psi\left(\psi_l^l(u)\right)+\sum_{k=1}^{l}T_l^{\psi}\left(\psi_l^l(w)\right)\geq 0$$

Now, by applying the feasibility condition of MPEC1, i.e. $g_i\left(g_j(u)\right)\leq 0, h_m\left(H_m(u)\right)=0$,
$\theta_l\left(\theta_l^l(u)\right)\geq 0,\ \psi_{\lambda}\left(\psi_l^l(u)\right)\geq 0,$ hence it follows that

$$b(u,w)\left\lfloor F\left(F^{l}(u)-F\left(F^{l}(w)\right)\right)\right\rfloor-\sum_{j\in I_g}\lambda_j^g g_j\left(G_j(u)\right)$$

$$-\sum_{m=1}^{q}\lambda_m^h\ h_m\left(H_m(u)\right)+\sum_{m=1}^{q}\gamma_m^h\ h_m\left(H_m(w)\right)+\sum_{k=1}^{l}T_l^{\theta}\ \theta\left(\theta_l^l(u)\right)+\sum_{k=1}^{l}T_l^{\psi}\psi_l\left(\psi_l^l(u)\right)\geq 0$$

Hence, it follows that

$$b(u,w)\left[F\left(F^{l}(u)\right)-F\left(F^{l}(w)\right)\geq\sum_{j\in I_g}g_j\left(G_j(u)\right)\right]+\sum_{m=1}^{q}\lambda_m\ h_m\left(H_m(u)\right)-\sum_{k=1}^{l}\left[T_l^{\theta}\ \theta_l\left(\theta_l^l(u)\right)+T_l^{\psi}\psi_l\left(\psi_l^l(u)\right)\right]$$

Hence proved.

Theorem 4.2 (Strong Duality)

If $\tilde{u}$ be a local optimal solution for the problem MPEC1 and also assume that $F\left(F^{l}\right)$ is locally Lipschitz near $\tilde{u}$. Let us suppose that $F\left(F^{l}\right), g_j\left(G_j\right),\left(j\in I_g\right)$, $\pm h_m\left(H_m\right)(m=1,2,......,q)$, $-\theta_l\left(\theta_l^l\right)(1\in I\cup\delta),-\psi_l(1\in I\cup K)$ admit bounded upper semi – regular convexificators and are $\partial^{*}-V-\rho-$ univex function at $\tilde{u}$ with respect to the same common Kernel η and θ_1. According to [16, 30], if Gs – AC on holds at $\tilde{u}$ then there exists $\tilde{\mu}=\left(\tilde{\mu}^{g},\tilde{\mu}^{h},\tilde{\mu}^{\theta},\tilde{\mu}^{\psi}\right)\in\ R^{k+p+2l},\ni\left(\tilde{u},\tilde{\mu}\right)$ is an optimal solution of the dual (CWD) and the corresponding objective values of MPEC1 and CWD are equal.

Proof:

If $\tilde{u}$ be local optimal solution for the problem MPEC1 and the Generalized Slater and ACQ is satisfied at $\tilde{u}$ and also by using corollary 4.6 of [1, 16], there exists $\tilde{\mu}=\left(\tilde{\mu}^{g},\tilde{\mu}^{h},\tilde{\mu}^{\theta},\tilde{\mu}^{\psi}\right)\in R^{K+P=2ll}\ni$ the $\tilde{\lambda}\in\left(\tilde{\lambda}^{h},\tilde{\lambda}^{\theta},\tilde{\lambda}^{\psi}\right)$ $\in R^{\ p+2l}\in$ the corresponding GS – stationary conditions for the problem MPEC1 are satisfied, it follows that, there exists
By applying theorem 4.1, we get

$\tilde{\xi}\in\operatorname{con}\partial^{*}F\left(F^{l}(\tilde{u})\right)$,

$\xi_j^g\in\operatorname{con}\partial^{*}g_j\left(G_j(\tilde{u})\right),\tilde{\lambda}_m\in\operatorname{con}\partial^{*}\ h_m\left(H_m(\tilde{u})\right)$

$\tilde{\mu}_m\in\operatorname{con}\partial^{*}\left(-h_m\left(H_m\right)(\tilde{u})\right)$,

$\tilde{\xi}_l^{\theta}\in\operatorname{con}\partial^{*}\left(-\theta_l\left(\theta_l^l(\tilde{u})\right)\right)$,

$$\tilde{\xi}+\sum_{j\in I_g}\tilde{\mu}_j^g\ \tilde{\xi}_l^g+\sum_{m=1}^{q}\left[\tilde{\mu}_j\tilde{\lambda}_m+\tilde{\mu}_m T_m^h\right]+\sum_{K=1}^{l}\left[T_l^{\theta}\tilde{\xi}_i^{\theta}+T_l^{\psi}\tilde{\mu}_m\right]=0$$

$\tilde{\mu}_j^g\geq 0,\ \tilde{\mu}_j^h\geq 0,\ m=1,2,........,p$

$T_k^{\theta}\geq 0\ ,T_k^{\psi}\geq 0\ \ k=1,............,l$ and the remaining
$T_k^{\theta}=T_k^{\psi}=\tilde{\xi}_k^{\theta}=\tilde{\xi}_k^{\theta}=\tilde{\xi}_k^{\psi}=0$

$\forall j\in I,\tilde{\xi}_k^{\theta}=T_k^{\psi}=0$

$\therefore\left(\tilde{u},\tilde{\mu}\right)$ is feasible for the (CWD).

$$b(u,\mu)\left(F\left(F^{1}(\tilde{u})\right)-F\left(F^{1}(\mu)\right)\right)\geq\sum_{j\in Ig}\tilde{\mu}_{j}^{g}g_{j}\left(G_{j}(\mu)\right)+\sum_{m=1}^{q}\lambda_{m}^{h}h_{m}\left(H_{m}(\mu)\right)-\sum_{k=1}^{l}T_{k}^{\theta}\theta_{K}\left(\theta_{k}^{l}(\mu)\right)+T_{k}^{\psi}\psi_{k}\left(\psi_{k}'(\mu)\right) \quad (4.9)$$

Here λ_m^h represents $\mu_m^h - T_m^h$ for any feasible solution (μ, T) for the considered dual (CWD).

Now by applying the feasibility condition of the problem MPEC1 and its corresponding dual CWD, $\exists j\in I_g,(\tilde{u}),g_j\left(G_j(\tilde{u})\right)=0,$

$$h_m\left(H_m(\tilde{u})\right)=0,(m=1,2,.....q),\theta_k\left(\theta_k^l(\tilde{u})\right)=0$$

$\forall k\in\delta\cup I,$ and $\psi_K\left(\psi_K^l(\tilde{u})\right)=0,\ \forall\ k\in I\cup K,$ it follows that

$$b(u,\tilde{u})\left(F\left(F^{1}(u)\right)-F\left(F^{1}(\tilde{u})\right)\right)\geq\sum_{j\in I_g}\tilde{\mu}_j^g\ g_j\left(G_j(\tilde{u})\right)$$

$$+\sum_{m=1}^{v}\tilde{\lambda}_m^h\ h_m\left(H_m(u)\right)-\sum_{k=1}^{l}\left[T_k^\theta\ \theta_k\left(\theta_k\left(\theta_k^l\right)\tilde{u}\right)\right]+T_k^\psi\psi_k\left(\psi_k^l(\tilde{u})\right) \quad (4.10)$$

Applying (4. 9) and (4.10) and the indices $\tilde{\lambda}_m^h\ T_m^h-\tilde{\mu}_m^h,$ we set

$$b(\tilde{u},\tilde{\mu})\left[\left(F\left(F^{1}\right)(\tilde{u})\right)-F\left(F^{1}\right)(\tilde{\mu})\right]+\sum_{j\in I_g}\tilde{\mu}_j^g g_j\left(G_j(\tilde{u})\right)+\sum_{m=1}^{q}\lambda_m^h\ h_m\left(H_m(\tilde{u})\right)-\sum_{k=1}^{l}\left(\tilde{T}_k^\theta\theta_k\left(\theta_k^l(\tilde{u})\right)\right)$$

$$+\tilde{T}_k^\psi\ \psi_k\left(\psi_k^l(\tilde{u})\right)+\eta^T(\tilde{u},\tilde{\mu})+\rho\,\|\,\theta_1(\tilde{u},\tilde{\mu})\,\|^2$$

$$\geq b(\tilde{u}_1,\tilde{v}_1)\left[F\left(F^{1}(\tilde{u}_1)-F\left(F^{1}(\tilde{v}_1)\right)\right)\right]$$

$$+\sum_{j\in I_g}\tilde{\mu}_j^g\ g_j\left(G_j(\tilde{u}_1)\right)+\sum_{m=1}^{q}\lambda_m^h\ h_m\left(H(\tilde{v}_1)\right)$$

$$-\sum_{k=1}^{l}\left[T_K^\theta\theta_k\left(\theta_k^l(\tilde{u}_1)+T_k^\psi\ \psi_k\left(\psi_k^l(\tilde{v}_1)\right)\right)\right]+\eta^T(\tilde{u}_1,\tilde{v}_1)+\rho\,\|\,\theta_1(\tilde{u}_1,\tilde{v}_1)\,\|^2$$

Thus $(\tilde{u}_1,\tilde{v}_1)$ is an optimal solution for the dual problem CND and respective objective values of the two problems MPEC1 and CWD are equal.

5. MOND-WEIR DUALITY

Here, we will formulate and discuss the Mond-Weir type dual problem (CMWD) for the chosen problem MPEC1. Finally we will discuss duality theorem by using convexificators.

(CMWD) $\max\left\{F\left(F^{1}\right)(u)\right\}$

subject to conditions:

$$O\in\operatorname{con}\partial^{*}F\left(F^{1}(u)\right)+\sum_{j\in I_g}\lambda_j^g\operatorname{con}\partial^{*}g_j\left(G_j(u)\right)$$

$$+\sum_{m=1}^{q}\left(\mu_m^h\ \operatorname{cond}\partial^{*}h_m\left(H_m(u)\right)+T_m^h\operatorname{con}\partial^{*}\left(-h_mH_m(u)\right)\right)$$

$$+\sum_{k=1}^{l}\lambda_k^Q\ \operatorname{con}\partial^{*}\left(\theta_k\left(-\theta_k^l\right)(u)+\lambda_k^\psi\operatorname{con}\partial^{*}\left(-\psi_k\left(\psi_k^l(u)\right)\right)\right)$$

$$g_j\left(G_j(u)\right)\geq 0\ \left(j\in I_g\right),h_m\left(H_m(u)\right)=0,\ m=1,2,.......,q$$

$$\theta_k\left(\theta_k^l(u)\right)\leq 0\left(k\in\delta\cup I\right),$$

$$\psi_k\left(\psi_k^l(u)\right)\leq 0\ \left(k\in I\cup K\right).$$

$$\lambda_j^g\geq 0,\ \mu_m^h,\ T_m^h\geq 0,m=1,2,......,q$$

$\lambda_k^{\theta}, \lambda_k^{\psi}, \mu_k^{\theta}, \mu_k^{\psi} \geq 0, \ k = 1,2,.....,l.$

$$\lambda_k^{\theta} = \lambda_g^{\psi} = T_k^{\theta} = T_g^{\psi} = 0, \ \forall i \in I, \tag{5.1}$$

Here $\lambda = \left(\lambda^{\theta}, \mu^{h}, T^{h}, T^{\psi}\right) \in R^{k+p+2l}$.

and $T = \left(T^{h}, T^{\theta}, T^{\psi}\right) \in R^{p+l+l} = R^{p+2l}$

Theorem 5.1 (Weak Dualtiy)

Suppose $\tilde{u}_1$ is a feasible solution for the problem MPEC, and (v_1, τ_1) be a feasible solution for the corresponding dual (CMWD) and also the index sets I_g, δ, I, K be defined accordingly. Also, suppose that the composite functions $F\left(F^{l}\right), g_j\left(G_j\right), \pm h_m\left(H_m\right), (m = 1,2,.......q)$, $-\theta_k\left(\theta_k^{l}\right)(k \in \delta \cup I), -\psi_k\left(\psi_k^{l}\right)(k \in I \cup K)$ admit bounded upper semi – regular convexificators and are $\partial^* - V - \rho -$ univex functions at v_1, with respect to the some common η and θ_1. If we denote $I_{\gamma}^{\theta} \cup I_{\gamma}^{\psi} \cup \delta_{\gamma}^{+} \cup K_{+}^{+} = \phi$, then for any u_1 is feasible for the problem MPEC1, then

$$F\left(F^{l}\left(u_1\right)\right) \geq F\left(F^{l}\left(\tilde{u}_1\right)\right)$$

Proof: Proof is similar to theorem 4.1

Theorem 5.2 (Strong Duality)

If $\tilde{u}_1$ is a local optimal solution for the problem MPEC1 and let $F\left(F^{l}\right)$ be locally Lipschitz nearer at $\tilde{u}_1$ Also let us suppose that $F\left(F^{l}\right), g_j\left(G_j\right), j \in I_g, \pm h_m\left(H_m\right)$ $m = 1,......,q, \ -\theta_k\left(\theta_k^{l}\right), \quad k \in \delta \cup I, -\psi_k\left(\psi_k^{l}\right)$ $(k \in I \cup K)$ admit bounded upper semi – regular convexificators and are $\partial^* - V - \rho -$ univex functions at $\tilde{u}_1$ with respect to the some common kernals η_1 and θ_1. According to [1, 16], if GS – ACQ holds at $\tilde{u}_1$ then $\exists$ $\tilde{\tau}_1, \ni \left(\tilde{u}_1, \tilde{\tau}_1\right)$ is an optimal solution of the CMWD and the respective objective function values of MPEC1 and (CMWD) are equal.

Proof: This theorem can be proved as in Theorem 4.2 by making use of assumptions in the statement.

Theorem 5.3 (Weak Duality)

Suppose $\tilde{u}_1$ is a feasible, solution for the considered problem MPEC, $\left(\tilde{v}_1, \tilde{\tau}_1\right)$ is another feasible solution for the corresponding dual CMWD and the index sets I_g, δ_1, W_1, K_1 defined accordingly. Also, let us suppose that $F\left(F^{l}\right)$ is $\partial^* - V - \rho$ pseudounivex at v_1. with respect to the some common kernals η_1 and θ_1 and $g_j\left(G_j\right)\left(j \in I_g\right), \pm h_m\left(H_m\right)$ admit bounded upper semi-regular convexificators and are $\partial^* - V - \rho$ quasi univex functions as with respect to the some common Kernals η_1 and θ_1 Let us suppose, if $w_{\gamma}^{\theta} \bigcup w_{\gamma}^{+} \cup K_{\gamma}^{+} = \psi$, then for any other u_2 feasible for the problem MPEC1, it follows that $F\left(F^{l}\left(\tilde{u}_1\right)\right) \geq F\left(F^{l}\right)\left(\tilde{v}_1\right)$.

Proof: Let us assume that $\tilde{u}_1$ as some feasible point, $\exists\left(F\left(F^{l}\right)\left(\tilde{u}_1\right)\right) \leq F\left(F^{l}\right)\left(\tilde{v}_1\right)$.

Then by definition of $\partial^* - V - \rho$ pseudounivexity of $F\left(F^{l}\right)$ at $\tilde{v}_1$ with respect to some common kernals η_1 and θ_1, we have

$$\left(\xi_1, \eta_1^{T}\right)\left(\tilde{u}_1, \tilde{v}_1\right) + \rho_1 \,||\, \theta_1\left(\tilde{u}_1, \tilde{v}_1\right) ||^2 < 0$$
$$\forall \tilde{\xi}_1 \in \partial^* F\left(F^{l}\left(\tilde{v}_1\right)\right) \tag{5.2}$$

But from (4.2), there exists $\tilde{\xi}_1$ con $\partial^* F\left(F^{l}\left(\tilde{v}_1\right)\right)$,

$\tilde{\xi}_j^{g} \in \text{con}\,\partial^* g_j\left(G_j\left(\tilde{v}_1\right)\right), \ \tilde{\lambda}_m \in \text{con}\ \partial^* h_m\left(H_m\left(\tilde{v}_1\right)\right)$, $\tilde{\mu}_m \in \text{con}\,\partial^*\left(-h_m\left(-H_m\right)\right)\left(\tilde{v}_1\right), \tilde{\xi}_k^{g} \in \text{con}\,\partial^*\left(-\theta_k\left(\theta_k^{l}\right)\right)\left(\tilde{v}_1\right)$ and $\tilde{\xi}_k^{\psi} \in \text{con}\,\partial^*\left(\psi_k\left(\psi_k^{l}\right)\left(\tilde{v}_1\right)\right), \ni$

$$-\sum_{j \in I_g} \tau_j^{g} \tilde{\xi}_j^{g} - \sum_{m=1}^{q}\left[\tau_m^{h} \ \tilde{\tau}_m + \gamma_m^{h} \tilde{\mu}_m\right]$$
$$-\sum_{\delta_1 \cup W_1} \tau_k^{\theta} \ \tilde{\xi}_k^{\theta} - \sum_{w_1 \cup k_1} \tau_l^{\psi} \ \tilde{\xi}_i^{\psi} \in \partial^* F\left(F^{l}\left(\tilde{v}_1\right)\right) \tag{5.3}$$

Applying (5.2) one can obtain

$$\left(\sum_{j \in I_g} \tau_j^{g} \ \tilde{\xi}_j^{g} + \sum_{m=1}^{q}\left[\tau_m^{h} \ \tilde{\tau}_m + \gamma_m^{h} \ \tilde{\mu}_m\right] + \sum_{\delta_1 \cup W_1} \tau_k^{\theta} \tilde{\xi}_k^{\theta} + \sum_{w_1 \cup k_1} \tau_k^{\psi} \tilde{\xi}_k^{\psi}\right),$$
$$\eta_1^{T}\left(\tilde{u}_1, \tilde{v}_1\right) + \rho_1 \,||\, \theta_1\left(\tilde{u}_1, \tilde{v}_1\right) ||^2 > 0 \tag{5.4}$$

Now for each $j \in I_g, g_j\left(G_j\left(\tilde{u}_1\right)\right) \leq 0 \leq g_j\left(G_j\right)\left(\tilde{v}_1\right)$.

By definition of $\partial^* - V - \rho -$ quasiunivexity, we get

$$\left[\tilde{\xi}_j^{g}, \eta_1^{T}\left(\tilde{u}_1, \tilde{v}_1\right) + \rho_1 \,||\, \theta_1\left(\tilde{u}_1, \tilde{v}_1\right) ||^2\right]$$
$$\leq 0, \ \forall \xi_j^{g} \partial^* g_j\left(G_j\left(\tilde{v}_1\right)\right), \forall j \in I_g \tag{5.5}$$

Similarly, we can write the remaining as,

$$\left\langle \tau_m, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle, \forall \tau_m \in \partial^* \tau_m, F\left(F^1(\tilde{v}_1)\right)$$

$$\left\langle \mu_m, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle \le 0,\ \forall \mu_m \in \partial^*\left(-h_m(H_m)(\tilde{v}_1)\right),$$

$$\forall m = 1,2,........,q \tag{5.6}$$

$$\left\langle \xi_j^\theta, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\| \right\rangle \le 0, \xi_j^\theta \in \partial^*\left(-\theta_k\left(\theta_k^l(v_1)\right)\right),\ \forall k \in \delta_1 \cup W_1. \tag{5.7}$$

$$\left\langle \xi_k^\psi, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0, \forall \in_K^\psi \in \partial^* - \psi_k\left(\psi_k^1(\tilde{v}_1)\right)\right\rangle, \forall k \in w_1 \cup K_1. \tag{5.8}$$

Above equations (5.3) – (5.8) given

$$\left\langle \tilde{\xi}_j^g, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0, \left(\forall j \in I_g\right)\right\rangle,$$

$$\left\langle \tau_m,\ \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0, \forall m = 1,2,.....,q\right\rangle,$$

$$\left\langle \tilde{\mu}_m, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0,\ \forall m = 1,2,.....q\right\rangle,$$

$$\left\langle \tilde{\xi}_K^\theta, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle, \forall l \in \delta_1 \cup W_1$$

$$\left\langle \tilde{\xi}_k^\theta, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0\right\rangle, \forall k \in k_1 \cup W_1$$

By hypothesis, since $w_\gamma^\theta \cup w_\gamma^\psi U \delta_\gamma^+ \cup K_\gamma^+ = \phi,$ we set

$$\left\langle \sum_{j \in I_g} \tau_j^q\, \tilde{\xi}_j^g, \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle \le 0$$

$$\left\langle \sum_{m=1}^{q} \left[\tau_m^h\ \tilde{\lambda}_m + \lambda_m^h\, \tilde{\mu}_m\right], \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle \le 0$$

$$\left\langle \sum_{\delta_1 \cup w_1} \tau_k^\theta\, \tilde{\xi}_k^\theta,\ \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \right\rangle \le 0$$

$$\left\langle \sum_{W_1 \cup K_1} \tau_k^\psi\, \tilde{\xi}_k^\psi,\ \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0 \right\rangle$$

Hence, we obtain as

$$\left\langle \left(\sum_{j \in I_g} \tau_j^\theta\ \tilde{\xi}_j^g + \sum_{m=1}^{q} \tau_m^h\, \tilde{\lambda}_m + \lambda_m^h \tilde{\mu}_m\right) + \sum_{\delta_1 \cup w_1} \tau_k^\theta\, \tilde{\xi}_k^q + \sum_{w_1 \cup K_1} \tau_k^\psi \tilde{\xi}_k^\psi \right\rangle,\ \eta_l^T(\tilde{u}_1,\tilde{v}_1) + \rho_1 \| \theta_1(\tilde{u}_1,\tilde{v}_1)\|^2 \le 0$$

Which is a contradiction to equation (5.8)

Hence $b(\tilde{u}_1,\tilde{v}_1)\left\{\left(F\left(F^1(\tilde{u}_1)\right)\right) - \left(F\left(F^1(\tilde{v}_1)\right)\right)\right\} \ge 0$

$$\Rightarrow F\left(F^1(\tilde{u}_1)\right) \ge \left(F\left(F^1(\tilde{v}_1)\right)\right)$$

Thus, the result is proved.

Theorem (5.4) (Straong Duality)

Suppose $\tilde{u}$ is a local optimal solution for the problem MPEC1 and also let F (F^1) be a locally Lipschitz near at $\tilde{u}_1$. Let in suppose that F (F^1) is $\partial^* - V - \rho -$ pseudounivex at $\tilde{u}$ with respect to the some common kernals at η_1 and $\theta_1, g_j(G_j)(j \in I_g), \pm h_m(H_m), (m = 1, ..., q), -\theta_k(\theta_k^1), (k \in \delta_1 \cup W_1), -\psi_k(\psi_k^1),$ $(K \in W_1 \cup K_1)$ admit bounded upper semi-regular convenificators and are $\partial^* - V - \rho$ quasi univex functions at $\tilde{u}_1$ with respect to the some common kernals η_1 and θ_1. Also, let us suppose that, if $GS - ACQ$ of [1, 16] holds at $\tilde{u}_1$ then $\exists \tilde{\tau}_1 \in (\tilde{u}_1, \tilde{\tau}_1)$ is an optimal solution of the dual (CMWD), therefore, the corresponding objective values are equal.

Proof: This theorem can be proved similar to the theorem 4.2 by making use of the assumption stated in theorem 5.4

CONCLUSION

In this paper we derived generalized duality theorems of Wolfe type and Mond-Weir type with respective to generalized univexity. These results are the generalizations of [16, 31].

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